Yellow MOP Lecture: 9–10 July 2002.
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1 Triangles

1.1 Facts

1. **Extended Law of Sines** $a/\sin A = 2R$.

2. $[ABC] = abc/4R$.

3. (Geometry Revisited, page 3.) Let p and q be the radii of two circles through A, touching BC at B and C, respectively. Then $pq = R^2$.

4. **Ceva** Given triangle ABC. Let $D \in BC$, $E \in CA$, and $F \in AB$. Suppose that:

$$\frac{AF}{FB}\frac{BD}{DC}\frac{CE}{EA} = 1.$$

Prove that AD, BE, and CF are concurrent.

5. **Trig Ceva** Given triangle ABC. Let $D \in BC$, $E \in CA$, and $F \in AB$. Suppose that:

$$\frac{\sin CAD}{\sin DAB}\frac{\sin ABE}{\sin EBC}\frac{\sin BCF}{\sin FCA} = 1.$$

Prove that AD, BE, and CF are concurrent.

6. Prove that the centroid of a triangle lies 2/3 of the way down each median.

7. **Steiner-Lehmus** Let ABC be a triangle such that the lengths of two angle bisectors are equal. Prove that ABC is isosceles.

8. (Geometry Revisited, page 13.) Prove that $abc = 4srR$.

9. (Geometry Revisited, page 13.) Let r_a, r_b, and r_c be the radii of the three excircles of triangle ABC. Prove that $1/r = 1/r_a + 1/r_b + 1/r_c$.

10. **Orthic Triangle** The feet of the altitudes of triangle ABC determine a triangle, called the *orthic triangle*. Prove that the orthocenter of ABC is the incenter of that triangle.

11. **Euler Line** Let O, G, and H be the circumcenter, centroid, and orthocenter of ABC, respectively. Prove that O, G, and H are collinear, and that $HG = 2GO$.

1.2 Problems

1. (Nordic 1998.2). Let C_1 and C_2 be two circles which intersect at A and B. Let M_1 be the center of C_1 and M_2 the center of C_2. Let P be a point on the segment AB such that $|AP| \neq |BP|$. Let the line through P perpendicular to M_1P meet C_1 at C and D, and let the line through P perpendicular to M_2P meet C_2 at E and F. Prove that C, D, E, F are the vertices of a rectangle.

   Solution: It is already clear that the diagonals of $CEDF$ bisect each other, so it suffices to show that they are the same length. But since P is on the radical axis of C_1 and C_2, it must have equal power with respect to the two circles; this implies that $PF = PD$, so we are done.

2. (Belarus 2000.1). Let M be the intersection point of the diagonals AC and BD of a convex quadrilateral $ABCD$. The bisector of angle ACD hits ray BA at K. If $MA \cdot MC + MA \cdot CD = MB \cdot MD$, prove that $\angle BKC = \angle CDB$.

   Solution: N is isection CK, BD. ABT on MCD means that $CD/DN = MC/MN$, or $CD = MC \cdot DN/MN$. Then $MB * MD = MA * MC + MA * MC * ND/MN = MA * MC * MD/MN$, and power of a point with M in $QBCN$. Now $KBD = ABN = ACN = NCD = KCD$ so K, B, C, D concyclic. Hence $BKC = CDB$.

3. (UK 1998.3). Let ABP be an isosceles triangle with $AB = AP$ and $\angle PAB$ acute. Let PC be the line through P perpendicular to BP, with C a point on the same side of BP as A (and not lying on AB). Let D be the fourth vertex of parallelogram $ABCD$, and let PC meet DA at M. Prove that M is the midpoint of DA.

   Solution: Let X be the intersection of the altitude from A and BC. We will find congruent triangles: $AB = CD$, $\angle BAX = \angle DCM$, and $\angle ABC = \angle ADC$ by the parallelogram. Therefore, $MD = BX$. But since the altitude is the midline of triangle CDB, and since $MA = XC$ by parallelogram, we are done.

2 Brutal Force

1. (Razvan, 6/19/98, *Quadrilaterals* #6) Prove that if in a convex quadrilateral two opposite angles are congruent, the bisectors of the other two angles are parallel.

   Solution: True for parallelogram; then tilt to a general case.

2. (Razvan, 6/19/98, *Quadrilaterals* #13) Prove that the interior bisectors of the angles of a parallelogram form a rectangle whose diagonals are parallel to the sides of the parallelogram.

   Solution: True for rectangle; tilt to general case.

3. (Rookie Contests 1998, 1999, 2000, Po's Star Theorem) Given two congruent circles, ω_1 and ω_2. Let them intersect at B and C. Select a point A on ω_1. Let AB and AC intersect ω_2 at A_1 and A_2. Let X be the midpoint of BC. Let A_1X and A_2X intersect ω_1 at P_1 and P_2. Prove that $AP_1 = AP_2$.

   Solution: True for symmetric case; perturb A by θ. Then A_1 and A_2 move by θ (vertical angles), and P_1 and P_2 also move by θ (symmetry through X). Therefore done.

4. (Russia 1998.14). A Circle S centered at O meets another circle S' at A and B. Let C be a point on the arc of S contained in S'. Let E, D be the second intersections of S' with AC, BC, respectively. Show that $DE \perp OC$.

   Solution: Clearly true for symmetric case; for perturbation, angles flow around as usual.

3 Collinearity and Concurrence

3.1 Definitions

Definition 1 *Let ω be a circle with center O and radius r, and let P be a point. Then the **power** of P with respect to ω is $OP^2 - r^2$. Note that the power can be negative.*

Definition 2 *Let ω_1 and ω_2 be two circles. Then the **radical axis** of ω_1 and ω_2 is the locus of points with equal power with respect to the two circles. This locus turns out to be a straight line.*

Definition 3 *Two triangles ABC and DEF are **perspective from a point** when AD, BE, and CF are concurrent.*

Definition 4 *Two triangles ABC and DEF are **perspective from a line** when $AB \cap DE$, $BC \cap EF$, and $CA \cap FD$ are collinear.*

3.2 Arsenal

Ceva Let ABC be a triangle, and let $D \in BC$, $E \in CA$, and $F \in AB$. Then AD, BE, and CF concur if and only if:

$$\frac{AF}{FB}\frac{BD}{DC}\frac{CE}{EA} = 1.$$

Trig Ceva Let ABC be a triangle, and let $D \in BC$, $E \in CA$, and $F \in AB$. Then AD, BE, and CF concur if and only if:

$$\frac{\sin CAD}{\sin DAB}\frac{\sin ABE}{\sin EBC}\frac{\sin BCF}{\sin FCA} = 1.$$

Radical Axis Let $\{\omega_k\}_1^3$ be a family of circles, and let ℓ_k be the radical axis of ω_k and ω_{k+1}, where we identify ω_4 with ω_1. Then $\{\ell_k\}_1^3$ are concurrent.

Brianchon Let circle ω be inscribed in hexagon $ABCDEF$. Then the diagonals AD, BE, and CF are concurrent.

Identification Three lines AB, CD, and EF are concurrent if and only if the points A, B, and $CD \cap EF$ are collinear.

Desargues Two triangles are perspective from a point if and only if they are perspective from a line.

Menelaus Let ABC be a triangle, and let D, E, and F line on the extended lines BC, CA, and AB. Then D, E, and F are collinear if and only if:

$$\frac{AF}{FB}\frac{BD}{DC}\frac{CE}{EA} = -1.$$

Pappus Let ℓ_1 and ℓ_2 be lines, let $A, C, E \in \ell_1$, and let $B, D, F \in \ell_2$. Then $AB \cap DE$, $BC \cap EF$, and $CD \cap FA$ are collinear.

Pascal Let ω be a conic, and let $A, B, C, D, E, F \in \omega$. Then $AB \cap DE$, $BC \cap EF$, and $CD \cap FA$ are collinear.

3.3 Problems

1. (Bulgaria 1996.2). The circles k_1 and k_2 with respective centers O_1 and O_2 are externally tangent at the point C, while the circle k with center O is externally tangent to k_1 and k_2. Let ℓ be the common tangent of k_1 and k_2 at the point C and let AB be the diameter of k perpendicular to ℓ. Assume that O_2 and A lie on the same side of ℓ. Show that the lines AO_1, BO_2, and ℓ have a common point.

   Solution: An equivalent way to specify AB is that AB is the diameter of k parallel to O_1O_2. Let $X = AO_1 \cap BO_2$ and note that O_1O_2X and ABX are similar. Now just use trig, where we let r_1, r_2, and R be the respective radii of the circles and ϕ be the angle O_1O_2O.

2. (USAMO 1997.2) Let ABC be a triangle, and draw isosceles triangles BCD, CAE, ABF externally to ABC, with BC, CA, and AB as the respective bases. Prove that the lines through A, B, C perpendicular to the (possibly extended) lines EF, FD, DE, respectively, are concurrent.

   Solution: Construct circles centered at D, E, and F such that they contain BC, CA, and AB as respective chords. Apply radical axis.

3. (Ireland 1996.9). Let ABC be an acute triangle and let D, E, F be the feet of the altitudes from A, B, C, respectively. Let P, Q, R be the feet of the perpendiculars from A, B, C to EF, FD, DE, respectively. Prove that the lines AP, BQ, CR are concurrent.

   Solution: Trig Ceva on orthic triangle.

4. (Po 1999.x). Let Γ be a circle, and let the line ℓ pass through its center. Choose points $A, B \in \Gamma$ such that A and B are on one side of ℓ, and let T_A and T_B be the respective tangents to Γ at A and B. Suppose that T_A and T_B are on opposite sides of O. Let A' and B' be the reflections of A and B across ℓ. Prove that A, B, A', B', $T_A \cap \ell$, and $T_B \cap \ell$ lie on an ellipse.

   Solution: Pascal on $AAB'BBA'$; then $AA \cap BB$, $AB' \cap A'B$, and $A'A' \cap B'B'$ are collinear. Pascal again on $T_A AB' T_B BB'$ yields result.

5. (Bulgaria 1997.10). Let $ABCD$ be a convex quadrilateral such that $\angle DAB = \angle ABC = \angle BCD$. Let H and O denote the orthocenter and circumcenter of the triangle ABC. Prove that H, O, D are collinear.

   Solution: Let M be the midpoint of B and N the midpoint of BC. Let $E = AB \cap CD$ and $F = BC \cap AD$. Then EBC and FAB are isosceles triangles, so $EN \cap FM = O$. By Pappus on $MCENAF$, we get that G, O, D collinear and by Euler Line we are done.