Red MOP Lecture (with solutions): July 16, 2013
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1 Definitions

Definition 1 *Let ω be a circle with center O and radius r, and let P be a point. Then the* ***power*** *of P with respect to ω is $OP^2 - r^2$. Note that the power can be negative.*

Definition 2 *Let ω_1 and ω_2 be two circles. Then the* ***radical axis*** *of ω_1 and ω_2 is the locus of points with equal power with respect to the two circles. This locus turns out to be a straight line.*

Definition 3 *Two triangles ABC and DEF are* ***perspective from a point*** *when AD, BE, and CF are concurrent.*

Definition 4 *Two triangles ABC and DEF are* ***perspective from a line*** *when $AB \cap DE$, $BC \cap EF$, and $CA \cap FD$ are collinear.*

2 Arsenal

Ceva Let ABC be a triangle, and let $D \in BC$, $E \in CA$, and $F \in AB$. Then AD, BE, and CF concur if and only if:

$$\frac{AF}{FB}\frac{BD}{DC}\frac{CE}{EA} = 1.$$

Trig Ceva Let ABC be a triangle, and let $D \in BC$, $E \in CA$, and $F \in AB$. Then AD, BE, and CF concur if and only if:

$$\frac{\sin CAD}{\sin DAB}\frac{\sin ABE}{\sin EBC}\frac{\sin BCF}{\sin FCA} = 1.$$

Radical Axis Let $\{\omega_k\}_1^3$ be a family of circles, and let ℓ_k be the radical axis of ω_k and ω_{k+1}, where we identify ω_4 with ω_1. Then $\{\ell_k\}_1^3$ are concurrent.

Brianchon Let circle ω be inscribed in hexagon $ABCDEF$. Then the diagonals AD, BE, and CF are concurrent.

Identification Three lines AB, CD, and EF are concurrent if and only if the points A, B, and $CD \cap EF$ are collinear.

Desargues Two triangles are perspective from a point if and only if they are perspective from a line.

Menelaus Let ABC be a triangle, and let D, E, and F line on the extended lines BC, CA, and AB. Then D, E, and F are collinear if and only if:

$$\frac{AF}{FB}\frac{BD}{DC}\frac{CE}{EA} = -1.$$

Pappus Let ℓ_1 and ℓ_2 be lines, let $A, C, E \in \ell_1$, and let $B, D, F \in \ell_2$. Then $AB \cap DE$, $BC \cap EF$, and $CD \cap FA$ are collinear.

Pascal Let ω be a conic, and let $A, B, C, D, E, F \in \omega$. Then $AB \cap DE$, $BC \cap EF$, and $CD \cap FA$ are collinear.

3 Problems

1. (Bulgaria 1996.2). The circles k_1 and k_2 with respective centers O_1 and O_2 are externally tangent at the point C, while the circle k with center O is externally tangent to k_1 and k_2. Let ℓ be the common tangent of k_1 and k_2 at the point C and let AB be the diameter of k perpendicular to ℓ. Assume that O_2 and A lie on the same side of ℓ. Show that the lines AO_1, BO_2, and ℓ have a common point.

   Solution. An equivalent way to specify AB is that AB is the diameter of k parallel to O_1O_2. Let $X = AO_1 \cap BO_2$ and note that O_1O_2X and ABX are similar. Now just use trig, where we let r_1, r_2, and R be the respective radii of the circles and ϕ be the angle O_1O_2O.

2. (USAMO 1997.2) Let ABC be a triangle, and draw isosceles triangles BCD, CAE, ABF externally to ABC, with BC, CA, and AB as the respective bases. Prove that the lines through A, B, C perpendicular to the (possibly extended) lines EF, FD, DE, respectively, are concurrent.

   Solution. Construct circles centered at D, E, and F such that they contain BC, CA, and AB as respective chords. Apply radical axis.

3. (Ireland 1996.9). Let ABC be an acute triangle and let D, E, F be the feet of the altitudes from A, B, C, respectively. Let P, Q, R be the feet of the perpendiculars from A, B, C to EF, FD, DE, respectively. Prove that the lines AP, BQ, CR are concurrent.

   Solution. Trig Ceva on orthic triangle.

4. (Po 1999.x). Let Γ be a circle, and let the line ℓ pass through its center. Choose points $A, B \in \Gamma$ such that A and B are on one side of ℓ, and let T_A and T_B be the respective tangents to Γ at A and B. Suppose that T_A and T_B are on opposite sides of O. Let A' and B' be the reflections of A and B across ℓ. Prove that A, B, A', B', $T_A \cap \ell$, and $T_B \cap \ell$ lie on an ellipse.

   Solution. Pascal on $AAB'BBA'$; then $AA \cap BB$, $AB' \cap A'B$, and $A'A' \cap B'B'$ are collinear. Pascal again on $T_AAB'T_BBB'$ yields result.

5. (Bulgaria 1997.10). Let $ABCD$ be a convex quadrilateral such that $\angle DAB = \angle ABC = \angle BCD$. Let H and O denote the orthocenter and circumcenter of the triangle ABC. Prove that H, O, D are collinear.

   Solution. . Let M be the midpoint of B and N the midpoint of BC. Let $E = AB \cap CD$ and $F = BC \cap AD$. Then EBC and FAB are isosceles triangles, so $EN \cap FM = O$. By Pappus on $MCENAF$, we get that G, O, D collinear and by Euler Line we are done.

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