Proof by contradiction

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1 Warm-up

Putnam 2004/A1. Basketball star Shanille O'Keal's team statistician keeps track of the number, $S(N)$, of successful free throws she has made in her first N attempts of the season. Early in the season, $S(N)$ was less than 80% of N, but by the end of the season, $S(N)$ was more than 80% of N. Was there necessarily a moment in between when $S(N)$ was exactly 80% of N?

2 Classical results

1. Prove that there are infinitely many prime numbers. (A number n is prime if its only divisors are 1 and n.)
2. Prove that $\sqrt{2}$ is irrational, i.e., that it cannot be expressed in the form a/b where both a, b are integers.
3. (Discrete Helly's Theorem.) Let d be a positive integer, and let $\mathcal{F} = \{E_1, \ldots, E_n\}$ be a family of d-element subsets of some set X, with $n > d$. Suppose that every choice of $d+1$ of the sets E_i has nonempty intersection. Show that the intersection of all E_i is nonempty.

3 Problems

VTRMC 2006/1. Find, and give a proof of your answer, all positive integers n such that neither n nor n^2 contain a 1 when written in base 3.

Putnam 2006/A2. Alice and Bob play a game in which they take turns removing stones from a heap that initially has n stones. The number of stones removed at each turn must be one less than a prime number. The winner is the player who takes the last stone. Alice plays first. Prove that there are infinitely many n such that Bob has a winning strategy. (For example, if $n = 17$, then Alice might take 6 leaving 11; then Bob might take 1 leaving 10; then Alice can take the remaining stones to win.)

Putnam 2003/B1. Do there exist polynomials $a(x)$, $b(x)$, $c(y)$, $d(y)$ such that

$$1 + xy + x^2y^2 = a(x)c(y) + b(x)d(y)$$

holds identically?

Putnam 2008/A1. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a function such that $f(x,y) + f(y,z) + f(z,x) = 0$ for all real numbers x, y, and z. Prove that there exists a function $g : \mathbb{R} \to \mathbb{R}$ such that $f(x,y) = g(x) - g(y)$ for all real numbers x and y.

Putnam 2006/B1. Show that the curve $x^3 + 3xy + y^3 = 1$ contains only one set of three distinct points, A, B, and C, which are vertices of an equilateral triangle, and find its area.