Course Homepage

`http://www.math.cmu.edu/~pikhurko/801/`

Instructor

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Office Hours: Wen 11:00–12:00 and Fri 10:00–11:00 on all class days for the university.

Course Description

The aim of this course is to introduce important non-combinatorial techniques (i.e., those based on tools from linear/higher algebra, topology, analysis, etc) that have been successfully applied to combinatorial problems. Here is a preliminary list of topics I will try to cover or touch upon:

- Lagrangian of hypergraphs
- higher incidence matrices
- intersection theorems
- chromatic number of $\mathbb{R}^n$
- Borsuk's conjecture, including Kahn-Kalai construction and Hinrich's counterexamples from the Leech lattice
- Shannon and Sperner capacities
- Alon's Combinatorial Nullstellensatz
- Wilson's constructions in design theory
- eigenvalues and expanders, including the zig-zag product of Reingold, Vadham and Wigderson
- Gurvits' proof of van der Waerden's permanent conjecture using hyperbolic polynomials
- graphons and Razborov's flag algebras
- Borsuk-Ulam Theorem

A part of the course (2-4 weeks) will be seminar-like: we will have student presentations covering parts of the book *"Using Borsuk-Ulam Theorem: Lectures on Topological Methods in Combinatorics and Geometry"* by Jiri Matousek (Springer, 2003).

Textbooks

Unfortunately, there is no single book available that covers the whole course. In fact, most of material will come from research publications. I will be supplying references for further reading as we go along. My previous lecture notes from 2001 are available at

`http://www.math.cmu.edu/~pikhurko/AlgMet.ps`

Some parts of them will be used in the current course.

Grade

- Homework: 20%
- Two in-class tests: 10+10% (on **March 17 and April 21**)
- Presentations: 20%
- Final exam: 40% (date TBA)

Homework

Each homework problem has a point value and a type (W or C) indicated in brackets:

W: Carefully write down your solutions and bring your work to the class on the due date for me to grade at home.

C: Be prepared to discuss and present your solutions during the class. If you wish, feel free to write sketches of your solutions and use them as a reference during your presentation.

Blackboard

The course Blackboard is available via `http://www.cmu.edu/blackboard/`

I plan to use only the gradebook (under "Tools") to enable students view their grades.

Homework 1. Due January 25

Feel free to search internet and scientific publications to find and use any known results related to homework questions (for this and all future assignments).

Exercise 1 (C2) Show that for every prime power q, we have $\mathrm{ex}(q^2+q+1, C_4) \geq q(q+1)^2/2$.

[Hint: the C_4-free graph we constructed in class has $2(q^2+q+1)$ vertices, so we have to reduce their number by a factor of 2 somehow.]

Exercise 2 (W2) Let a k-graph F have the following property: if $G \not\supset F$ then cloning any vertex of G does not create an F-subgraph. Prove that the Turán density $\pi(F)$ equals the maximum of $k!\,\Lambda_G$ over all F-free k-graphs G.

Open Questions

Open Problem 1 *How many non-isomorphic projective planes of order* 10 *are there?*

Open Problem 2 *Does there exist a projective plane of order* 12*?*

Complementary Reading

`AlgMet` (Sec 12): constructing projective planes.

`Furedi96jctb`: computing the exact value of $\mathrm{ex}(n, C_4)$ for infinitely many n.

`MotzkinStraus65cjm`: $\mathrm{ex}(n, K_{r+1})$ via Lagrangian.

`Sidorenko87mnasussr`: Turan function of $\mathcal{T}_k$ via Lagrangian.

Homework 2. Due February 1

Exercise 3 (W1) Prove that for k-uniform set systems, s-independence implies t-independence, provided $s \le t \le k$.

Open Questions

Open Problem 3 *Prove/disprove that for every k and L there is a real r such that $m(n,k,L) = \Theta(n^r)$. Find at least one example of k and L with r irrational.*

Complementary Reading

`AlgMet` (Sec 9): constructive lower bounds on Ramsey numbers from intersection theorems.

`Frankl86misc`: an old (but nice) survey of intersection theorems with many open questions.

Homework 3. Due February 15

Exercise 4 (C2) Give an explicit coloring showing that for every $t \geq 3$ the Ramsey number $R(t,t)$ is greater than $\binom{t-1}{3}$.

Open Questions

Open Problem 4 Improve the bounds $4 \leq \chi(\mathbb{R}^2) \leq 7$ pointed by Nelson yet in 1950s.

Tons of open questions related to the chromatic number of $\mathbb{R}^n$ and Borsuk's problem.

Complementary Reading

`AlgMet` (Sec 6 & 7): Chromatic number of $\mathbb{R}^n$ and Borsuk's conjecture.

`Raigorodskii01rms`: nice survey of known result on the chromatic number of $\mathbb{R}^n$ and Borsuk's conjecture.

`Hinrichs02dm`: disproving Borsuk's conjecture using the Leech lattice.

Homework 4. Due February 22

Exercise 5 (W2+1) i) Call a function $f : \mathbb{N} \to \mathbb{R}_{\geq 0}$ *sup-additive* if for any $i, j \in \mathbb{N}$ we have $f(i + j) \geq f(i) + f(j)$. Prove that for any sup-additive function the limit $\lim_{n\to\infty} f(n)/n$ exists (it may be possibly infinity).

ii) Let G be a graph. Define $f(i) = \log_2(\alpha(G^i))$. Show that f is sup-additive. Conclude that the Shannon capacity $\Theta(G) = \sup_{n\geq 1}(\alpha(G^n))^{1/n}$ can be equivalently defined as $\lim_{n\to\infty}(\alpha(G^n))^{1/n}$.

Open Questions

Open Problem 5 Alon'98: Show that whp the Shannon capacity of random graph $G_{n,1/2}$ is $O(\log n)$.

Complementary Reading

`Alon98c`: Shannon capacity of union.

`BohmanHolzman03ieeetit`: A lower bound on the Shannon capacity of odd cycles.

Homework 5. Due February 29

Exercise 6 (C2) Let $A = (a_{ij})$ be an $n \times n$ matrix over a field $\mathbb{F}$. The *permanent* of A is $\mathrm{per}(A) = \sum_\sigma \prod_{i=1}^n a_{i,\sigma(i)}$, where the sum is taken over all $n!$ permutations σ of $[n]$. Prove that if $\mathrm{per}(A) \neq 0$, then for any vector $\mathbf{b} \in \mathbb{F}^n$, there is a subset of the rows of A whose sum differs from b in all coordinates.

Complementary Reading

`Alon99cpc`: the paper where the Combinatorial Nullstellensatz is proved (with some applications).

`KezdySnevily02cpc`: application of the Combinatorial Nullstellensatz to tree embedding.

Presentations on Topological Methods in Combinatorics

We will use the book *"Using Borsuk-Ulam Theorem: Lectures on Topological Methods in Combinatorics and Geometry"* by Jiri Matousek (Springer, 2003).

Here is the suggested list of sections to present:

Speaker 1: 2.1–2.2

Speaker 2: one of the two proofs of Tucker's Lemma from 2.3–2.4, 3.1

Speaker 3: 3.2–3.3

Speaker 4: 3.4–3.5

Each speaker will probably need two 50-minute classes. We shall start presentations on April 12 (which would give us 8 full classes). I recommend that everyone revises Section 1 from the book independently before the presentations start. The talks may include some background from Section 1 if this would make them clearer.

Important

There are no classes on March 7 and, after the Spring Break, the classes resume on March 17. Exam 1 will take place during the regular class hour on March 17. Covered material: from the very beginning to the Combinatorial Nullstellensatz (inclusive).

Complementary Reading

`Pyber85c`: upper bound on $\mathrm{ex}(n, k\text{-reg})$.

Homework 6. Due April 11

Exercise 7 (W2) Let G be a d-regular graph. Prove that $-d$ is an eigenvalue of G if and only if G has a component that is a bipartite graph.

Exercise 8 (C3) Let d and $c > 0$ be fixed, $n \to \infty$, and G_n be an (n, d, c)-expander. Show that $\limsup_{n\to\infty} \lambda_n/d < 1$, where λ_n is the second eigenvalue of G_n.

Complementary Reading

`AlonSpencer01`: Section 9.2 presents basic results on eigenvalues and expension property.

`ReingoldVadhanWigderson02am`: The definition and properties of the zig-zag product.

`Gurvits07arxiv`: A new proof of Van der Waerden's Conjecture on permanents.