Course Webpages

http://www.math.cmu.edu/~pikhurko/701/

The course Blackboard is available via http://www.cmu.edu/blackboard/ I plan to use the gradebook (where you can see your current grades) and the discussion board (for viewing/posting messages related to the course).

Course Description

The aim of this course is to provide a graduate-level introduction into main areas of Discrete Mathematics. The topics will include

• Ramsey Theory
  – Ramsey’s Theorem
  – Compactness Principle
  – Van der Waerden’s Theorem
  – Hales–Jewett Theorem

• Generating Functions and Other Counting Techniques
  – Formal power series
  – Exponential Formula
  – Cayley’s Formula

• Extremal Combinatorics
  – Sperner’s Theorem
  – Erdős–Ko–Rado Theorem
  – Kruskal–Katona Theorem
  – Fischer’s Inequality
  – Ray-Chadhuri–Wilson Theorem (and the linear algebra method)
  – Turán Problem

• The Probabilistic Method
  – The basic method
  – First moment/alterations (and Markov’s inequality)
  – Second Moment Method

Office Hours (Wean 7105): Mon 14:00–15:00, Wen 13:05-14:00, Thurs 13:30-15:30
– Chernoff Bounds

• (Time permitting) Some possible further topics such as
  – Razborov’s flag algebras
  – Stability theorem for intersecting families (following Friedhut (2008))
  – Approximating volume of convex bodies (Dyer, Frieze and Kannan (1991))

This is not a complete list of the material covered in 21-701. It is simply a list of the major themes I hope to cover, meant to help in the preparation for the exam.

Books

Unfortunately, there is no single book available that covers the whole course. The following books may be helpful for some parts of the course.

• AS: Alon and Spencer “Probabilistic Method”, 3d edition, Wiley
• D: Diestel “Graph Theory”, 3d edition, Springer;
• GRS: Graham, Rothschild, and Spencer “Ramsey Theory”, 2d edition, Wiley;
• GKP: Graham, Knuth, Patashnik “Concrete Mathematics”, 2d edition, Addison-Wesley;
• J: Jukna “Extremal Combinatorics”, Springer;
• LW: Lint and Wilson “A Course in Combinatorics”, CUP;

NB: Electronic copies of Wilf’s and Diestel’s books (earlier editions) should be available for free from authors’ homepages.

Office Hours and Getting Help

My office hours (Wean Hall 7105) are: Mon 14:00–15:00, Wen 13:05–14:00 and Thurs 13:30–15:30 on all class days for the university.

Normally, I would discourage contacting me by email directly. Instead, you can post your questions to the discussion group at the blackboard. I will try to reply promptly. This way all questions and replies are accessible to all students. Feel free to post not only questions but any comments, follow-ups, etc, related to the course.

Other ways to find me: see me after class or after any ACO seminar. It this does not work, try calling my office phone (412-268-9782) to arrange an individual meeting.
Grade

The TOTAL score consists of

- Homework: 20%
- Two tests: 20+20% (during the class on Oct’13 & Nov’22)
- Presentations 10% (during the last 1-2 weeks)
- Final exam: 30% (Preliminary date: 3:30–5:30pm, Dec’3)

All exams are closed book and notes. The final grade will be determined as follows:

A: TOTAL ≥ 80%
B: 65% ≤ TOTAL < 80%
C: 55% ≤ TOTAL < 65%
D: 45% ≤ TOTAL < 55%

Homework

You should bring written homework solutions on the due date to the class. Please work individually on each assignment: no cooperation for homework is allowed.

Returning Graded Material

I will be distributing the graded homeworks and exams by passing it after a class (and keeping all unclaimed copies in my office). If you prefer that your work is not distributed this way, please let me know and I will individually accommodate such requests.
Some Reference Material on Generating Functions

An introduction to generating functions is written by Wilf [W] and Graham, Knuth and Patashnik [GKP: Concrete Mathematics, Addison, 1989]. Some theory of formal power series can be found, for example, in Niven [“Formal Power Series”, Amer. Math. Monthly 76 (1969) 871–889]. The following useful tables are copied from [GKP].

<table>
<thead>
<tr>
<th>Table 3.34 Generating function manipulations.</th>
</tr>
</thead>
<tbody>
<tr>
<td>[ \alpha f(z) + \beta G(z) = \sum_n (\alpha f_n + \beta g_n) z^n ]</td>
</tr>
<tr>
<td>[ z^m G(z) = \sum_n g_{n-m} z^n, \quad \text{integer } m \geq 0 ]</td>
</tr>
<tr>
<td>[ \frac{G(z) - g_0 - g_1 z - \cdots - g_{m-1} z^{m-1}}{z^m} = \sum_{n \geq 0} g_{n+m} z^n, \quad \text{integer } m \geq 0 ]</td>
</tr>
<tr>
<td>[ G(cz) = \sum_n c^n g_n z^n ]</td>
</tr>
<tr>
<td>[ G'(z) = \sum_n (n+1) g_{n+1} z^n ]</td>
</tr>
<tr>
<td>[ zG'(z) = \sum_n n g_n z^n ]</td>
</tr>
<tr>
<td>[ \int_0^z G(t) , dt = \sum_{n \geq 1} \frac{1}{n} g_{n-1} z^n ]</td>
</tr>
<tr>
<td>[ F(z) G(z) = \sum_n \left( \sum_k f_k g_{n-k} \right) z^n ]</td>
</tr>
<tr>
<td>[ \frac{1}{1-z} G(z) = \sum_n \left( \sum_{k \leq r} g_k \right) z^n ]</td>
</tr>
</tbody>
</table>

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Table 335 Simple sequences and their generating functions.

<table>
<thead>
<tr>
<th>sequence</th>
<th>generating function</th>
<th>closed form</th>
</tr>
</thead>
<tbody>
<tr>
<td>({1,0,0,0,0,0,\ldots})</td>
<td>(\sum_{n \geq 0} [n = 0] z^n)</td>
<td>1</td>
</tr>
<tr>
<td>({0,\ldots,0,1,0,0,\ldots})</td>
<td>(\sum_{n \geq 0} [n = m] z^n)</td>
<td>(z^m)</td>
</tr>
<tr>
<td>({1,1,1,1,1,1,\ldots})</td>
<td>(\sum_{n \geq 0} z^n)</td>
<td>(\frac{1}{1 - z})</td>
</tr>
<tr>
<td>({1,-1,1,-1,1,-1,\ldots})</td>
<td>(\sum_{n \geq 0} (-1)^n z^n)</td>
<td>(\frac{1}{1 + z})</td>
</tr>
<tr>
<td>({1,0,1,0,1,0,\ldots})</td>
<td>(\sum_{n \geq 0} [2 \backslash n] z^n)</td>
<td>(\frac{1}{1 - z^2})</td>
</tr>
<tr>
<td>({1,0,\ldots,0,1,0,\ldots,0,1,0,\ldots})</td>
<td>(\sum_{n \geq 0} [m \backslash n] z^n)</td>
<td>(\frac{1}{1 - z^m})</td>
</tr>
<tr>
<td>({1,2,3,4,5,6,\ldots})</td>
<td>(\sum_{n \geq 0} (n + 1) z^n)</td>
<td>(\frac{1}{(1 - z)^2})</td>
</tr>
<tr>
<td>({1,2,4,8,16,32,\ldots})</td>
<td>(\sum_{n \geq 0} 2^n z^n)</td>
<td>(\frac{1}{1 - 2z})</td>
</tr>
<tr>
<td>({1,4,6,4,1,0,0,\ldots})</td>
<td>(\sum_{n \geq 0} \binom{4}{n} z^n)</td>
<td>((1 + z)^4)</td>
</tr>
<tr>
<td>({1,c,\left(\frac{c}{2}\right),\left(\frac{c}{3}\right),\ldots})</td>
<td>(\sum_{n \geq 0} \binom{c}{n} z^n)</td>
<td>((1 + z)^c)</td>
</tr>
<tr>
<td>({1,c,\left(c+\frac{1}{2}\right),\left(c+\frac{2}{3}\right),\ldots})</td>
<td>(\sum_{n \geq 0} \binom{c+n-1}{n} z^n)</td>
<td>(\frac{1}{(1 - z)^c})</td>
</tr>
<tr>
<td>({1,c,c^2,c^3,\ldots})</td>
<td>(\sum_{n \geq 0} c^n z^n)</td>
<td>(\frac{1}{1 - cz})</td>
</tr>
<tr>
<td>({1,\left(\frac{m+1}{m}\right),\left(\frac{m+2}{m}\right),\left(\frac{m+3}{m}\right),\ldots})</td>
<td>(\sum_{n \geq 0} \binom{m+n}{m} z^n)</td>
<td>(\frac{1}{(1 - z)^{m+1}})</td>
</tr>
<tr>
<td>({0,1,\frac{1}{2},\frac{1}{3},\frac{1}{4},\ldots})</td>
<td>(\sum_{n \geq 1} \frac{1}{n} z^n)</td>
<td>(\ln \frac{1}{1 - z})</td>
</tr>
<tr>
<td>({0,1,-\frac{1}{2},\frac{1}{3},-\frac{1}{4},\ldots})</td>
<td>(\sum_{n \geq 1} \frac{(-1)^{n+1}}{n} z^n)</td>
<td>(\ln(1 + z))</td>
</tr>
<tr>
<td>({1,1,\frac{1}{2},\frac{1}{6},\frac{1}{24},\frac{1}{120},\ldots})</td>
<td>(\sum_{n \geq 0} \frac{1}{n!} z^n)</td>
<td>(e^z)</td>
</tr>
</tbody>
</table>
Homework 1. Due September 15

The point value of each exercise is stated in the brackets. Attempt all questions. Please bring written solutions on the due date to the class.

Problem 1 [2+1+1] Let $\mathbb{N}$ be the set of natural numbers and $\mathbb{R}_{\geq 0}$ be the set of non-negative reals. Let us call a function $f : \mathbb{N} \rightarrow \mathbb{R}_{\geq 0}$ super-additive if $f(m) + f(n) \leq f(m + n)$ for every $m, n \in \mathbb{N}$.

i) Prove that for every super-additive function $f$, the ratio $f(n)/n$ tends to a limit (possibly $+\infty$) as $n \rightarrow \infty$.

ii) Deduce from Item i) that if a function $f$ satisfies $f(m + n) \geq f(m) + f(n) - 1000$ for all $m, n \in \mathbb{N}$, then the ratio $f(n)/n$ tends to a limit (possibly $+\infty$) as $n \rightarrow \infty$.

iii) Let $f(k) = R(3, 3, \ldots, 3)$ be the $k$-color Ramsey number of triangles. Deduce from Item i) that $f(k)^{1/k}$ tends to a limit as $k \rightarrow \infty$.

Problem 2 [3+1] Let $T$ be a tree having $a$ vertices (a tree is a connected graph containing no cycles).

i) If $n = (a - 1)(b - 1) + 1$ and $\binom{n}{2} = R \cup B$ then either $R$ contains a copy of $T$ or $B$ contains a copy of $K_b$ (that is, any blue-red coloring of edges of $K_n$ gives a red $T$ or a blue $K_b$).

ii) This is not true if $n = (a - 1)(b - 1)$.

Problem 3 [1] Let $C(n)$ be the minimum $N$ such that any set $X$ of $N$ points in the plane, no three of which lie on a line, contains $n$ points in convex position (these are the Erdős-Szekeres numbers). Show that $C(n) \leq R_3(n, n)$ by coloring a triple $\{x, y, z\}$ of $X$ red if and only if the number of points of $X$ inside the interior of the convex hull of $\{x, y, z\}$ is even.

Problem 4 [2] We define a two-dimensional arithmetic progression of order $k$ to be a subset of $\mathbb{Z}^2$ of the form

$$\{(a_1 + id_1, a_2 + jd_2) : 0 \leq i, j \leq k - 1\}.$$  

Deduce from van der Waerden’s theorem that for all $k, r$ and any $f : \mathbb{N}^2 \rightarrow [r]$, there exists a monochromatic two-dimensional arithmetic progression of order $k$. (Note: you are not allowed to use Hales-Jewett’s or Gallai’s theorems.)

Problem 5 [2] Let $n, m, t$ be positive integers and

$$[n] = I_0 \cup I_1 \cup \ldots \cup I_m$$
be a partition in which \( I_1, \ldots, I_m \) are nonempty. Let \( f(j) : I_0 \to [t] \) be any function. Recall that the combinatorial \( m \)-space defined by \( I_0, I_1, \ldots, I_m, f \) is the set of vectors \( v \in [t]^n \) such that

1. \( i, j \in I_k, k \neq 0 \) implies \( v_i = v_j \), and
2. \( i \in I_0 \) implies \( v_i = f(i) \).

So, the coordinates in the set \( I_0 \) are fixed throughout the \( m \)-space and the coordinates in the set \( I_k, 1 \leq k \leq m \), vary together. Note that a combinatorial 1-space is simply a combinatorial line. Use the Hales-Jewett Theorem to prove that for every triple of positive integers \( m, t, c \) there exists an \( n \) such that any coloring of \( [t]^n \) with \( c \) colors has a monochromatic combinatorial \( m \)-space.

**Extra Credit Problem**

**Problem 6 [3]** Let

\[
f(k) = R(3,3,\ldots,3) - 1.
\]

Prove that \( f(k + 1) \geq 3f(k) + f(k - 2) \). [Hint: begin by showing that \( f(3) \geq 16 \).]

Use this to find constants \( c \) and \( r \) such that \( f(k) \geq cr^k \) for all \( k \) (of course, you should aim for \( r \) as large as possible).
Homework 2. Due October 4

Problem 7 [3] Let $t_n$ denote the number of non-congruent triangles whose sides have integer length and whose perimeter is $n$. For instance, $t_9 = 3$, corresponding to $3 + 3 + 3$, $2 + 3 + 4$, $1 + 4 + 4$. Find $\sum_{n \geq 3} t_n x^n$.

Problem 8 [1] Find a closed form for $\sum_{k \geq 0} x^{2k+1} \frac{2k+1}{2k+1}$, which is convergent for $|x| < 1$.

Problem 9 [1+2] Let $\alpha \neq 1$ be a constant. Let $s_n = \sum_{i=1}^{n} i \alpha ^i$.

i) Show that the generating function $S(z) = \sum_{n \geq 0} s_n z^n$ is equal to $\frac{\alpha z}{(1-z)(1-\alpha z)}$.

ii) Find a formula for $s_n$. Check your answer by induction.

Problem 10 [1+1+1] For a random variable $\xi$ taking values in $\mathbb{N}$, we write $P(z) = P_\xi(z)$ for the probability generating function for $\xi$:

$$P(z) = \sum_{n \geq 0} Pr(\xi = n) z^n.$$ 

1. Express the expectation $E[\xi]$ in terms of $P(z)$.

2. Let $\xi_1, \ldots, \xi_t$ be independent random variable and $\xi = \sum_{i=1}^{t} \xi_i$. Express $P_\xi(z)$ in terms of the $P_{\xi_i}(z)$'s.

3. Suppose $\xi$ takes values in $\{0, 1, \ldots, n\}$ (so $P$ is a polynomial). Show that if the roots of $P$ are real, then $\xi = \sum_{i=1}^{n} \xi_i$ where $\xi_1, \ldots, \xi_n$ are independent (but not necessarily identical) 0-1 random variables.

Extra Credit Problem

Problem 11 [3] Let $u_n$ be the number of unlabeled trees on $n$ vertices. (Unlabeled means that we do not distinguish isomorphic trees. Thus $u_1 = u_2 = u_3 = 1$, $u_4 = 2$, $u_5 = 3$, etc.)

Let $r_n$ be the number of rooted unlabeled tree on $n$ vertices. (Here, two trees are regarded the same if there is an isomorphism that maps the root to the root.)

Prove that the corresponding o.g.f. $U(z) = \sum_{n \geq 1} u_n z^n$ and $R(z) = \sum_{n \geq 1} r_n z^n$ satisfy the following relation:

$$U(z) = R(z) - \frac{(R(z))^2 - R(z^2)}{2}.$$ 

Here are some useful facts/ideas:

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1. \( r_n = \sum p(T) \), where the sum is taken over all unlabeled trees of order \( n \) and \( p(T) \) is the number of equivalence classes of vertices. (Two vertices of \( T \) are in the same class if there is an automorphism of \( T \) which maps one to the other.)

2. What is \( p(T) - q(T) \), where \( q(T) \) is the number of equivalence classes of edges?
Homework 3. Due October 11

Problem 12 [1+1] Let $a_n$ be the number of involutions of $[n]$ (where we agree that $a_0 = 1$) and let $A(x) = \sum_{n=0}^{\infty} a_n x^n/n!$. (Thus we know that $A(x) = \exp(x + x^2/2)$.)

i) Find via generating functions a simple closed form for $
\sum_{i=0}^{n} (-1)^i \binom{n}{i} a_i.$

ii) Using the Principle of Inclusion-Exclusion find a combinatorial proof of your formula from Item i).

Problem 13 [2] Find a formula for $a_n$, the number of permutations $\pi \in S_n$ containing no pattern 132, that is, for which there do not exist $i < j < k$ with $\pi_i < \pi_k < \pi_j$.

Problem 14 [2+2] [Combinatorial trigonometry?] A permutation $a_1a_2\cdots a_n$ of $[n]$ is alternating if $a_1 < a_2 > a_3 < a_4 > \cdots$. Let $b_n$ be the number of alternating permutations if $n$ is odd, $e_n$ the number if $n$ is even ($e_0 = 1$). Let $b(z)$ and $e(z)$ be the exponential generating functions of $b_n$ and $e_n$ respectively. (The numbers $b_n$ and $e_n$ are called Bernoulli and Euler numbers respectively.)

i) Show that if $n \geq 1$, then

$$b_{n+1} = \sum_{k \text{ odd}} \binom{n}{k} b_k b_{n-k}, \text{ if } n+1 \text{ is odd},$$

$$e_{n+1} = \sum_{k \text{ even}} \binom{n}{k} e_k b_{n-k}, \text{ if } n+1 \text{ is even}.$$ 

Deduce that $b(z) = \tan(z)$, $e(z) = \sec(z)$. [Hint: If you show that e.g. $b(z)$ and $\tan(z)$ satisfy the same differential equation and the same initial condition at $z = 0$, you are allowed to assume for free that they have the same Taylor expansion around 0.]

ii) Give a combinatorial proof of $1 + \tan^2(z) = \sec^2(z)$, that is, give a bijection between the structures counted by the e.g.f. $1 + b^2(z)$ and the structures counted by the e.g.f. $e^2(z)$.

A Remark on Quicksort

Here is a simple proof of the formula for $c_n$, the average number of comparisons in Quicksort. Suppose that the unknown order is $y_1 < \ldots < y_n$. For $i < j$, let the random variable $Y_{ij}$ be 1 if we compare $y_i$ and $y_j$ and 0 otherwise. Note that the probability of $Y_{ij} = 1$ is exactly...
Indeed, consider for the first time when one of \( y_i, y_{i+1}, \ldots, y_j \) becomes the pivot; then we compare \( y_i \) and \( y_j \) if and only if the pivot is \( y_i \) or \( y_j \). Then, by the linearity of expectation,

\[
c_n = \sum_{1 \leq i < j \leq n} \frac{2}{j - i + 1} = \sum_{k=2}^{n} (n - k + 1) \frac{2}{k},
\]

where we group all pairs \( i < j \) by \( k = j - i + 1 \).

**Homework 4. Due October 25**

**Problem 15 [1+2+1]**

i) Deduce from Hall’s Marriage Theorem that if \( G \) is a non-empty bipartite graph with parts \( V(G) = A \cup B \) such that \( d(x) \geq d(y) \) for every \( x \in A \) and \( y \in B \), then \( G \) has a matching of size \( |A| \).

ii) Prove that every \( r \)-regular graph \( G \) (not necessarily bipartite), where \( r \) is even and positive, has a 2-factor (that is, there is a 2-regular subgraph \( H \subseteq G \) with \( V(H) = V(G) \)).

iii) A chain is a collection of nested sets. Prove that for every \( n \) we can partition \( 2^{[n]} \) into \( \binom{n}{\lfloor n/2 \rfloor} \) chains.

**Problem 16 [3]** Let \( r, c \leq n \) be positive integers. A Latin rectangle is an \( r \times c \) matrix with entries from \( [n] \) such that each \( i \in n \) appears at most once in each row and column. Such a rectangle with \( r = c = n \) is a Latin square. Show that an \( r \times c \) Latin rectangle extends (in the obvious sense) to a Latin square iff it uses each symbol at least \( r + c - n \) times.

**Problem 17 [2]** By analysing the shifting proof presented in class, deduce the following. If \( n > 2k \) and \( \mathcal{F} \subseteq \binom{[n]}{k} \) is an intersecting family of size \( \binom{n-1}{k-1} \), then \( \mathcal{F} \) is a star (that is, there is some element \( x \in [n] \) such that every element of \( \mathcal{F} \) contains \( x \)).

**Problem 18 [3]** Let \( \mathcal{A} = \{A_1, \ldots, A_m\} \) be a collection of subsets of \( [n] \) and suppose that it is convex in the following sense: If \( A_i \subseteq B \subseteq A_j \) for some \( i, j \) then \( B \in \mathcal{A} \). Prove

\[
\left| \sum_{i=1}^{m} (-1)^{|A_i|} \right| \leq \binom{n}{\lfloor n/2 \rfloor}.
\]

**Extra Credit Problem**

**Problem 19 [1]** Prove that if \( \mathcal{F} \subseteq 2^{[n]} \) is an antichain and

\[
\sum_{i=0}^{n} \frac{|\mathcal{F} \cap \binom{[n]}{i}|}{\binom{n}{i}} = 1,
\]

then \( \mathcal{F} = \binom{[n]}{i} \) for some \( i \).

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Homework 5. Due November 3

Problem 20 [2] Suppose that $\mathcal{F} \subseteq \binom{[n]}{k}$, $m = |\mathcal{F}|$, and the (unique) real $x \geq k$ was chosen so that $\binom{x}{k} = m$. Prove the following bound on the size of the shadow of $\mathcal{F}$:

$$|\partial(\mathcal{F})| \geq \binom{x}{k - 1}.$$

Problem 21 [2] Show that the Kruskal–Katona Theorem implies that for $n \geq 2k$ the maximum size of an intersecting family $\mathcal{F} \subseteq \binom{[n]}{k}$ is $\binom{n - 1}{k - 1}$ (where we do not need to characterize cases of equality).

[Hint: Consider complements.]

Problem 22 [2] Let $A_1, \ldots, A_m$ be finite sets having cardinality $a$ and $B_1, \ldots, B_m$ be finite sets having cardinality $b$. Show that if

$$A_i \cap B_j = \emptyset \iff i = j.$$

then $m \leq \binom{a+b}{a}$.

[Hint: For every ordering of the vertices, there is at most one index $i$ such that all elements of $A_i$ come before all elements of $B_i$.]

Problem 23 [3] Prove that the maximum value of $m$ such that for any collection of $k$-sets $A_1, \ldots, A_m$, there are distinct $(k-1)$-sets $B_1, \ldots, B_m$ with $B_i \subseteq A_i$ for each $i \in [m]$ is

$$m(k) = \binom{2k - 1}{k} + \binom{2k - 3}{k - 1} + \ldots + \binom{3}{2} + \binom{1}{1}.$$
Homework 6. Due November 17

Problem 24 [3] Prove by induction on \( r \geq 2 \) that if \( G = (V, E) \) is a \( K_{r+1} \)-free graph on \( n \) vertices with \( t_r(n) - k \) edges, then there is a partition \( V = V_1 \cup \ldots \cup V_r \) such that \( \sum_{i=1}^{r} e(G[V_i]) \leq k \) (that is, at most \( k \) edges of \( G \) lies inside the parts).

Problem 25 [2+1+2] Let \( 2 \leq k < n \), and \( T \) be a fixed tree with \( k \) edges (and \( k+1 \) vertices). The Erdős-Sós conjecture is that \( \text{ex}(n, T) \leq (k-1)n/2 \)

1. Show that \( \text{ex}(n, T) \leq (k-1)n \).
   [Hint: Let \( d(H) = \frac{1}{v(H)} \sum_{x \in V(H)} d(x) \) be the average degree of a graph \( H \). One way is to start by showing that \( d(G-v) \geq d(G) \) iff \( d(v) \leq d(G)/2 \) and use this to build a high degree subgraph of \( G \).]

2. Prove the conjecture when \( T \) is a star.

3. Prove the conjecture when \( T \) is a path.
   [Hint: First prove that if \( G \) is connected then \( G \) contains a path of length \( \min(2\delta(G), v(G)-1) \).]

Problem 26 [2] Show that for every \( k \) there exists a tournament \( T = (V, A) \) such that for all \( X \subseteq V \) of cardinality \( k \) and all \( Y \subseteq X \) there exists \( v \in V \) such that

\( (x \in X \setminus Y \Rightarrow (x, v) \in A) \) and \( (x \in Y \Rightarrow (v, x) \in A) \).

(Informally, \( v \) beats every element of \( Y \) while every element of \( X \setminus Y \) beats \( v \)).

Extra Credit Problem

Problem 27 [2+2] Prove the asymptotic version of Turán’s Theorem, i.e. show that for any fixed \( s \geq 2 \),

\[
\text{ex}(n, K_{s+1}) \leq \left( \frac{s-1}{s} + o(1) \right) \binom{n}{2},
\]

in the following two ways. Let \( G = (V, E) \) be an arbitrary graph.

1. Consider the (quadratic programming) problem

\[
\max \sum_{xy \in E} \lambda(x)\lambda(y)
\]
subject to
\[ \lambda(x) \geq 0 \quad \forall x \in V \]
\[ \sum_{x \in V} \lambda(x) = 1. \]

2. By taking a random ordering of \( V \) and using some greedy algorithm prove that the independence number \( \alpha(G) \) satisfies the following inequality
\[ \alpha(G) \geq \sum_{x \in V} \frac{1}{d(x) + 1}. \]

Homework 7. Due December 1

**Problem 28** [1] Show that Chebyshev’s inequality is generally best possible. That is, for any reals \( \lambda \geq 1, \sigma > 0 \) and \( \mu \) give an example of a random variable \( X \) such that \( E(X) = \mu \), \( \text{Var}(X) = \sigma^2 \), and
\[ Pr(|X - \mu| \geq \lambda \sigma) \geq \frac{1}{\lambda^2}. \]

**Problem 29** [3] Prove that every 3-uniform hypergraph \( G \) on \( n \) vertices and \( m \geq n/3 \) edges contains an independent set (i.e. a set of vertices that spans no edge) of size at least \( \frac{2m^{3/2}}{3\sqrt{3}\sqrt{m}} \).

[Hint: Use an alteration.]

**Problem 30** [2] Fix \( c > 0 \), let \( n \to \infty \), and let \( X \) denote the number of isolated edges in random graph \( G_{n,p} \), where \( p = c/n \). Find the asymptotic formula for \( \mu = E(X) \) and prove that for every fixed \( \varepsilon > 0 \), we have \( Pr(|X - \mu| > \varepsilon \mu) = o(1) \).

**Extra Credit Problem**

**Problem 31** [2] Let \( X \) be a random variable with expectation \( E(X) = 0 \) and variance \( \sigma^2 \). Prove that for all \( \lambda > 0 \) we have
\[ Pr(X \geq \lambda) \leq \frac{\sigma^2}{\sigma^2 + \lambda^2}. \]
(Thus if we need “one-sided” estimate only, then Chebyshev’s inequality can be slightly improved.)

Office Hours (Wean 7105): Mon 14:00–15:00, Wen 13:05–14:00, Thurs 13:30–15:30