

# Hamilton Cycles in Random Lifts of Graphs

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August 21, 2005

## Abstract

An  $n$ -lift of a graph  $K$ , is a graph with vertex set  $V(K) \times [n]$  and for each edge  $(i, j) \in E(K)$  there is a perfect matching between  $\{i\} \times [n]$  and  $\{j\} \times [n]$ . If these matchings are chosen independently and uniformly at random then we say that we have a random  $n$ -lift. We show that there are constants  $h_1, h_2$  such that if  $h \geq h_1$  then a random  $n$ -lift of the complete graph  $K_h$  is hamiltonian **whp** and if  $h \geq h_2$  then a random  $n$ -lift of the complete bipartite graph  $K_{h,h}$  is hamiltonian **whp**.

## 1 Introduction

For a graph  $K$ , an  $n$ -lift  $G$  of  $K$  has vertex set  $V(K) \times [n]$  where for each vertex  $v \in V(K)$ ,  $\{v\} \times [n]$  is called the *fiber* above  $v$  and will be denoted by  $\Pi_v$ . The edge set of a an  $n$ -lift  $G$  consists of a perfect matching between fibers  $\Pi_u$  and  $\Pi_w$  for each edge  $(u, w) \in E(K)$ . The set of  $n$ -lifts will be denoted  $\mathcal{L}_n(K)$ . In this paper we discuss random  $n$ -lifts, chosen uniformly from  $\mathcal{L}_n(K)$ . In this case, the matchings between fibers are chosen independently and uniformly at random.

Lifts of graphs were introduced by Amit and Linial in [1] where they proved that if  $K$  is a connected, simple graph with minimum degree  $\delta \geq 3$ , and  $G$  is chosen randomly from  $\mathcal{L}_n(K)$  then  $G$  is  $\delta$ -connected **whp**, where the asymptotics are for  $n \rightarrow \infty$ . They continued the study of random lifts in [2] where they proved expansion properties of lifts. Together with Matoušek, they gave bounds on the independence number and chromatic number of random lifts in [3]. Linial and Rozenman [4] give a tight analysis for when a random  $n$ -lift has a perfect matching.

In this paper we discuss the probability that a random  $n$ -lift is hamiltonian. In particular we study the case where  $K$  is the complete graph  $K_h$  or the complete bipartite graph  $K_{h,h}$ . We use the notation  $y \stackrel{r}{\in} Y$  for “ $y$  is chosen uniformly at random from  $Y$ ”.

**Theorem 1.** *There exists a constant  $h_1$  such that if  $h \geq h_1$  and  $G \stackrel{r}{\in} \mathcal{L}_n(K_h)$  then  $G$  is hamiltonian **whp**.*

**Theorem 2.** *There exists a constant  $h_2$  such that if  $h \geq h_2$  and  $G \stackrel{r}{\in} \mathcal{L}_n(K_{h,h})$  then  $G$  is hamiltonian **whp**.*

Theorem 1 is proved in the next section. Theorem 2 is proved in Section 3.

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## 2 Proof of Theorem 1

### 2.1 Structural Properties of $\mathcal{L}_n(K_h)$

The vertices of  $\mathcal{L}_n(K_h)$  will be denoted by  $V$  and its edges will be denoted  $E$ .

We will use the coloring argument of Fenner and Frieze [7] to show  $G$  is hamiltonian **whp**. For  $G \in \mathcal{L}_n(K_h)$  we choose a set  $H_1 = H_1(G) \subseteq E(G)$  as follows: Each vertex of  $G$  arbitrarily chooses 12 edges of  $G$  incident with it. Thus the number of distinct edges chosen is between  $6hn$  and  $12hn$  and the minimum degree of the graph induced by  $H_1$  is at least 12. Next let  $P_0 = P_0(G)$  be a specific longest path in  $G$ . Let  $F(G) = P_0 \cup H_1$  be the fixed edges of  $G$ .

The analysis uses an *unspecified*, sufficiently small, positive constant  $\beta < 1$ .

Let  $\mathcal{B} = \mathcal{B}(G)$  be the set of subsets of  $E(G)$  of size  $\beta \binom{h}{2} n$ . We say that a subset of edges  $H$  is *acceptable* if  $H = B \cup F$  for some  $B \in \mathcal{B}(G)$ . Let  $\mathcal{H}(G)$  be the collection of acceptable subgraphs of  $G$ . For a lift  $G$ , each  $B \in \mathcal{B}(G)$  defines a coloring of the edges of  $G$  in which the edges of  $H = B \cup F$  are colored blue and the edges of  $R = G \setminus H$  are colored green.

Let  $S \subseteq V$  be of size  $s$  and let  $S_i$  be the intersection of  $S \subseteq V$  with fiber  $\Pi_i$  for  $i \in [h]$ . The number of choices for  $S$  is  $\binom{hn}{s}$  and by considering the number of choices for the  $S_i$  we see that

$$\sum_{s_1 + \dots + s_h = s} \prod_i \binom{n}{s_i} = \binom{hn}{s} \leq \left(\frac{hne}{s}\right)^s. \quad (1)$$

For a graph  $G = (V, E)$  and  $S \subseteq V$  let  $N(S) = \{v \in V \setminus S : \exists u \in S \text{ such that } (u, v) \in E(G)\}$  be the disjoint neighborhood of  $S$ .

For  $G \in \mathcal{L}_n(K_h)$  and sets  $S \subseteq \Pi_i$  and  $T \subseteq \Pi_j$ ,  $|S| = s, |T| = t$ ,

$$\Pr(N(S) \cap \Pi_j \subseteq T) = \frac{t(t-1)\dots(t-s+1)}{n(n-1)\dots(n-s+1)} \leq \left(\frac{t}{n}\right)^s. \quad (2)$$

Throughout this section all statements hold for  $n$  and  $h$  sufficiently large.

**Lemma 1.** For  $G \in \mathcal{L}_n(K_h)$ ,

$$\Pr(\exists S \subseteq V : |S| \leq \frac{n}{10h} \text{ and } S \text{ contains at least } 2|S| \text{ edges}) = o(1)$$

**Proof** Using (1) we see that the expected number of sets  $S$  of size  $s$  that contain at least  $2s$  edges is no more than

$$\begin{aligned} \phi(s) &= \binom{hn}{s} \binom{\binom{s}{2}}{2s} \left(\frac{1}{n-2s}\right)^{2s} \\ &\leq \left(\frac{hne}{s}\right)^s \left(\frac{s^2 e}{4s}\right)^{2s} \left(\frac{1}{n(1-\frac{1}{5h})}\right)^{2s} \\ &\leq \left(\frac{hne}{s}\right)^s \left(\frac{se}{4}\right)^{2s} \left(\frac{2}{n}\right)^{2s} \\ &\leq \left(\frac{e^3 hs}{4n}\right)^s \end{aligned}$$

Then

$$\sum_{s=5}^{n/10h} \phi(s) = o(1).$$

□

**Lemma 2.** *If  $G \in \mathcal{L}_n(K_h)$  and  $H \in \mathcal{H}(G)$ , then **whp**  $H$  satisfies*

$$S \subseteq V, |S| \leq hn/4 \text{ implies } |N_H(S)| \geq 2|S|. \quad (3)$$

**Proof** Assume first that  $|S| \leq n/10h$  and let  $U = S \cup N(S)$ . Let  $a$  be the number of edges contained in  $S$  and let  $b$  be the number of edges from  $S$  to  $N(S)$ . The degree sum of  $S$  in  $H_1$  is at least  $12|S|$  and so  $2a + b \geq 12|S|$ . But then  $U$  contains at least  $a + b \geq 6|S|$  edges and we can assume by Lemma 1 that  $|U| > 3|S|$ . This completes the argument for  $|S| \leq n/10h$ .

Let  $H'$  be defined by including an edge of  $G$  in  $H'$  independently with probability  $\beta'$  where  $\beta' < \beta$ . Then  $|H'|$  is a binomial random variable whose expected value is less than  $\beta \binom{h}{2} n$ . The Chernoff bound implies that for a monotone increasing property of lifts  $\mathcal{Q}$ , if  $H' \in \mathcal{Q}$  **whp**, then  $H \in \mathcal{Q}$  **whp**.

For  $n/10h < |S| \leq hn/4$ , let  $T = N(S)$  and  $t = |T|$ . Using (1) and (2), the expected number  $Z$  of sets  $S$  with  $|N_{H'}(S)| < 2|S|$  is bounded as follows: In the first line of the following display, the notation  $j \succ i$  denotes  $s_j + t_j > s_i + t_i$  or  $s_j + t_j = s_i + t_i$  and  $j > i$ .

$$\begin{aligned} Z &\leq \sum_{s=n/10h}^{hn/4} \sum_{t=0}^{2s-1} \sum_{s_1+\dots+s_h=s} \sum_{t_1+\dots+t_h=t} \prod_i \binom{n}{s_i} \prod_j \binom{n}{t_j} \prod_{i=1}^h \prod_{j \succ i} \left( \beta' \frac{s_j + t_j}{n} + (1 - \beta') \right)^{s_i + t_i} \\ &\leq \sum_{s=n/10h}^{hn/4} \sum_{t=0}^{2s-1} \sum_{s_1+\dots+s_h=s} \prod_i \binom{n}{s_i} \sum_{t_1+\dots+t_h=t} \prod_j \binom{n}{t_j} \prod_{i=1}^h \prod_{j \neq i} \left( \beta' \frac{s_j + t_j}{n} + (1 - \beta') \right)^{(s_i + t_i)/2} \\ &= \sum_{s=n/10h}^{hn/4} \sum_{t=0}^{2s-1} \sum_{s_1+\dots+s_h=s} \prod_i \binom{n}{s_i} \sum_{t_1+\dots+t_h=t} \prod_j \binom{n}{t_j} \prod_{j=1}^h \left( \beta' \frac{s_j + t_j}{n} + (1 - \beta') \right)^{(s+t - (s_j + t_j))/2} \\ &\leq \sum_{s=n/10h}^{hn/4} \sum_{t=0}^{2s-1} \sum_{s_1+\dots+s_h=s} \prod_i \binom{n}{s_i} \sum_{t_1+\dots+t_h=t} \prod_j \binom{n}{t_j} \left( \sum_{j=1}^h \left( \beta' \frac{s_j + t_j}{(h-1)n} + (1 - \beta') \right) \right)^{(h-1)(s+t)/2} \\ &= \sum_{s=n/10h}^{hn/4} \sum_{t=0}^{2s-1} \sum_{s_1+\dots+s_h=s} \prod_i \binom{n}{s_i} \sum_{t_1+\dots+t_h=t} \prod_j \binom{n}{t_j} \left( \beta' \frac{s+t}{(h-1)n} + (1 - \beta') \right)^{(h-1)(s+t)/2} \\ &\leq \sum_{s=n/10h}^{hn/4} \sum_{t=0}^{2s-1} \left( \frac{neh}{s} \right)^s \left( \frac{neh}{t} \right)^t \left( 1 - \beta' \left( 1 - \frac{s+t}{(h-1)n} \right) \right)^{(h-1)(s+t)/2} \quad \text{using (1)} \\ &\leq \sum_{s=n/10h}^{hn/4} \sum_{t=0}^{2s-1} \left( \frac{neh}{s} \right)^s \left( \frac{neh}{t} \right)^t \exp \left\{ -\beta' \left( 1 - \frac{s+t}{(h-1)n} \right) (h-1)(s+t)/2 \right\} \\ &\leq \sum_{s=n/10h}^{hn/4} \left( \frac{neh}{s} \right)^{3s} \exp \left\{ -\frac{\beta'hs}{10} \right\} \\ &\leq e^{-\beta n/199}. \end{aligned} \quad (4)$$

□

**Lemma 3.** *If  $G \stackrel{r}{\in} \mathcal{L}_n(K_h)$  and  $H \stackrel{r}{\in} \mathcal{H}(G)$ , then **whp**  $H$  is connected.*

**Proof** If  $H$  is not connected, Lemma 2 implies that **whp**  $H$  is the union of a constant number of components of size at least  $hn/4$ . We will again work under the assumption that edges are included in  $H'$  independently with probability  $\beta'$  where  $\beta' < \beta$ .

Assume without loss of generality that  $|S| \leq hn/2$ . The expected number of sets  $S$  of size  $|S| \in [hn/4, hn/2]$  with no edges between  $S$  and its complement is no more than

$$\begin{aligned}
& \sum_{s=hn/4}^{hn/2} \left( \sum_{s_1+\dots+s_h=s} \prod_i \binom{n}{s_i} \right) \prod_{i=1}^h \prod_{j>i} \left( \beta' \left( \frac{s_j}{n} \right) + (1-\beta') \right)^{s_i} \\
\leq & \sum_{s=hn/4}^{hn/2} \left( \frac{neh}{s} \right)^s \left( \beta' \left( \frac{s}{(h-1)n} \right) + (1-\beta') \right)^{(h-1)s/2} \\
\leq & \sum_{s=hn/4}^{hn/2} \left( \frac{neh}{s} \right)^s \exp \left\{ -\frac{\beta' s}{2} (h/2 - 1) \right\} \\
\leq & e^{-\beta h^2 n/5}
\end{aligned} \tag{5}$$

□

Let  $P = (v_0, \dots, v_k)$  be a longest path in graph  $H$ . A *Pósa rotation* of  $P$  [10] with  $v_0$  fixed gives another longest path  $P' = (v_0, \dots, v_i v_k \dots v_{i+1})$  created by adding edge  $(v_k, v_i)$  and deleting edge  $(v_i, v_{i+1})$ . Let  $END_H(v_0, P)$  be the set of endpoints obtained by a sequence of Pósa rotations starting with  $P$ , keeping  $v_0$  fixed and using an edge  $(v_k, v_i)$  of  $H$ .

Each vertex  $v_j \in END_H(v_0, P)$  can then be used as the initial vertex of another set of longest paths  $END_H(v_j, P)$ , this time using  $v_j$  as the fixed vertex, but again only adding edges from  $H$ . Let  $END_H(P) = \{v_0\} \cup END_H(v_0, P)$ .

The Pósa condition

$$|N(END(v, P))| \leq 2|END(v, P)| - 1$$

for  $v \in END_H(P)$  together with Lemma 2 implies the following.

**Lemma 4.** *If  $G \stackrel{r}{\in} \mathcal{L}_n(K_h)$  and  $H \stackrel{r}{\in} \mathcal{H}(G)$ , then **whp**  $|END_H(v, P)| \geq hn/4$  for all  $v \in END_H(P)$ ,  $P = P(G)$ .*

We say next that an *ordered* pair of fibers  $(\Pi_k, \Pi_l)$  is *good* w.r.t. a longest path  $P$  if

$$|\{(u, v) \in \Pi_k \times \Pi_l : u \in END_H(P), v \in END_H(u, P)\}| \geq n^2/32. \tag{6}$$

**Lemma 5.** *If (3) holds then  $G$  has at least  $h(h-1)/32$  good fiber pairs.*

**Proof** For  $k, l \in [h]$ ,  $k \neq l$  let

$$X = X(k, l) = |\{(u, v) \in \Pi_k \times \Pi_l : u \in END_H(P), v \in END_H(u, P)\}|.$$

Suppose next that  $k, l$  are chosen uniformly at random without replacement. Then

$$\begin{aligned}
\mathbf{E}(X) &= \frac{1}{h(h-1)} \sum_{k=1}^h \sum_{u \in \Pi_k \cap \text{END}_H} \sum_{\substack{l=1 \\ l \neq k}}^h |\{v \in \Pi_l \cap \text{END}_H(P, u)\}| \\
&\geq \frac{1}{h(h-1)} \sum_{k=1}^h \sum_{u \in \Pi_k \cap \text{END}_H} \left(\frac{h}{4} - 1\right) n \\
&\geq \frac{n^2}{16}.
\end{aligned}$$

Suppose then that there are  $\alpha h(h-1)$  good fiber pairs. Then since  $X \leq n^2$  we have

$$\alpha n^2 + (1 - \alpha) \frac{n^2}{32} \geq \frac{n^2}{16}$$

and the lemma follows.  $\square$

## 2.2 The Proof

For a lift  $G$ , let  $\mathcal{D}(G)$  be the subset of  $\mathcal{H}(G)$  in which  $H$  is connected and satisfies (3) for  $|S| > n/10h$  and let  $\mathcal{D} = \cup_G \mathcal{D}(G)$ . Let  $\mathcal{A}$  be the subset of  $\mathcal{L}_n(K_h)$  such that for  $G \in \mathcal{A}$  and  $H$  chosen randomly from  $\mathcal{H}(G)$ ,

$$\Pr(H \in \mathcal{D}(G)) \geq 1 - \alpha.$$

where  $\alpha = e^{-\beta n/400}$ .

Let  $\mathcal{C}$  be the subset of  $\mathcal{L}_n(K_h)$  that is not hamiltonian and let  $\mathcal{F} = \mathcal{A} \cap \mathcal{C}$ . To show that  $\Pr(\mathcal{C}) \rightarrow 0$ , we will first show that  $|\mathcal{A}| = (1 - o(1)) |\mathcal{L}_n(K_h)|$  and then use the coloring argument of Fenner and Frieze [7] to show that  $\Pr(\mathcal{F}) \rightarrow 0$ .

**Lemma 6.**  $|\mathcal{A}| = (1 - o(1)) |\mathcal{L}_n(K_h)|$

**Proof** If  $G \overset{r}{\in} \mathcal{L}_n(K_h)$  and  $H \overset{r}{\in} \mathcal{H}(G)$  then

$$\begin{aligned}
\Pr(H \in \mathcal{D}) &= \sum_{G \in \mathcal{L}_n(K_h)} \Pr(H \in \mathcal{D}|G) \Pr(G) \\
&= \sum_{G \in \mathcal{A}} \Pr(H \in \mathcal{D}|G) \Pr(G) + \sum_{G \notin \mathcal{A}} \Pr(H \in \mathcal{D}|G) \Pr(G) \\
&\leq \Pr(\mathcal{A}) + (1 - \alpha)(1 - \Pr(\mathcal{A})) \\
&= 1 - \alpha + \alpha \Pr(\mathcal{A})
\end{aligned} \tag{7}$$

and (4) and (5) imply that

$$\Pr(H \in \mathcal{D}) \geq 1 - \alpha^2. \tag{8}$$

Putting (7) and (8) together, we get

$$1 - \alpha + \alpha \Pr(\mathcal{A}) \geq 1 - \alpha^2.$$

so that

$$\Pr(\mathcal{A}) \geq 1 - \alpha.$$

□

To get an upper bound on the number of graphs  $G \in \mathcal{L}_n(K_h)$  such that  $G \in \mathcal{F}$ , we construct a 0-1 matrix  $A = ||a_{i,j}||$ . Row index  $i$  corresponds to a graph  $G_i \in \mathcal{L}_n(K_h)$  and index  $j$  ranges over all acceptable subgraphs  $H \in \mathcal{H}(G_i)$ . Subgraph  $j$  of  $G_i$  will be denoted by  $H_{i,j}$ . Let

$$a_{i,j} = 1 \text{ if } \begin{cases} (i) & S \subseteq V, |S| \leq hn/4 \text{ implies } |N_{H_{i,j}}(S)| \geq 2|S| \\ (ii) & H_{i,j} \text{ is connected} \\ (iii) & H_{i,j} \supseteq P_0(G_i) \\ (iv) & G_i \text{ is not Hamiltonian} \\ (v) & |E_{H_{i,j}}(\Pi_k, \Pi_l)| \in [(1 \pm n^{-1/3})\beta n], \forall k \neq l \in [h] \end{cases} \quad (9)$$

Note that (ii), (iii) and (iv) imply

$$\exists \text{ longest path } P \text{ of } H_{i,j}, (u, v) \in E(R_{i,j}) : u \in \text{END}_{H_{i,j}}(P), v \in \text{END}_{H_{i,j}}(u, P) \quad (10)$$

Now let

$$N_1 = \sum_i \sum_j a_{i,j}$$

be the number of ones in  $A$ .

**Lemma 7.** *If  $G_i \in \mathcal{F}$  then*

$$\sum_j a_{i,j} \geq (1 - o(1)) \binom{\binom{h}{2}n - 13hn}{(1 - \beta)\binom{h}{2}n - 13hn}.$$

**Proof**  $G_i \in \mathcal{F}$  and  $H_{i,j} \in \mathcal{H}(G_i)$  implies that  $H_{i,j}$  satisfies (i), (ii), (iii) and (iv) **whp**. Now  $B_1, B_2 \in \mathcal{B}(G)$  may give rise to the same subgraph  $H$  if the edges not in  $B_1 \cap B_2$  are all in  $F$ . So we count the number of ways to select  $R$  as a lower bound on  $|\mathcal{H}(G_i)|$ . We have  $|H| \leq \beta \binom{h}{2}n + 13hn$  since there are at most  $13hn$  edges in  $P_0$  and  $H_1$ . Then the number of choices for  $R$  is at least the number of ways to select a set of  $(1 - \beta)\binom{h}{2}n - 13hn$  edges from the  $\binom{h}{2}n - 13hn$  not in  $F$ . Condition (v) holds through the Chernoff bound. □

It follows immediately from Lemma 7 that

$$N_1 \geq (1 - o(1)) \binom{\binom{h}{2}n - 13hn}{(1 - \beta)\binom{h}{2}n - 13hn} |\mathcal{F}|. \quad (11)$$

We now obtain an upper bound on  $N_1$ . Let

$$\mathcal{X} = \{H : \exists i, j \text{ for which } H_{i,j} = H \text{ and } a_{i,j} = 1\}$$

The following bound follows from the definition and a concentration inequality for sampling without replacement, see Hoeffding [9], Theorem 4:

$$|\mathcal{X}| \leq \binom{\binom{h}{2}n}{13hn} \left( (1 + o(1)) \binom{n}{\beta n} (\beta n)! \right)^{\binom{h}{2}}. \quad (12)$$

For a fixed  $H \in \mathcal{X}$  let

$$\mathcal{G}_H = \{G_i : H_{i,j} = H \text{ and } a_{i,j} = 1\}.$$

Thus,

$$N_1 = \sum_{H \in \mathcal{X}} |\mathcal{G}_H|.$$

**Lemma 8.**

$$H \in \mathcal{X} \text{ implies } |\mathcal{G}_H| \leq e^{-ch^2n} \left( ((1 - \beta + O(n^{-1/3}))n)! \right)^{\binom{h}{2}} \quad (13)$$

for some absolute constant  $c > 0$ .

**Proof** We begin with  $H$  and count the number of ways to add back the edges of  $R$  to form a lift  $G_i \in \mathcal{G}_H$ . The number of edges in  $R(k, l)$  between two fibers of  $G_i$  is no more than  $(1 - \beta + O(n^{-1/3}))n$ . Thus there are at most  $((1 - \beta + O(n^{-1/3}))n)!$  possible matchings to add back between each pair of fibers.

Suppose now that we consider choosing one of these possible matchings at random. When adding back new edges to  $H$  we must avoid edges  $(u, v)$  where  $u \in \text{END}_H$  and  $v \in \text{END}(u, H)$  so that  $a_{i,j} = 1$  in the resulting graph. For a good fiber pair  $(\Pi_k, \Pi_l)$  as defined in (6), there are at least  $n^2/32$  pairs of vertices  $u \in \Pi_k, v \in \Pi_l$  that must be avoided and at least  $n^2/32 - (2 + o(1))\beta n^2$ . The probability that we avoid all such edges between a good fiber pair is at most

$$\left( \frac{31/32 + 2\beta}{(1 - \beta + o(1))^2} \right)^{1 - \beta - o(1)} n \leq e^{-n/33}$$

As there are at least  $\binom{h}{2}/32$  good fiber pairs, the probability that a set of new edges avoids all required edges in  $G_i$  is at most  $(e^{-n/33})^{h^2/16}$ .  $\square$

It follows from (11), (12) and (13) that  $\frac{|\mathcal{F}|}{|\mathcal{L}_n(K_h)|}$  is bounded above by

$$\begin{aligned} & \frac{e^{-ch^2n} ((1 - \beta + O(n^{-1/3}))n)!^{\binom{h}{2}} \binom{\binom{h}{2}n}{13hn} \left( (1 + o(1)) \binom{n}{\beta n}^2 (\beta n)! \right)^{\binom{h}{2}}}{(1 - o(1))(n!)^{\binom{h}{2}} \binom{\binom{h}{2}n - 13hn}{(1 - \beta) \binom{h}{2}n - 13hn}} \\ & \leq \frac{e^{-ch^2n/2} \binom{n}{\beta n}^{\binom{h}{2}}}{\binom{\binom{h}{2}n}{\beta \binom{h}{2}n} \beta^{14hn}} \\ & \leq e^{-ch^2n/2 + 14hn \ln(1/\beta)} \\ & = o(1) \end{aligned}$$

where the second line uses  $\binom{a-x}{b-x} \geq \left(\frac{b-x}{a-x}\right)^x \binom{a}{b}$ .  $\square$

## 3 Proof of Theorem 2

### 3.1 Structural Properties of $\mathcal{L}_n(K_{h,h})$

Let  $V_1, V_2$  be the bipartition of  $K_{h,h}$  and let  $W_1, W_2$  be the bipartition of the lifts of  $K_{h,h}$  that it induces.

We now prove similar properties to those in Section 2.1. Let  $H_1, P_0$  be sets of edges defined as in Section 2.1 and let  $F = P_0 \cup H_1$ . Again we use an unspecified, suitably small constant  $\beta < 1$ , let  $B$  be a set of  $\beta \binom{h}{2} n$  edges in  $G$  and  $\mathcal{B}(G)$  the collection of subgraphs  $B$ . A set of edges  $H$  in  $G$  is acceptable if  $H = B \cup F$  for some  $B \in \mathcal{B}(G)$ . Let  $\mathcal{H}(G)$  be the collection of acceptable subgraphs of  $G$  and let  $R = G \setminus H$ .

Throughout this section all statements hold for  $n$  and  $h$  sufficiently large. The proof is similar to that for  $K_h$  and so we will omit calculations that are almost identical to those of the previous sections.

The main difficulty with using a Posá type argument is that if a longest path  $P$  in  $G$  is even then it cannot be closed to a cycle, connectivity notwithstanding i.e. we gain nothing from avoiding choosing edges to join  $v$  to  $END(v)$ . In this case, there are no edges to avoid. We therefore have to modify the argument. We follow Bollobás and Kohayakawa [6] who considerably simplified the argument of [8].

**Lemma 9.** For  $G \stackrel{r}{\in} \mathcal{L}_n(K_{h,h})$

$$\Pr(\exists S \subseteq V : |S| \leq \frac{n}{20h} \text{ and } S \text{ contains at least } 2|S| \text{ edges}) = o(1)$$

□

**Lemma 10.** If  $G \stackrel{r}{\in} \mathcal{L}_n(K_{h,h})$  and  $H \stackrel{r}{\in} \mathcal{H}(G)$ , then **whp**  $H$  satisfies

$$S \subseteq W_i, |S| \leq hn/4 \text{ implies } |N_H(S)| \geq 2|S|. \quad (14)$$

□

**Lemma 11.** If  $G \stackrel{r}{\in} \mathcal{L}_n(K_{h,h})$  and  $H \stackrel{r}{\in} \mathcal{H}(G)$  then **whp**  $H$  is connected.

□

**Lemma 12.** If  $K$  has a 2-factor and  $G \in \mathcal{L}_n(K)$ , then  $G$  has a 2-factor.

**Proof** Let  $C \subseteq V(K)$  be one of the cycles of a 2-factor of  $K$  and let  $G[C]$  the subgraph of  $G$  induced by the fibers above the vertices of  $C$ . Let  $v_1, \dots, v_k$  be an ordering of the vertices of  $C$  such that  $(v_i, v_{i+1})$  is an edge of  $C$  (where  $v_1 = v_{k+1}$ ) and let  $\Pi_i$  be the fiber of  $G$  above  $v_i \in C$ . Let  $\sigma_i$  be the permutation that defines the matching from fiber  $\Pi_i$  to  $\Pi_{i+1}$  for each  $\Pi_i \in G[C]$ . For each  $j \in \Pi_1$ , define  $\sigma(j) = \sigma_k \sigma_2 \cdots \sigma_1(j)$  to be the permutation on the vertices of  $\Pi_1$  that results from following the permutations  $\sigma_1$  through  $\sigma_k$  back to  $\Pi_1$ . Then a cycle of  $\sigma$  is a cycle of  $G$  so that the cycles of  $\sigma$  define a 2-factor of  $G[C]$ . This process can be repeated for all cycles of a 2-factor of  $K$  to obtain a 2-factor of  $G \in \mathcal{L}_n(K)$ . □

We now describe an extension-rotation process which attempts to transform the 2-factor  $F$  of Lemma 12 into a Hamilton cycle.

**General Step:** Given the current 2-factor (initially  $F$ ) choose an edge  $e = (x, y)$  of  $G$  which joins two distinct cycles  $C, C'$ . This is possible because  $G$  is connected **whp**. Let  $f$  be an edge of  $C$  incident with  $x$  and  $f'$  be an edge of  $C'$  incident with  $y$ . Let  $P$  be the path  $C \cup C' \cup \{e\} \setminus \{f, f'\}$ . There are now several possibilities.

(a): There is an endpoint  $u$  say, of  $P$  which has a neighbour  $v$  in a cycle  $C''$  disjoint from  $P$ . We *extend*  $P$  by replacing  $P, C''$  by  $P \cup C'' \cup \{(u, v)\} \setminus f''$  where  $f''$  is an edge of  $C''$  incident with  $v$ . We repeat this operation as long as we can. We then carry out (b) or (c).

(b) The endpoints  $u, v$  of  $P$  are connected by an edge in  $H$ . Adding  $(u, v)$  to  $P$  creates a 2-factor with at least one less cycle than at the start of the General Step and completes it.

(c) Carry out rotations on  $P$  until either (i) we construct a path  $Q$  with an endpoint  $x$  which is adjacent to a vertex  $y$  on cycle  $C$  outside  $Q$  or (ii) we satisfy the condition of (b). In the latter case we proceed as in (b) above. In the former case we extend  $Q$  by adding the edge  $(x, y)$  and deleting an edge of  $C$  incident with  $y$ .

We continue the above operations until we either obtain a Hamilton cycle or obtain a path  $P_0 = P_0(G) = (v_0, v_1, \dots, v_p)$  that cannot be extended or closed to a cycle via a sequence of rotations. Note that this path is necessarily of odd length.

We therefore let  $P_0$  be a longest path of *odd* length which (i) cannot be extended by rotations and (ii) for which there are a set of vertex disjoint cycles covering the vertices not in  $P$ .

We use the Pósa condition (which still holds) and Lemma 10 to get the following.

**Lemma 13.** *If  $G \overset{r}{\in} \mathcal{L}_n(K_{h,h})$  and  $H \overset{r}{\in} \mathcal{H}(G)$ , then **whp**  $|END_H(v, P_0)| \geq hn/4$  for all  $v \in END_H(P_0)$ ,  $P_0 = P_0(G)$ .*

We say next that an *ordered* pair of fibers  $(\Pi_k, \Pi_l)$  is *good* w.r.t. a longest path  $P$  if  $\Pi_k \in W_x$ ,  $\Pi_l \in W_{3-x}$ ,  $x = 1, 2$  and

$$|\{(u, v) \in \Pi_k \times \Pi_l : u \in END_H(P), v \in END_H(u, P)\}| \geq n^2/32. \quad (15)$$

**Lemma 14.** *If (14) holds then  $G$  has at least  $h^2/32$  good fiber pairs.*

**Proof** We first note that  $P_0$  and the paths obtained by rotations are of odd length and so each has one endpoint in each of  $W_1, W_2$ .

The rest of the proof is identical to that of Lemma 5. □

### 3.2 The Proof

Define the sets  $\mathcal{A}, \mathcal{C}, \mathcal{F}$  as in the proof of Theorem 1. We have  $|\mathcal{A}| \geq (1 - o(1)) |\mathcal{L}_n(K_{h,h})|$  using the argument in Lemma 6 with the results from Lemmas 10 and 11. Define also the matrix  $A$  and  $N_1$  as in the proof of Theorem 1. The proofs of the following Lemmas are similar to the proofs of Lemmas 7 and 8.

**Lemma 15.** *If  $G_i \in \mathcal{F}$  then*

$$\sum_j a_{i,j} \geq (1 - o(1)) \binom{h^2n - 25hn}{(1 - \beta)h^2n - 25hn}.$$

It follows immediately from Lemma 15 that

$$N_1 \geq (1 - o(1)) \binom{h^2n - 25hn}{(1 - \beta)h^2n - 25hn} |\mathcal{F}|. \quad (16)$$

We now obtain an upper bound on  $N_1$ . Let

$$\mathcal{X} = \{H : \exists i, j \text{ for which } G_{i,j} = H \text{ and } a_{i,j} = 1\}$$

It follows from the definition that

$$|\mathcal{X}| \leq \binom{h^2 n}{25hn} \left( (1 + o(1)) \binom{n}{\beta n}^2 (\beta n)! \right)^{h^2}. \quad (17)$$

For a fixed  $H \in \mathcal{H}$  let

$$\mathcal{G}_H = \{G_{i,j} : H_{i,j} = H \text{ and } a_{i,j} = 1\}.$$

Thus,

$$N_1 = \sum_{H \in \mathcal{X}} |\mathcal{G}_H|.$$

**Lemma 16.**

$$H \in \mathcal{X} \text{ implies } |\mathcal{G}_H| \leq e^{-ch^2 n} \left( (1 - \beta + O(n^{-1/3})n)! \right)^{h^2}. \quad (18)$$

for some absolute constant  $c > 0$ .

It follows from (16), (17) and (18) that  $\frac{|\mathcal{F}|}{|\mathcal{L}_n(K_h)|}$  is bounded above by

$$\begin{aligned} & \frac{e^{-ch^2 n} \left( (1 - \beta + O(n^{-1/3})n)! \right)^{h^2} \binom{h^2 n}{25hn} \left( (1 + o(1)) \binom{n}{\beta n}^2 (\beta n)! \right)^{h^2}}{(1 - o(1))(n!)^{h^2} \binom{h^2 n - 25hn}{(1-\beta)h^2 n - 25hn}} \\ & \leq \frac{e^{-ch^2 n/2} \binom{n}{\beta n}^{h^2}}{\binom{h^2 n}{\beta h^2 n} \beta^{24hn}} \\ & \leq e^{-ch^2 n/2 + 24hn \ln(1/\beta)} \\ & = o(1). \end{aligned}$$

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