PREPARATION FOR ORAL EXAM
MAIN THEOREM AND USEFUL COROLLARY

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Contents

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Part 1. The One Dimensional Cases - Sobolev Functions, BV Functions, and AC Fucntions

1. Sobolev Functions v.s. AC Functions v.s. BV Functions

Theorem 1.1 (The Relationship Between $W^{1,1}$, BV, and AC)**.** *We discuss this for I bounded and unbounded:*

(1) *Let I be open bounded, then we have*

$$W^{1,1}(I) = AC(I) \subsetneq BV(I)$$

(2) *Let I be unbounded, we have*

$$W^{1,1}(I) \subsetneq AC(I) \text{ and } W^{1,1}(I) \subsetneq BV(I),$$

but there is no relationship between $AC(I)$ and $BV(I)$, because AC is more or less a local property but BV is always a global property.

Theorem 1.2 (The Algebraical Properties of $AC(I)$, $W1,p(I)$, and $BV(I)$)**.** *We assume I is bounded, then*

(1) *Let $u, v \in AC(I)$, we have*
   (a) *$uv \in AC(I)$,*
   (b) *$u/v \in AC(I)$ provided that $v > 0$*
   (c) ~~*??AC(I) is separable??*~~ *Yes, on bounded interval it is separable.*
   (d) *$f \circ u \in AC(I)$ if and only if f is locally Lipschitz*

(2) *Let $u, v \in W^{1,p}(I)$, we have*
   (a) *$uv \in W^{1,p}(I)$,*
   (b) *$u/v \in W^{1,p}(I)$ provided $v \geq c > 0$,*

(3) *Let $u, v \in BV(I)$, we have*
   (a) *$uv \in BV(I)$,*
   (b) *$u/v \in BV(I)$ provided $v \geq c > 0$,*
   (c) *$BV(I)$ is NOT separable under norm $\|u\|_{BV(I)} := \|u\|_{L^1(I)} + Var(u)$.*
   (d) *$f \circ u \in BV(I)$ if and only if f is locally Lipschitz*
   (e) *The smallest vector space of functions $u : I \to \mathbb{R}$ that contains all monotone functions (respectively, bounded monotone functions) is given by the space $BPV_{loc}(I)$ (respectively, $BPV(I)$). Moreover, every function in $BPV_{loc}(I)$ (respectively, $BPV(I)$) may be written as a difference of two increasing functions (respectively, two bounded increasing functions).*

Theorem 1.3 (The Embedding in 1-D)**.** *Generally, for unbounded $I \subset \mathbb{R}$, we have, for $1 \leq p \leq \infty$*

$$\|u\|_{C_0(I)} \leq C \|u\|_{W^{1,p}(I)}.$$

So $W^{1,p}(I)$ is necessarily a bounded continuous function.
Also, we have that the bounded variation function is necessarily bounded, i.e.,

$$BV(I) \subset L^\infty(I)$$

Next, for I bounded, we have

(1) *for $1 < p \leq \infty$, we have*

$$W^{1,p}(I) \subset\subset C^{0,\alpha}(I)$$

for any $\alpha < 1 - 1/p$.

(2) *for $p = 1$, we have*

$$W^{1,1}(I) \subset\subset L^q(I), \quad 1 \leq q < \infty$$

and

$$BV(I) \subset\subset L^q(I), \quad 1 \leq q < \infty$$

Theorem 1.4 (Some Theorems)**.** *We list some important theorems in 1-D special cases.*

(1) *The space $BV([0,1])$ armed with norm $\|u\|_{BV} := |u(c)| + Var[u]$, is a Banach space.*

(2) *As* $V(x) := Var_{[0,x]}u$, *we have* $V'(x) = |u'(x)|$ *a.e.,*

$$V(x) \geq \int_0^x |u'(t)|\, dt$$

and the equality hold if and only if $u \in AC(0,1)$

(3) *This is the Helly's selection theorem. Assume* $\mathcal{F}$ *is an infinite collection of function* $u \in BV([0,1])$ *and* $\|u\|_{BV} \leq C < \infty$ *for all* $u \in \mathcal{F}$. *Then there exists* $\{u_n\}_{n=1}^{\infty} \subset \mathcal{F}$ *and a function* $v \in BPV(I)$ *such that* $u_n(x) \to v(x)$ *for all* $x \in I$.

(4) *This is l.s.c. in 1-D. If* $u_n \to u$ *everywhere, then* $u \in BV(I)$ *and*

$$\liminf_{n\to\infty} Var[u_n] \geq Var[u]$$

(5) *Having* $V(x)$ *is defined, we can write* $u = V - (V - u)$ *as the combination of two increasing functions. Also, all* V, $V - u$, *and* $V + u$ *are increasing.*

Theorem 1.5 (The Relationship Between Hölder continuous, AC, UC, and BV)**.** *We assume* $I \subset \mathbb{R}$ *is bounded, then*

(1) u *is Hölder then* u *is* UC.

(2) u *is Hölder can not imply* u *is* AC, *nor* u *is* AC *can NOT imply* u *is Hölder neither.*

Remark 1.6. A very interesting approach to study Sobolev and BV space is to study 1-D first and try to extend the result to Muti-Dimensions.

Hence, we find some properties hold by case $N = 1$ also hold for $N < p$.

Part 2. Sobolev Spaces

2. Basic Definition and Properties

Remark 2.1. In this section, unless specific, $\Omega \subset \mathbb{R}^N$ will always denotes an open set, not necessary bounded. By $V \subset\subset \Omega$ we mean V is open and $\bar{V} \subset \Omega$ is compact.

Remark 2.2. Before we formally talk about Sobolev space. We remark that what Sobolev do, more then L^p, is L^p only capture the width and height of a function but Sobolev capture the scale and oscillation as well.

Definition 2.3. (The Weak Partial Derivative)
Assume $\Omega \subset \mathbb{R}^N$ is open and $u \in L^1_{\text{loc}}(\Omega)$. We say $g \in L^1_{\text{loc}}(\Omega)$ is the *weak partial derivative* of u if

$$\int_\Omega u \cdot \partial_i \phi \, dx = -\int_\Omega g \, \phi \, dx,$$

for all $\phi \in C_c^\infty(\Omega)$. or C_c^1 is enough.

Definition 2.4. (The Definition of Sobolev Space) Here we provide 3 equivalent definitions of Sobolev space. The 1st is the general definition, the 2nd one use Risze representation and the 3rd one use weak compactness of L^p space. Therefore, the last two ONLY work at the case $1 < p \leq \infty$.

(1) We say $u \in W^{1,p}(\Omega)$ if $u \in L^p(\Omega)$ and its weak partial derivative belongs to $L^p(\Omega)$ as well.
(2) For $1 < p \leq \infty$, there exists $C > 0$ such that

$$\left| \int_\Omega u \, \partial_i \varphi \, dx \right| \leq C \, \|\varphi\|_{L^{p'}(\Omega)}$$

for all $\varphi \in C_c^1(\Omega)$
(3) For $1 < p \leq \infty$, there exists $C > 0$ such that for all $\Omega' \subset\subset \Omega$, and $|h| < \text{dist}(\Omega', \partial\Omega)$,

$$\|\tau_h u - u\|_{L^p(\Omega')} \leq C \, |h|,$$

where $\tau_y u(x) = u(x + y)$.

Definition 2.5. (The Sobolev Space)

(1) We say $u \in W^{1,p}(\Omega)$ if $u \in L^p(\Omega)$ and its weak partial derivative belongs to $L^p(\Omega)$ as well.
(2) We say $u \in W^{1,p}_{\text{loc}}(\Omega)$ if $u \in W^{1,p}(V)$ for any $V \subset\subset \Omega$.
(3) We say u is a Sobolev function if $u \in W^{1,p}_{\text{loc}}(\Omega)$ for some $1 \leq p \leq \infty$.

Remark 2.6. $W^{1,p}(\Omega)$ is uniformly convex for any $1 < p < \infty$ with respect to norm

$$\|u\|_{W^{1,p}(\Omega)} := \left(\int_\Omega |u|^p \, dx + \sum_{i=1}^N \int_\Omega |\partial_i u|^p \, dx \right)^{\frac{1}{p}}$$

Theorem 2.7 (The Interpolation by Smoothness, an equivalent norm)**.** *We have, for any $\varepsilon > 0$, that*

$$\|u\|_{W^{m,p}} \leq \varepsilon \, \|D^\alpha u\|_{L^p} + C_\varepsilon \, \|u\|_{L^q},$$

where $|\alpha| = m$ and $q = p$ will work, but usually if the domain is good enough and by embedding we could extend q to where embedding would go.
And this is also provide an equivalent norm in $W^{m,p}$.

Comment: This theorem also gives us a general idea: some times the extreme term (The highest and lowest) in a sum often already suffice to control the intermediate terms. For example, C^k norm and $C^{k,\alpha}$ norm has similar properties.

Definition 2.8. (The Definition by Fourier Transform)
A function $u \in H^k(\mathbb{R}^N)$ if and only if

$$(1+|y|^2)^{\frac{s}{2}}\hat{u}(y) \in L^2(\mathbb{R}^N).$$

Moreover, there exists a constant C_1 and C_2 such that

$$C_1 \|u\|_{H^k(\mathbb{R}^N)} \leq \left\|(1+|y|^2)^{\frac{s}{2}}\hat{u}(y)\right\|_{L^2(\mathbb{R}^N)} \leq C_2 \|u\|_{H^k(\mathbb{R}^N)}.$$

In addition, we extend our definition to any $-\infty < s < \infty$, could be non-integer, to define $H^s(\mathbb{R}^N)$ for

$$\|u\|_{H^s(\mathbb{R}^N)} = \left\|(1+|y|^2)^{\frac{s}{2}}\hat{u}(y)\right\|_{L^2(\mathbb{R}^N)}.$$

Keep in mind that we only use Fourier Transform to define Sobolev space for $p = 2$ on entire space $\mathbb{R}^N$, that is, they are all Hilbert.

Theorem 2.9 (The Weak Compactness in $W^{1,p}(\Omega)$)**.** *Let $\Omega \subset \mathbb{R}^N$ be open, suppose the sequence $\{u_n\}_{n=1}^\infty$ is uniformly bounded in $W^{1,p}(\Omega)$, then, up to a subsequence, we have*

$$u_n \rightharpoonup u \text{ in } W^{1,p}(\Omega)$$

and

$$\nabla u_n \rightharpoonup \nabla u \text{ in } [W^{1,p}(\Omega)]^N$$

Moreover, we can future extract a subsequence such that $u_n \to u$ a.e., no matter Ω is bounded of not. Ω can even be $\mathbb{R}^N$.

Definition 2.10 (Mollifier)**.** The standard mollifier we will take all the way is defined as

$$\eta(x) := \begin{cases} c\exp\left(\frac{1}{|x|^2-1}\right) & 0 < |x| < 1 \\ 0 & |x| \geq 1 \end{cases}$$

and adjust constant c such that

$$\int_{B(0,1)} \eta(x)dx = 1.$$

Then we define our mollifier to be

$$\eta_\varepsilon(x) := \frac{1}{\varepsilon^N}\eta\left(\frac{x}{\varepsilon}\right).$$

We notice that η_ε is radially symmetric and compact supported in $B(0,\varepsilon)$.

Theorem 2.11. [The properties of mollification]
We define, for $u \in L^1_{loc}(\mathbb{R}^N)$

$$u_\varepsilon(x) := \int_{\mathbb{R}^N} u(y)\eta_\varepsilon(x-y)dy = \int_{\mathbb{R}^N} u(x-y)\eta_\varepsilon(y)dy.$$

1. *$u_\varepsilon \in C^\infty(\mathbb{R}^N)$ as long as u is $L^1_{loc}(\mathbb{R}^N)$*
2. *$u_\varepsilon \to u$ uniformly on a compact set if u is continuous.*
3. *$u_\varepsilon(x) \to u(x)$ on every Lebesgue point of u.*
4. *For any $V \subset\subset \Omega$, we have $u_\varepsilon \to u$ in $L^p(V)$ of $u \in L^p(\Omega)$, for $1 \leq p < \infty$. ~~??what happens for $p = \infty$??~~ For $p = \infty$, we have uniform norm, which leads u_ε never leave $C(V)$, but u can be out of $C(V)$ for sure.*
5. *Suppose $u \in W^{1,p}(\Omega)$, we have*

$$\partial_i u_\varepsilon(x) = \int_{\mathbb{R}^N} \frac{\partial}{\partial x_i}\eta_\varepsilon(x-y)u(y)dy = \int_{\mathbb{R}^N} \eta_\varepsilon(x-y)\frac{\partial}{\partial y_i}u(y)dy = \eta_\varepsilon * \partial_i u.$$

6. *For any $V \subset\subset \Omega$, we have $u_\varepsilon \to u$ in $W^{1,p}(V)$ for $u \in W^{1,p}(\Omega)$. Or, in other word, $u_\varepsilon \to u$ in $W^{1,p}_{loc}(\Omega)$*
7. *If $u_n \rightharpoonup u$ in $L^p(\Omega)$, then for $x \in \Omega$,*

$$(u_n)_\varepsilon(x) \to u_\varepsilon(x),$$

i.e., η_ε is a good test function for any p.

Theorem 2.12. [The Local Approximation by C^∞ functions]
For any $u \in W^{1,p}(\Omega)$, $1 \le p < \infty$, there exists $\{u_n\}_{n=1}^\infty \subset C^\infty(\Omega) \cap W^{1,p}(\Omega)$ such that

$$\lim_{n\to\infty} \|u_n - u\|_{W^{1,p}(\Omega)} = 0.$$

note that p can NOT take value $+\infty$. Because if we do, u_n will end up with a C^1 function.

Corollary 2.13. *Take $p = \infty$ and Ω is finite, we can still have $\{u_n\}_{n=1}^\infty \subset C^\infty(\Omega) \cap W^{1,\infty}(\Omega)$ but only with*

$$\lim_{n\to\infty} \|u_n - u\|_{L^\infty(\Omega)} = 0,$$

and

$$\lim_{n\to\infty} \|\nabla u_n\|_{L^\infty(\Omega)} = \|\nabla u\|_{L^\infty(\Omega)} \text{ and } \nabla u_n \to \nabla u \text{ a.e.}$$

Comment: this Corollary implies that if $u \in W^{1,\infty}(\Omega)$, then $u \in C^0(\Omega)$, no matter what ugly boundary we have. Of course, we could have embedding such that $W^{1,\infty}(\Omega) \subset C^0(\Omega)$, or, in another word, trivially,

$$\|u\|_{C^0(\Omega)} \le \|u\|_{W^{1,\infty}}$$

Theorem 2.14. [The Global Approximation by C_c^∞ function]
Here we assume that Ω is open, bounded, and Lipschitz boundary (Actually an extension domain will do). Then we have for any $u \in W^{1,p}(\Omega)$, $1 \le p < \infty$, there exists $\{u_n\}_{n=1}^\infty \subset C^\infty(\overline{\Omega}) \cap W^{1,p}(\Omega)$ such that

$$\lim_{n\to\infty} \|u_n - u\|_{W^{1,p}(\Omega)} = 0.$$

Again, note that p can NOT take value $+\infty$.
~~??can we have same result as in Corollary above??~~ For $p = \infty$, we have same result as in Corollary above.
Moreover, we could do the same for unbounded C^0 domain.

Theorem 2.15 (The Fake Global Approximation by C_c^∞ function on arbitrary domain)**.** *Let $\Omega \subset \mathbb{R}^N$ be given, no other assumptions, then for $u \in W^{1,p}(\Omega)$, there exists $\{u_n\}_{n=1}^\infty \subset C_c^\infty(\mathbb{R}^N)$ such that*

$$u_n \to u \text{ in } L^p(\Omega),$$

and

$$\nabla u_n \to \nabla u \text{ in } L^p(\Omega')$$

for any $\Omega' \subset\subset \Omega$.

Corollary 2.16 (Better Approximation in $\mathbb{R}^N$)**.** *Let $u \in W^{1,p}(\mathbb{R}^N)$, $1 \le p < \infty$, there exists $\{v_n\}_{n=1}^\infty \subset W^{1,p}(\mathbb{R}^N) \cap C_c^\infty(\mathbb{R}^N)$ such that*

$$\lim_{n\to\infty} \|v_n - u\|_{W^{1,p}(\mathbb{R}^N)} = 0.$$

This is the special view of $W_0^{1,p}(\mathbb{R}^N) = W^{1,p}(\mathbb{R}^N)$

Theorem 2.17 (The Algebraical Properties of $W^{k,p}(\Omega)$)**.** *We claim that when $N < kp$, $W^{k,p}(\Omega)$ is an algebra, i.e., for u, $v \in W^{k,p}(\Omega)$ we have $uv \in W^{k,p}(\Omega)$ as well. The key is that embedding should be good enough so that $W^{k,p}(\Omega) \subset L^\infty(\Omega)$. This observation also matches what we required for Chain rule in Sobolev space.*

Theorem 2.18 (The Extension Domain)**.** *Suppose Ω is open bounded with Lipschitz boundary (or a bounded extension domain). Let $\Omega \subset\subset V$, there exists a linear bounded operator $\mathcal{E}$: $W^{1,p}(\Omega) \to W^{1,p}(\mathbb{R}^N)$ such that*

(1) *$spt(\mathcal{E}(u)) \subset V$.*
(2) *$\mathcal{E}(u) = u$ for every $x \in \Omega$.*
(3) *$\|\mathcal{E}(u)\|_{W^{1,p}(\mathbb{R}^N)} \le C \|u\|_{W^{1,p}(\Omega)}$, where $C = C(p, N, \Omega, U)$*

Remark 2.19. Any open set Ω, not necessary bounded, could be extension domain if it has smooth, Lipschitz, boundary.

Theorem 2.20. [Trace Operator]
Suppose Ω is open bounded with Lipschitz boundary and $1 \le p < \infty$. Then there exists a bounded linear operator T : $W^{1,p}(\Omega) \to L^p(\partial\Omega)$ such that

(1) *$T[u] = u$ on $\partial\Omega$ if $u \in C^0(\bar{\Omega}) \cap W^{1,p}(\Omega)$.*

(2) $\|T[u]\|_{L^p(\partial\Omega)} \le C\,\|u\|_{W^{1,p}(\Omega)}$ *where* $C = C(\Omega, p, N)$
(3) *Moreover, we have* $\|u\|_{L^p(\Omega)} \le C(\|T[u]\|_{L^p(\partial\Omega)} + \|\nabla u\|_{L^p(\Omega)})$ *where* $C = C(\Omega, p, N)$
(4) *The integration by part formula is valid, i.e. for any* $\varphi \in C^1(\mathbb{R}^N; \mathbb{R}^N)$.

$$\int_\Omega u\, div\, \varphi\, dx = -\int_\Omega \langle \nabla u, \varphi\rangle\, dx + \int_{\partial\Omega} T[u]\,\varphi \cdot \nu\, dS\, x,$$

where ν *is the outer normal vector of* Ω.
Moreover, by approximation, we have, for $u,\ v \in W^{1,p}(\Omega)$

$$\int_\Omega u\,\partial_i v\, dx = -\int_\Omega \partial_i u\, v\, dx + \int_{\partial\Omega} T[u]\,T[v]\,\sigma_i\, dS\, x.$$

Comment: for the case $p = \infty$ *there is not too much to talk about. Because as* Ω *process a nice boundary,* $W^{1,\infty}(\Omega)$ *is necessary a Lipschitz function and can be extended unique up to the boundary.*
Comment: we never have trace operator in L^p *space, as showed in Q8 in* [1].

Theorem 2.21. [The Line Properties of $W^{1,p}$ Functions]

(1) *If* $u \in W^{1,p}_{loc}(\mathbb{R}^N)$, *then for each* $n = 1, \dots, N$ *the functions*

$$u_n(x', t) \equiv u(\dots, x_{n-1}, t, x_{n+1}, \dots)$$

are absolutely continuous in t *on compact subsets of* $\mathbb{R}$, *for* $\mathcal{L}^{N-1}$ *a.e. point* $x' = (\dots, x_{n-1}, x_{n+1}, \dots) \in \mathbb{R}^{N-1}$. *In addition* $(u_n)' \in L^p_{loc}(\mathbb{R}^N)$.
(2) *Conversely, suppose* $f \in L^p_{loc}(\mathbb{R}^N)$ *and* $f = g$ $\mathcal{L}^N$ *a..e, where for each* $n = 1, \dots, N$ *the functions*

$$g_n(x', t) \equiv g(\dots, x_{n-1}, t, x_{n+1}, \dots)$$

are absolutely continuous in t *on compact subsets of* $\mathbb{R}$ *for* $\mathcal{L}^{n-1}$ *a.e. point* $x' = (\dots, x_{n-1}, x_{n+1}, \dots) \in \mathbb{R}^{N-1}$, *and* $g_n' \in L^p_{loc}(\mathbb{R}^N)$. *Then* $u \in W^{1,p}_{loc}(\mathbb{R}^N)$

Theorem 2.22 ($W^{1,\infty}$ and Lipschitz Function)**.** *In short,* $W^{1,\infty}_{loc}(\Omega)$ *is equal to Locally Lipschitz function in* Ω.
$W^{1,\infty}(\Omega)$ *is equal to Lipschitz function in* Ω *for* Ω *open bounded extension domain. This is result in* [1], *p279.*
But for arbitrary domain, especially the domain lies on the both side of boundary, this result will fail.

Theorem 2.23 (The Póincare Inequality in $B(x, r)$)**.** *For any* $u \in W^{1,p}_0(\Omega)$, $1 \le p < \infty$, *we have*

$$\int_\Omega |u - u_B|^p\, dx \le r^p \int_\Omega |\nabla u|^p\, dx.$$

~~*??For* $p = \infty$, *this is still true.??*~~ *Yes, as same reason as above.*

Theorem 2.24 (The General Póincare Inequality)**.** *Let* Ω *be an open connected extension domain with finite measure, then for any* $u \in W^{1,p}(\Omega)$, $1 \le p < \infty$ *and* $|E| \neq 0$,

$$\int_\Omega |u - u_E|^p\, dx \le C \int_\Omega |\nabla u|^p\, dx,$$

where $C = C(\Omega, E, p, n)$. *Thus, if we have* $u \equiv 0$ *in a positive measure set, we could obtain the same type of Póincare Inequality in* $W^{1,p}_0(\Omega)$ *case.*
For $p = \infty$, *we have the same result, namely*

$$\|u - u_\Omega\|_{L^\infty(\Omega)} \le C\,\|\nabla u\|_{L^\infty(\Omega)}.$$

This result from p274 in [1]. ~~*??does case E hold in* L^∞ *as well??*~~ *Yes, the embedding works just fine. The key is the embedding for* $W^{1,\infty} \subset\subset C^{0,\alpha}(\Omega)$ *for any* $\alpha < 1$.

Corollary 2.25. *Let* Ω *be an open bounded extension domain, then for any* $u \in W^{1,p}(\Omega)$, $1 \le p < N$,

$$\left(\int_\Omega |u - u_E|^{p^*}\, dx\right)^{\frac{1}{p^*}} \le C\left(\int_\Omega |\nabla u|^p\, dx\right)^{\frac{1}{p}},$$

where $C = C(\Omega, E, p, n)$.

Theorem 2.26. [The Differential Quotient]
We define the differential quotient $D_i^h(u)(x)$ to be

$$D_i^h(u)(x) = \frac{u(x+he_i) - u(x)}{h},$$

for any $x \in \Omega_h$. Then we have

(1) *For $1<p \leq \infty$, $u \in L^p(\Omega)$, if, for all $1 \leq i \leq N$,*

$$\sup_{h>0} \left\| D_i^h(u) \right\|_{L^p(V)} \leq M < \infty,$$

for all $0 < |h| < \frac{1}{2} dist(V, \partial\Omega)$. Then we have $u \in W^{1,p}(V)$ with $\|\nabla u\|_{L^p(V)} \leq M$.
This is false for $p = 1$ because we may end up in $BV(\Omega)$. For example, take any continuous function in 1-D which converges to a jump function

(2) *Suppose $1 \leq p < \infty$ and $u \in W^{1,p}(\Omega)$. Then for each $V \subset\subset U$,*

$$\left\| D^h u \right\|_{L^p(V)} \leq C \|Du\|_{L^p(\Omega)},$$

for some constant C and all $0 < |h| < \frac{1}{2} dist(V, \partial\Omega)$. ??counter-example for $p = \infty$??

(3) *We future have*

$$\lim_{h\to 0} \left\| D_i^h u \right\|_{L^p(\Omega_h)} = \|\partial_i u\|_{L^p(\Omega)}$$

and

$$\left\| D_i^h u - \partial_i u \right\|_{L^p(\mathbb{R}^N)} = 0$$

for $1 \leq p < \infty$.

Note: for case $1 < p < \infty$ we can prove easily by using uniformly convex space property, but for $p = 1$ we need to do computation directly.

Theorem 2.27 (The Dual Space of $W^{1,p}(\Omega)$)**.** *Given $\Omega \subset \mathbb{R}^N$ is open, then we define $W^{-1,p'}(\Omega) = \left(W_0^{1,p}(\Omega)\right)^*$ and we have:*

(1) *For any $f \in \left(W^{1,p}(\Omega)\right)^*$, there exists f^0, $f^1 \ldots$, $f^n \in L^{p'}(\Omega)$ such that*

$$\langle f, v \rangle = \int_\Omega f^0 v\, dx + \sum_{i=1}^n \int_\Omega f^i \partial_i v\, dx;$$

and

$$\|f\|_{W^{-1,p}(\Omega)} = \max\left\{\|f_i\|_{L^{p'}(\Omega)}\right\}$$

(2) *If we assume $f \in W^{-1,p'}(\Omega)$, we may take $f_0 \equiv 0$.*

Moreover, if Ω is bounded in one direction, we could take $f^0 = 0$ because now by Poincáre that we have equivalent norm $\|\nabla u\|_{L^p(\Omega)}$.

Remark 2.28. We denote by $H^{-1}(\Omega)$ the dual of $H_0^1(\Omega)$. The dual of $L^2(\Omega)$ is identified with $L^2(\Omega)$, but we do NOT identify $H_0^1(\Omega)$ with its dual, and we have inclusions: $H_0^1(\Omega) \subset L^2(\Omega) \subset H^{-1}(\Omega)$, where the injection are cont's and dense.
Also, if Ω is bounded then we have $W_0^{1,p}(\Omega) \subset L^2(\Omega) \subset W^{-1,p}(\Omega)$ if $2N/(N+2) \leq p \leq 2$, and if Ω is not bounded, the same holds but only for the range $2N/(N+2) \leq p \leq 2$.

3. Sobolev Embedding & Compact Embedding

Theorem 3.1 ($p < N$)**.** *Suppose $u \in W^{1,p}(\mathbb{R}^N)$ and $p < N$, then we have*

$$\left(\int_{\mathbb{R}^N} |u|^{p^*} dx\right)^{\frac{1}{p^*}} \leq C \left(\int_{\mathbb{R}^N} |\nabla u|^p dx\right)^{\frac{1}{p}},$$

where $C = C(N, p)$ and

$$p^* = \frac{Np}{N-p}.$$

Moreover, by L^p interpolation, we have $\|u\|_{L^q(\mathbb{R}^N)} \leq C\|\nabla u\|_{L^p(\mathbb{R}^N)}$ for any $p \leq q \leq p^*$.
note that we have no information for $q < p$.

Comment: This is an poincare-like inequality, and this is the only place we have it. We lose this poincare-like properties in the case of $N = p$ or extension domain.

Corollary 3.2 ($p < N$ for Bounded Extension Domain)**.** *Suppose $u \in W^{1,p}(\Omega)$ and $p < N$, and we in addition assume that Ω is open, bounded, extension domain, then we have $W^{1,p}(\Omega)$ is continuous embedded in $L^{p^*}(\Omega)$, i.e.,*

$$\|u\|_{L^{p^*}(\Omega)} \leq C\|u\|_{W^{1,p}(\Omega)}$$

where $C = C(N, p, \Omega)$ and

$$p^* = \frac{Np}{N-p}.$$

Moreover, since Ω is bounded, we have the theorem hold for any $1 \leq q \leq p^*$.

Theorem 3.3 ($p = N$)**.** *Suppose $u \in W^{1,p}(\mathbb{R}^N)$ and $p = N$, then we have*

$$\|u\|_{L^q(\mathbb{R}^N)} \leq C\|u\|_{W^{1,p}(\mathbb{R}^N)}.$$

where $C = C(N, p)$ and $p \leq q < \infty$.
Note that we have no information for $q < p = N$.
The same result will be hold in bounded extension domain.

Theorem 3.4 ($p > N$)**.** *Suppose $u \in W^{1,p}(\mathbb{R}^N)$ and $p > N$ ($p = \infty$), then we have the space $W^{1,p}(\mathbb{R}^N)$ is continuous embedded in $C^{0,1-\frac{n}{p}}(\mathbb{R}^N)$ and*

$$\lim_{|x|\to\infty} u(x) = 0.$$

and also

$$|u(x) - u(y)| \leq C|x-y|^{1-\frac{N}{p}}\|\nabla u\|_{L^p(\mathbb{R}^N)}.$$

The same result will be hold in bounded extension domain but with

$$|u(x) - u(y)| \leq C|x-y|^{1-\frac{N}{p}}\|u\|_{W^{1,p}(\Omega)}$$

The extreme case for $p = \infty$, we get nothing but

$$|u(x) - u(y)| \leq C|x-y|\|u\|_{L^\infty(\Omega)}$$

i.e., an Lipschitz function, and this match with the observation that $W^{1,\infty}(\Omega) = C^{0,1}(\bar{\Omega})$ if Ω is an extension domain.

Corollary 3.5. *[The Algebraical Property of $W^{1,p}(\mathbb{R}^N)$ ($W^{1,p}(\Omega)$)] Assume $p > N$, then we have space $W^{1,p}(\mathbb{R}^N)$ is an algebra, i.e., $uv \in W^{1,p}(\mathbb{R}^N)$ if u, $v \in W^{1,p}(\mathbb{R}^N)$. The same thing hold for Ω is an extension domain.*
Notice that this observation is consistent with the product rule in $W^{1,p}$, as product rule require that u, $v \in W^{1,p}(\Omega) \cap L^\infty(\mathbb{R}^N)$

Corollary 3.6 (A Special Case)**.** *We have*

$$\|u\|_{C^0(\mathbb{R}^N)} \leq C\|u\|_{W^{N,1}(\mathbb{R}^N)}$$

This is the case $N = kp$. But we actually get L^∞ estimation, even better, C^0.

Comment: before we talk about compact embedding, we provide the general idea behind it. Actually, sequence bounded in a high regularity space, and constrained to lie in a compact domain, will tend to have convergent subsequence in low regularity spaces (regularity with respect to derivative). As a by-product, we have the compact embedding in Hölder space.

Theorem 3.7 (The Compact Embedding in Hölder Space)**.** *Let $\Omega \subset \mathbb{R}^N$ be open bounded, not necessarily have a good boundary, we have*

$$C^{0,\alpha}(\bar{\Omega}) \subset\subset C^{0,\beta}(\bar{\Omega})$$

for any $\alpha < \beta$.

Theorem 3.8 (The Compact Embedding in Bounded Extension Domain for $1 \le p < N$)**.** *Suppose Ω is an open bounded extension domain then we have*

$$W^{1,p}(\Omega) \subset\subset L^q, \text{ for any } 1 \le q < p^*$$

Notice that Ω is finite is enough, not necessary to be bounded. See Leoni's Theorem 11.10
Notice that we can say NOTHING about $q = p^*$. See example 13.7
Notice that for $p = 1$, we may end up in $BV(\Omega)$.??counterexample??

Theorem 3.9 (The Compact Embedding in Extension Domain for $p = N$)**.** *Suppose Ω is an open bounded extension domain then we have*

$$W^{1,p}(\Omega) \subset\subset L^q, \text{ for any } 1 \le q < \infty$$

note that we can say NOTHING about $q = \infty$. ??counterexample??

Theorem 3.10 (The Compact Embedding in Bounded Extension Domain for $p > N$)**.** *Suppose Ω is an open bounded extension domain then we have*

$$W^{1,p}(\Omega) \subset\subset C^{0,\alpha}(\bar{\Omega}), \text{ for any } 0 \le \alpha < 1 - \frac{N}{p}$$

note that we can say NOTHING about $\alpha = 1 - \frac{N}{p}$. ??counterexample??
Also, if $p = \infty$, we have

$$W^{1,\infty}(\Omega) \subset\subset C^{0,\alpha}(\bar{\Omega}), \text{ for any } 0 \le \alpha < 1$$

Theorem 3.11. [Compact Embedding in Bounded Extension Domain for Arbitrary p]
In summary, if Ω be an open bounded Extension domain, we have the compact embedding

$$W^{1,p}(\Omega) \subset\subset L^p(\Omega),$$

is always compact for any $1 \le p \le \infty$.
Again, for $p = 1$ we may end up in BV, but the embedding is still compact.

Theorem 3.12 (Some Special Case for Unbounded, or Non-Extension Domain)**.** *Notice that in usual embedding theorem, boundness(finiteness) of domain and regularity of boundary is essential. However, we do have some exception.*

(1) *Generally, if Ω is unbounded (infinite), we have no compact embedding, and a moving triangle example would do. However a special case will happen for radial function. See Leoni's Exercise 11.19.*
(2) *Generally, if Ω is non-extensionable, we can not define mollification and hence the usual prove will not work. However, if we only consider the case $q < p$ and use uniformly integrability, we can overcome this difficult and obtain the following result:*
Suppose Ω is finite, not necessary extension-able, we still have that

$$W^{1,p}(\Omega) \subset\subset L^q, \text{ for any } 1 \le q < p.$$

Moreover, we have

$$W^{1,p}(\Omega) \subset\subset L^p$$

if and only if

$$\lim_{n\to\infty} \sup_{u \in W^{1,p}(\Omega)} \int_{\Omega \setminus \Omega_n} |u|^p \, dx = 0$$

where $\Omega_n := \{x \in \Omega,\ dist(x, \partial\Omega) > 1/n\}$

4. The $W_0^{1,p}(\Omega)$ Space

Here we comment out some special properties of $W_0^{1,p}$ functions.

Definition 4.1 (The space $W_0^{1,p}(\Omega)$, for $1 \le p < \infty$)**.** Given $\Omega \subset \mathbb{R}^N$ be open, we define $W_0^{1,p}(\Omega)$ to be the closure of $C_c^\infty(\Omega)$ in $W^{1,p}$ norm. Note that we do NOT define for $p = \infty$ in this way.

Remark 4.2 (The space $W_0^{1,\infty}(\Omega)$, for $1 \leq p < \infty$)**.** We usually define $W_0^{1,\infty}(\Omega)$ by define the space

$$W_0^{1,\infty}(\Omega) := \left\{u \in W^{1,\infty}(\Omega),\ T[u] = 0\right\}$$

and hence in this definition $W_0^{1,\infty}(\Omega)$ will contain the piece-wise affine functions. Also, we could identify $W_0^{1,\infty}(\Omega)$ by

$$W_0^{1,\infty}(\Omega) = \left\{\text{The weak star closure of } C_c^\infty(\Omega) \text{ under } W_0^{1,\infty} \text{ norm }\right\}$$

Lemma 4.3. *Suppose $u \in W^{1,p}(\Omega)$ and $spt(u) \subset\subset \Omega$, then $u \in W_0^{1,p}(\Omega)$.*

Theorem 4.4. *(The Equivalent Properties of $W_0^{1,p}(\Omega)$) Here we assume $1 < p < \infty$ and Ω is open only*

(1) $u \in W_0^{1,p}(\Omega)$

(2) *There exists $C > 0$ such that*

$$\left|\int_\Omega u\, \partial_i \varphi \, dx\right| \leq C \left\|\varphi\right\|_{L^{p'}(\Omega)}$$

for all $\varphi \in C_c^1(\mathbb{R}^N)$

(3) *The canonical extension*

$$\bar{u} := \begin{cases} u & x \in \Omega \\ 0 & x \notin \Omega \end{cases}$$

belongs to $W^{1,p}(\mathbb{R}^N)$. Moreover, we have

$$\nabla \bar{u} = \overline{\nabla u}$$

Remark 4.5. The implication from (1) to (2)&(3) is hold for $p = 1$ and require nothing on the boundary of Ω. However, as $(3) \to (1)$, we need $\partial\Omega$ to be regular.

Remark 4.6. The implication $(1) \to (3)$ is THE MOST IMPORTANT conclusion in $W_0^{1,p}(\Omega)$. It says that any $W_0^{1,p}(\Omega)$ function can be extended no matter what $\partial\Omega$ is. Hence, the embedding and compact embedding will work on $W_0^{1,p}(\Omega)$ no matter how ugly $\partial\Omega$ will be.

Theorem 4.7 (Some Useful but not Trivial result)**.** *Suppose $u \in W_0^{1,p}(\Omega)$, $1 \leq p \leq \infty$, $\Omega \subset \mathbb{R}^N$ is open, but not necessary have a nice boundary. Then we have*

(1) $u^+,\ u^- \in W_0^{1,p}(\Omega)$

(2) *if $u^+,\ u^- \in W_0^{1,p}(\Omega)$, we have $(u+v)^+ \in W_0^{1,p}(\Omega)$. (Here we do not assume $u,\ v \in W_0^{1,p}$)*

Theorem 4.8 (The Póincare Inequality in $W_0^{1,p}(\Omega)$)**.** *Suppose Ω is bounded in one direction, i.e., it lies between two parallel hyperplanes, then we have for any $u \in W_0^{1,p}(\Omega)$, $1 \leq p < \infty$, that*

$$\int_\Omega |u|^p \, dx \leq \frac{d^p}{p} \int_\Omega |\nabla u|^p \, dx,$$

where d is the one direction bound for Ω.
~~*??what is the analogous result for $p = \infty$??*~~ *For $p = \infty$ we can either prove directly or notice that $\lim_{p\to\infty} \|u\|_{L^p(\Omega)} = \|u\|_{L^\infty(\Omega)}$, we have*

$$\|u\|_{L^\infty(\Omega)} \leq d \left\|\nabla u\right\|_{L^\infty(\Omega)}.$$

Theorem 4.9 ($W_0^{1,p}(\Omega)$ vs $T(u) = 0$)**.** *Let Ω be open bounded and Lipschitz domain. Then $u \in W^{1,p}(\Omega)$ if and only if $T[u] = 0$. That is, the kernel of the trace operator is $W_0^{1,p}(\Omega)$.*

Remark 4.10. $W_0^{1,p}(\Omega)$ is strongly closed subspace by definition or the estimation of trace. Hence $W_0^{1,p}(\Omega)$ is weakly closed, so does the space $u_0 + W_0^{1,p}(\Omega)$, where $u_0 \in W^{1,p}(\Omega)$

Part 3. **BV Spaces**

5. Basic Definition and Properties

Remark 5.1. In this section, unless specific, $\Omega \subset \mathbb{R}^N$ will always denotes an open set, not necessary bounded.

Definition 5.2. (The Weakest Partial Derivative)
Assume $\Omega \subset \mathbb{R}^N$ is open and $u \in L^1_{\text{loc}}(\Omega)$. We say a Radon measure μ_i is the *weak partial distributional derivative* of u if

$$\int_\Omega u \cdot \partial_i \phi \, dx = -\int_\Omega \phi \, d\mu_i,$$

for all $\phi \in C_c^\infty(\Omega)$. or C_c^1 is enough.

Definition 5.3. Here again we provide 3 equivalent definitions of BV space. The 1st is the general definition, the 2nd one use Risze representation and the 3rd one use weak star compactness in Radon measure. Now, we actually take care the case $p = 1$.

(1) We say $u \in BV(\Omega)$ if μ_i is a finite Radon measure for $1 \leq i \leq N$
(2) Let $\Omega \subset \mathbb{R}^N$ be a given open set, $u \in L^1(\Omega)$ ($L^1_{\text{loc}}(\Omega)$ is not good enough). Then if

$$\sup_\varphi \left\{ \int_\Omega u \operatorname{div} \varphi \, dx, \;\; \varphi \in C_c^1(\Omega), \;\; \|\varphi\|_{L^\infty(\Omega)} \leq 1 \right\} < \infty,$$

we say $u \in BV(\Omega)$.
Moreover, we say a subset E (not necessary open) has finite perimeter in Ω if $\chi_E \in BV(\Omega)$.
~~??What if we have Ω be closed set?? ??$C_c^1(\Omega) = C_c^1(\bar{\Omega})$??~~ The space $C_c(\bar{\Omega})$ is not well-defined.
(3) There exists $C > 0$ such that for all $\Omega' \subset\subset \Omega$, and $|h| < \operatorname{dist}(\Omega', \partial\Omega)$,

$$\|\tau_h u - u\|_{L^1(\Omega')} \leq C\,|h|,$$

where $\tau_y u(x) = u(x + y)$.

Remark 5.4. We take 2nd definition and we show they are actually the same thing.

Theorem 5.5. [Basic Properties of $BV(\Omega)$ as a Algebracial Space] *Here we take $\Omega \subset \mathbb{R}^N$ as an open set.*

(1) *The space $BV(\Omega)$ is not separable, although the space L^1 is separable. See example 13.5*
(2) *The space $BV(\Omega)$ is not an algebra, as $u,\ v \in BV(\Omega)$, but we may not have $uv \in BV(\Omega)$. The problem is $BV(\Omega)$ is not good enough, i.e., $BV(\Omega) \not\subset L^\infty(\Omega)$, even if Ω is bounded. Compare with Corollary 3.5.*

Definition 5.6. We say $u \in BV_{\text{loc}}(\Omega)$ if $u \in BV(\tilde{\Omega})$ for any $\tilde{\Omega} \subset\subset \Omega$.

Theorem 5.7. [Risez Representation - The Structure Theorem for BV_{loc} Functions]
For any $u \in BV_{loc}(\Omega)$, there exists an Radon measure μ and a μ-measurable function $\sigma : \mathbb{R}^N \to \mathbb{R}^N$ such that

(1) $|\sigma(x)| = 1$ *for a.e.* $x \in \Omega$
(2)

$$\int_\Omega u \, div \, \varphi \, dx = -\int_\Omega \langle \varphi \cdot \sigma \rangle \, d\mu,$$

for every $\varphi \in C_c^1(\Omega)$.
(3) *If $u \in BV(\Omega)$, we have*

$$\mu(\Omega) = \sup_\varphi \left\{ \int_\Omega u \, div \, \varphi \, dx, \;\; \varphi \in C_c^1(\Omega), \;\; \|\varphi\|_{L^\infty(\Omega)} \leq 1 \right\}$$

Notation. We use $\|Du\|$ to denote μ and Du as $\|Du\| \lfloor \sigma$

Theorem 5.8 (Basic Properties of Du)**.** *Assume $u \in BV_{loc}(\Omega)$, we have*

(1) *If $Du = 0$, $u \equiv C$.*

(2) *For any locally Lipschitz function* $\Phi\colon \Omega \to \mathbb{R}$, *we have* $u\Phi \in BV_{loc}(\Omega)$ *and*

$$D(u\Phi) = \Phi Du + u \otimes \nabla\Phi$$

(3) *Let* $I_\varepsilon := \{x \in \Omega,\ dist(x, \partial\Omega) > \varepsilon\}$, *then* $\nabla(u * \eta_\varepsilon) = Du * \eta_\varepsilon$

Remark 5.9. We say Du is the weakest sense that we can talk about derivative, and we write

$$\int_\Omega u\, \partial_i \phi\, dx = -\int_\Omega \phi\, \sigma_i\, d\, \|Du\|\,,\ \ \phi \in C_c^1(\Omega).$$

Moreover, we use $\|\partial E\|$ to denote $\|D\chi_E\|$ and we say E has finite perimeter in Ω if $\chi_E \in BV(\Omega)$.

Corollary 5.10. *For any* Ω *open, not necessary bounded, we have*

(1) $W^{1,p}_{loc}(\Omega) \subset W^{1,1}_{loc}(\Omega) \subset BV_{loc}(\Omega)$, *for any* $1 \le p \le \infty$.

(2) ~~*??* $W^{1,p}(\Omega) \subset W^{1,1}(\Omega) \subset BV(\Omega)$*, for any* $1 \le p \le \infty$*.??*~~ *We do have* $W^{1,1}(\Omega) \subset BV(\Omega)$, *but* $W^{1,p} \subset W^{1,1}$ *won't be true if* $|\Omega| = \infty$. *??we need a counterexample??*

(3) $W^{1,1}(\Omega) \neq BV(\Omega)$ *as* $\chi_E \in BV(\Omega)$. *A set* E *with* C^1 *boundary would do.*

Corollary 5.11. *For any* E *has finite perimeter in* Ω, *we have*

(1) *If* E *has* C^1 *boundary, then* $\|\partial E\|(\Omega) = \mathcal{H}^{N-1}(\partial E \cap \Omega)$

(2) ~~*??*$\|\partial E\|$ *is supported in* ∂E*??*~~ *Yes, it is.*

Definition 5.12. We define

$$\|u\|_{BV(\Omega)} := \|u\|_{L^1(\Omega)} + \|Du\|(\Omega).$$

Definition 5.13. The usual topology induced by norm of BV is not really useful. Here we define the so call weak convergence and strict convergence in BV space.

(1) We say $\{u_n\}_{n=1}^\infty \subset BV(\Omega)$ weakly convergences in BV to u if $u_n \to u$ in $L^1(\Omega)$ and Du_n weakly star convergence to Du in Ω, i.e.,

$$\lim_{n\to\infty} \int_\Omega \phi\, dDu_n = \int_\Omega \phi\, dDu$$

for all $\phi \in C_c(\Omega; \mathbb{R}^N)$

(2) We say $\{u_n\}_{n=1}^\infty \subset BV(\Omega)$ strictly convergences in BV to u if $u_n \to u$ in $L^1(\Omega)$ and

$$\|Du_n\|(\Omega) \to \|Du\|(\Omega).$$

Theorem 5.14 (The Lower Semi-continuous of $\|Du\|$)**.** *For* Ω *open, if* $\{u_n\}_{n=1}^\infty \subset BV(\Omega)$ *and* $u_n \to u$ *in* $L^1_{loc}(\Omega)$, *we have*

$$\liminf_{n\to\infty} \|Du_n\|(\Omega) \ge \|Du\|(\Omega).$$

~~*??what is the analogous result in* $W^{1,p}(\Omega)$*??*~~ *For* $W^{1,p}(\Omega)$, *we don't know unless* $|\Omega| < \infty$. *However, we do have same result for* $W^{1,1}(\Omega)$ *as* $W^{1,1}(\Omega) \subset BV(\Omega)$, *and again, the strict greater can happen even for* $W^{1,1}(\Omega)$. *See Example 13.3.*

Theorem 5.15 (The Approximation by C^∞ functions)**.** *For* $\Omega \subset \mathbb{R}^N$ *open and* $u \in BV(\Omega)$, *there exists* $\{u_n\}_{n=1}^\infty \subset BV(\Omega) \cap C^\infty(\Omega)$ *such that*

$$\lim_{n\to\infty} \|u_n - u\|_{L^1(\Omega)} = 0.$$

However, we only have

$$\lim_{n\to\infty} \|Du_n\|(\Omega) = \|Du\|(\Omega).$$

~~*??for* Ω *is a Lipschitz domain, do we have* $u_n \in BV(\Omega) \cap C^\infty(\bar{\Omega})$*??*~~ *Yes, next corollary.*

Corollary 5.16 (Better Approximation in $\mathbb{R}^N$)**.** *Let* $u \in BV(\mathbb{R}^N)$, *there exists* $\{v_n\}_{n=1}^\infty \subset BV(\mathbb{R}^N) \cap C_c^\infty(\mathbb{R}^N)$ *such that*

$$\lim_{n\to\infty} \|v_n - u\|_{L^1(\mathbb{R}^N)} = 0.$$

However, we only have

$$\lim_{n\to\infty} \|Dv_n\|(\mathbb{R}^N) = \|Du\|(\mathbb{R}^N).$$

Corollary 5.17 (The Approximation by $C^\infty(\bar{\Omega})$ functions)**.** *For $\Omega \subset \mathbb{R}^N$ open bounded with extension domain, if $u \in BV(\Omega)$, then there exists $\{u_n\}_{n=1}^\infty \subset BV(\Omega) \cap C^\infty(\bar{\Omega})$ such that*

$$\lim_{n\to\infty} \|u_n - u\|_{L^1(\Omega)} = 0.$$

and

$$\lim_{n\to\infty} \|Du_n\|(\Omega) = \|Du\|(\Omega).$$

Corollary 5.18 (The Subset in Convergence)**.** *For $\Omega \subset \mathbb{R}^N$ open and $u \in BV(\Omega)$, assume $\{u_n\}_{n=1}^\infty \subset BV(\Omega)$ and*

$$\lim_{n\to\infty} \|u_n - u\|_{L^1(\Omega)} = 0, \ \lim_{n\to\infty} \|Du_n\|(\Omega) = \|Du\|(\Omega).$$

Then for any $\Omega' \subset \Omega$, we have

$$\lim_{n\to\infty} \|Du_n\|(\Omega') = \|Du\|(\Omega'),$$

provided that $\|Du\|(\partial\Omega' \cap \Omega) = 0$.
~~??what is the analogous result in $W^{1,p}(\Omega)$??~~ The Sobolev case is trivial because for any $u \in W^{1,p}(\Omega)$, $Du << \mathcal{L}^N$ and we never worry about Lebesgue measure 0 set.

Theorem 5.19 (Weak Convergence of Du)**.** *Assume u and u_n is obtained from Theorem 5.15, define*

$$\mu_n(B) := \int_{B\cap\Omega} dDu_n, \ \mu(B) := \int_{B\cap\Omega} dDu$$

Then, we have $\mu_n \overset{*}{\rightharpoonup} \mu$, i.e., for any $\phi \in C_c^0(\mathbb{R}^N)$, we have

$$\lim_{n\to\infty} \int_\Omega \phi \, d\mu_n = \int_\Omega \phi \, d\mu.$$

~~??should be $\phi \in C_c(\mathbb{R}^N)$??~~ Yes, this is the density result. See Theorem 11.26

Theorem 5.20 (Compactness in $BV(\Omega)$)**.** *Let Ω be an open bounded extension domain and*

$$\sup_{n\in\mathbb{N}} \|u_n\|_{BV(\Omega)} < \infty.$$

Then, up to a subsequence, we have

$$\lim_{n\to\infty} \|u_n - u\|_{L^q}(\Omega) = 0,$$

for any $1 \le q < 1^*$. Also, by the compactness of Radon measure, $Du_n \overset{*}{\rightharpoonup} Du$, up to a subsequence.

Theorem 5.21 (Trace Operator)**.** *Suppose Ω is open bounded with Lipschitz boundary. Then there exists a bounded linear operator $T : BV(\Omega) \to L^1(\partial\Omega; \mathcal{H}^{N-1})$ such that*

1. *$T[u] = u$ on $\partial\Omega$ a.e. if $u \in C^0(\bar{\Omega}) \cap W^{1,1}(\Omega)$.*
2. *$\|T[u]\|_{L^1(\partial\Omega)} \le C \|u\|_{BV(\Omega)}$ where $C = C(\Omega, N)$. This is the result of T being a linear bounded operator. But it is not clear why T is bounded from prove.*
3. *Also, we have another direction of trace estimation*

$$\|u\|_{L^1(\Omega)} \le C(\Omega, \partial\Omega) \left(\|T[u]\|_{L^(\partial\Omega)} + \|Du\|(\Omega) \right).$$

4. *The integration by part formula is valid, i.e. for any $\varphi \in C^1(\mathbb{R}^N; \mathbb{R}^N)$.*

$$\int_\Omega u \, div \, \varphi \, dx = -\int_\Omega \langle \sigma, \varphi \rangle \, d\|Du\| + \int_{\partial\Omega} T[u] \, \varphi \cdot \nu \, d\mathcal{H}^{N-1},$$

where ν is the outer normal vector of Ω.

5. *T[u](x) is a lebesgue point for a.e. $x \in \partial\Omega$, i.e.*

$$\lim_{r\to0} \frac{1}{\mathcal{L}^N(B(x,r)\cap\Omega)} \int_{B(x,r)\cap\Omega} |u - T[u](x)| \, dy = 0$$

Theorem 5.22 (Trace Operator - Guess Result and After Reduced Boundary)**.** .

(1) *Still assume Ω is open bounded with Lipschitz boundary, but ??By approximation, can we have, for $u, v \in BV(\Omega)$??*

$$\int_\Omega u\,\sigma(v)_i\,d\,\|Dv\| = -\int_\Omega \sigma(u)_i\,v\,d\,\|Du\| + \int_{\partial\Omega} T[u]\,T[v]\,\nu_i\,dS\,x.$$

(2) *Now we only assume E has ??locally?? finite perimeter in $\mathbb{R}^N$, ??do we have??*

$$\int_E u\,div\,\varphi\,dx = -\int_E \langle\sigma,\varphi\rangle\,d\,\|Du\| + \int_{\partial^* E} T[u]\,\varphi\cdot\nu_E\,d\mathcal{H}^{N-1},$$

where $\varphi \in C_c^\infty(\mathbb{R}^N;\mathbb{R}^N)$.

(3) *In the end, ??do we have??*

$$\int_E u\,\sigma(v)_i\,d\,\|Dv\| = -\int_E \sigma(u)_i\,v\,d\,\|Du\| + \int_{\partial^* E} T[u]\,T[v]\,(\nu_E)_i\,d\mathcal{H}^{N-1}.$$

Theorem 5.23 (The Fake Extension Operator)**.** *Given Ω open bounded with Lipschitz boundary, assume $u_1 \in BV(\Omega)$ and $u_2 \in BV(\mathbb{R}^N \setminus \Omega)$. Define*

$$\bar{u} := \begin{cases} u_1 & x \in \Omega \\ u_2 & x \in \mathbb{R}^N \setminus \Omega \end{cases}$$

Then we have $\bar{u} \in BV(\mathbb{R}^N)$ and

$$\|D\bar{u}\|(\mathbb{R}^N) = \|Du_1\|(\Omega) + \|Du_2\|(\Omega) + \int_{\partial\Omega} |T[u_1] - T[u_2]|\,dS$$

Definition 5.24 (The Extension Domain)**.** We say that an open set $\Omega \subset \mathbb{R}^N$ is an extension domain if $\partial\Omega$ is bounded and for any open $A \supset \bar{\Omega}$, there exists a linear and continuous extension operator $E\colon BV(\Omega) \to BV(\mathbb{R}^N)$ satisfying

(1) $Eu = 0$ a.e. in $\mathbb{R}^N \setminus A$ for any $u \in BV(\Omega)$
(2) $\|D(Eu)\|(\partial\Omega) = 0$ for any $u \in BV(\Omega)$
(3) for any $p \in [1,\infty]$ the restriction of E to $W^{1,p}(\Omega)$ induces a linear continuous map between this space and $W^{1,p}(\mathbb{R}^N)$.

Comment: the 2nd property is the most important one. It states that the measure $\|D(Eu)\|$ never charge on the boundary.

Theorem 5.25. [The Line Properties in BV function]
Assume $u \in L^1_{loc}(\mathbb{R}^N)$. Then $u \in BV_{loc}(\mathbb{R}^N)$ if and only if

$$\int_K ess V_a^b u_n dx' < \infty$$

for each $n = 1,\dots,N$, $a < b$, and compact set $K \subset \mathbb{R}^{N-1}$

6. BV Embedding and Coarea Formula

Theorem 6.1 (Sobolev-Type Embedding for BV Functions)**.** *Assume $u \in BV(\mathbb{R}^N)$, we have*

$$\left(\int_{\mathbb{R}^N} |u|^q\,dx\right)^{\frac{1}{q}} \le C\,\|Du\|(\mathbb{R}^N),$$

for any $1 \le q \le 1^*$.
~~??Similar result in extension domain??~~ Yes, if extension domain we have

$$\left(\int_\Omega |u|^q\,dx\right)^{\frac{1}{q}} \le C\,\|u\|_{BV(\Omega)},$$

Theorem 6.2 (BV-Type Póincare Inequalities)**.** *??Is there a similar define in BV for $W_0^{1,p}$??*

(1) *The general case:*
Let Ω be open bounded extension domain and $u \in BV(\Omega)$, we have for any positive measure $E \subset \Omega$,

$$\int_\Omega |u - u_E|\, dx \le C \|Du\|(\Omega).$$

In addition, by embedding, we have

$$\left(\int_\Omega |u - u_E|^q\, dx\right)^{\frac{1}{q}} \le C \|Du\|(\Omega),$$

for any $1 \le q \le 1^*$.

(2) *The ball $B(x,r)$ case: Let $u \in BV(B(x,r))$*

$$\left(\int_B |u - u_B|^q\, dx\right)^{\frac{1}{q}} \le C \|Du\|(B),$$

for any $1 \le q \le 1^*$.

(3) *For each $0 < \alpha \le 1$, there exists a constant $C_3(\alpha)$ such that*

$$\|u\|_{L^{n/n-1}(B(x,r))} \le C_3(\alpha) \|Du\|(B(x,r))$$

for all $B(x,r) \subset \mathbb{R}^N$, $u \in BV_{loc}(\mathbb{R}^N)$ satisfying

$$\frac{\mathcal{L}^n(B(x,r) \cap \{f = 0\})}{\mathcal{L}^N(B(x,r))} \ge \alpha$$

This is actually a direct result of (1).

Theorem 6.3. [The Isoperimetric Inequality]
Let E be a bounded set has finite perimeter in $\mathbb{R}^N$.Then we have

$$\left(\mathcal{L}^N(E)\right)^{\frac{N-1}{N}} \le C \|\partial E\|(\mathbb{R}^N).$$

Theorem 6.4 (The Relative Isoperimetric Inequality)**.** *Let E be a bounded set has finite perimeter in $\mathbb{R}^N$, then for any $B(x,r)$, we have*

$$\min\left\{\mathcal{L}^N(B(x,r) \cap E), \mathcal{L}^N(B(x,r) - E\right\}^{1-\frac{1}{N}} \le C \|\partial E\|(B(x,r)).$$

This is just the application of Poincare inequality above.

Notation. We define the set E_t to be

$$E_t := \{x \in \Omega, u(x) > t\}$$

Theorem 6.5. [The Coarea Formula]
For Ω open and $u \in BV(\Omega)$, we have

$$\|Du\|(\Omega) = \int_{-\infty}^{+\infty} \|\partial E_t\|(\Omega)\, dt,$$

and hence $\|\partial E_t\|(\Omega)$ is finite for a.e. $t \in \mathbb{R}$.

On the other hand, if

$$\int_{-\infty}^{+\infty} \|\partial E_t\|(\Omega)\, dt < \infty,$$

we have $u \in BV(\Omega)$.
??Do we have analogous result of $BV_{loc}(\Omega)$??
??Can we use reduced boundary to rewrite this?? ~~??what is the analogous result in $W^{1,p}(\Omega)$, $W^{1,1}(\Omega)$, Lipschitz??~~

Comment: for Lipschitz function, this is just the special case of co-area formula, namely, we have if u is Lipschitz, then

$$\int_{\Omega} |\nabla u| \, dx = \int_{-\infty}^{\infty} \mathcal{H}^{N-1}(\{x \in \Omega, \ u(x) = t\}) \, dt$$

However, for the $W^{1,1}(\Omega)$ case, we stay with BV coarea formula, since we have no special information of the boundary of the level set (super-level set).

7. The Reduced Boundary

Definition 7.1. Assume E has locally finite perimeter in $\mathbb{R}^N$, we say $x \in \partial^* E$, the reduced boundary, if

(1) For every $r > 0$, we have $\|\partial E\| (B(x, r)) > 0$.
(2) $|\nu_E(x)| = 1$
(3) $\nu(x)$ is the lebesgue point w.r.t. ∂E, i.e.,

$$\lim_{r \to 0} \frac{1}{\|\partial E\| (B(x, r))} \int_{B(x,r)} \nu_E \, d \|\partial E\| = \nu_E(x).$$

Comment: condition (3) is the key because Lebesgue point means measure theoretically continuous and this is why we have the reduced boundary is measure theoretically C^1.

Theorem 7.2 (The Fake Integration by Parts). *Let E has locally finite perimeter in $\mathbb{R}^N$, then we have for any $B(x, r)$, $B(x, r) \cap E \neq \emptyset$,*

$$\int_{B \cap E} div \, \varphi \, dx = \int_{B \cap E} \varphi \cdot \nu_E \, d \|\partial E\| + \int_{\partial B \cap E} \varphi \cdot \nu \, d\mathcal{H}^{N-1}$$

Theorem 7.3. [The Properties of the Points in Reduced Boundary]
We say from this theorem that a point belongs to reduced boundary is reasonably good enough. It takes a fair share on the boundary. Here we take $x_0 \in \partial^* E$ where E has locally finite perimeter.

(1)
$$\liminf_{r \to 0} \frac{\mathcal{L}^N(B(x_0, r) \cap E)}{r^N} \geq A_1 > 0$$
(2)
$$\limsup_{r \to 0} \frac{\mathcal{L}^N(B(x_0, r) - E)}{r^N} \geq A_2 > 0$$
(3)
$$\liminf_{r \to 0} \frac{\|\partial E\| (B(x, r)))}{r^{N-1}} \geq A_3 > 0$$
(4)
$$\limsup_{r \to 0} \frac{\|\partial E\| (B(x, r)))}{r^{N-1}} \leq A_4$$
(5)
$$\limsup_{r \to 0} \frac{\|\partial(E \cap B(x, r))\| (\mathbb{R}^N)}{r^{N-1}} \leq A_5$$

Comment: We can actually write $A_1 = A_2 = 1/2$ and $A_3 = A_4 = A_5 = 1$.

Theorem 7.4 (The Blow-up). *Assume E has locally finite perimeter in $\mathbb{R}^N$, then for $x \in \partial^* E$, we have*

$$\chi_{E_r(x)} \to \chi_{H^-(x)} \text{ in } L^1_{loc}(\mathbb{R}^N)$$

where $E_r := E \cap B(x, r)$ and $H^-(x) := \left\{ y \in \mathbb{R}^N, \ \nu_E(x) \cdot (y - x) \leq 0 \right\}$

Corollary 7.5 (Being in reduced boundary has an outer normal vector). *Assume E has locally finite perimeter in $\mathbb{R}^N$, then for $x_0 \in \partial^* E$, we have*

(1)
$$\limsup_{r\to 0} \frac{\mathcal{L}^N(B(x,r)\cap E\cap H^+(x))}{r^N} = 0$$
$$\limsup_{r\to 0} \frac{\mathcal{L}^N(B(x,r)-E)\cap H^-(x))}{r^N} = 0$$
(2)
$$\lim_{r\to 0} \frac{\|\partial E\|(B(x,r))}{\alpha(N-1)r^{N-1}} = 1$$
Comment: This is somehow saying that ∂E is almost C_1.

Theorem 7.6 (The Reduced Boundary)**.** *Let E has locally finite perimeter in $\mathbb{R}^N$, we have*

1. *$\partial^* E$ is almost good, i.e.,*
$$\partial^* E = \left(\bigcup_{n=1}^{\infty} K_n\right) \cup N,$$
where $\|\partial E\|(N) = 0$ and K_n is a compact subset of C^1 hyper surface S_n in $\mathbb{R}^N$.
2. *For $x \in K_n$, we have $\nu_E(x) = \nu(x)$ where ν is the regular outer vector of K_n.*
3. *$\|\partial E\| = \mathcal{H}^{N-1}\lfloor\partial^* E$*

Definition 7.7 (The Measure Theoretical Boundary)**.** We define $x \in \partial_* E$, the measure theoretical boundary, if
$$\limsup_{r\to 0} \frac{\mathcal{L}^N(B(x,r)\cap E)}{r^N} > 0,$$
and
$$\limsup_{r\to 0} \frac{\mathcal{L}^N(B(x,r)\setminus E)}{r^N} > 0.$$
Note that $\partial_* E$ is purely a topological property, but $\partial^* E$ is more a measure-like property.

Theorem 7.8. *All in one, the measure theoretical boundary is almost the reduced boundary, i.e.,*
$$\mathcal{H}^{N-1}(\partial_* E \setminus \partial^* E) = 0.$$

Theorem 7.9 (The Generalized Gauss-Green Theorem)**.** *Let E has locally finite perimeter in $\mathbb{R}^N$, then for any $\varphi \in C^1_c(\mathbb{R}^N;\mathbb{R}^N)$, we have $\mathcal{H}^{N-1}(\partial_* E\cap K) < \infty$ for each K compact.*
$$\int_E div\,\varphi\, dx = \int_{\partial_* E} \varphi\cdot\nu_E\, d\mathcal{H}^{N-1},$$
as well as,
$$\int_E div\,\varphi\, dx = \int_{\partial^* E} \varphi\cdot\nu_E\, d\mathcal{H}^{N-1}.$$
??Moreover??, if E has locally finite perimeter in Ω, then for any $\varphi \in C^1_c(\Omega;\mathbb{R}^N)$, we have
$$\int_E div\,\varphi\, dx = \int_{\partial^* E\cap\Omega} \varphi\cdot\nu_E\, d\mathcal{H}^{N-1}.$$

Definition 7.10. Given $u \in BV_{\mathrm{loc}}(\mathbb{R}^N)$, we define
$$\mu(x) := \inf\left\{t,\ \lim_{r\to 0}\frac{\mathcal{L}^N(B(x,r)\cap\{u>t\})}{r^n} = 0\right\}$$
and
$$\lambda(x) := \sup\left\{t,\ \lim_{r\to 0}\frac{\mathcal{L}^N(B(x,r)\cap\{u<t\})}{r^n} = 0\right\}.$$
We define
$$J := \left\{x\in\mathbb{R}^N,\ \lambda(x) < \mu(x)\right\}$$

Lemma 7.11. *Of course we have $\lambda(x) = \mu(x)$ $\mathcal{L}^N$ a.e. In addition we have $-\infty < \lambda(x) \le \mu(x) < \infty$ $\mathcal{H}^{N-1}$ a.e.*

Theorem 7.12 (The Fine Properties of BV Function)**.** *Given $u \in BV_{loc}(\mathbb{R}^N)$, we have $\mathcal{H}^{N-1}$ a.e. $x \in J$ such that there exists vector $\nu(x)$ such that*

$$\lim_{r\to 0} \fint_{B(x,r)\cap H^+_{\nu(x)}} |u - \lambda(x)|^{\frac{N}{N-1}}\, dx = 0$$

and

$$\lim_{r\to 0} \fint_{B(x,r)\cap H^-_{\nu(x)}} |u - \mu(x)|^{\frac{N}{N-1}}\, dx = 0.$$

Moreover, setting $U(x) = (\mu(x) + \lambda(x))/2$, we have

$$\lim_{r\to 0} \fint_{B(x,r)} |u - U(x)|^{\frac{N}{N-1}}\, dx = 0.$$

$\mathcal{H}^{N-1}$ a.e., and same hold in $W^{1,p}$ as well.

Theorem 7.13 (The)**.** *If $\mathcal{H}^{N-1}(\partial E \cap K) < \infty$ for every compact set $K \subset \mathbb{R}^N$, then E has locally finite perimeter.*

??if uniform bounded, can we have finite perimeter instead of finite??

Part 4. **Elliptic PDEs**

In this Part, Ω will always denote an open bounded set with smooth boundary.

8. Laplace Equation

Definition 8.1 (Harmonic Function)**.** A function $u \in C^2(\Omega)$, $\Omega \subset \mathbb{R}^N$, is harmonic if $\triangle u = 0$.

Theorem 8.2. [The Properties of Laplace Operator]
We collect some short but useful properties of Laplace operator.

(1) *The Laplace operator $\triangle$ is Homogeneous (independent of position) and isotropic (independent of direction). That is, it is Rotation invariant, and also translation invariant.*
(2) *The Laplace operator is Radial symmetric*

Theorem 8.3. [The Fundamental Solution of Laplace Equation in $\mathbb{R}^N$] *The function $-\triangle u = 0$ on $\mathbb{R}^N$ has the fundamental solution*

$$\Phi(x) = \begin{cases} -\frac{1}{2\pi}\log(|x|), & N = 2 \\ \frac{1}{\alpha(N)(N-2)}\frac{1}{|x|^{N-2}}, & N \geq 3 \end{cases}$$

Theorem 8.4. [The Solution of $-\triangle u = f$ on $\mathbb{R}^N$]
Suppose $f \in C_c^2(\mathbb{R}^N)$, then the function

$$u(x) := \int_{\mathbb{R}^N} \Phi(y) f(x-y)dy = \int_{\mathbb{R}^N} \Phi(x-y) f(y)dy$$

solves the equation $-\triangle u = f$, and $u \in C^2(\mathbb{R}^N)$. Notice that $u(x) \to 0$ as $|x| \to \infty$ as $n \geq 3$ and growth in the rate of $\log(|x|)$ *as $N = 2$*
The condition of f can be weaken to f is Höoder continuous and integrable.
Intuitively, we write $-\triangle\Phi = \delta_x$

Theorem 8.5. [The Mean Value Theorem]
This is the most important theorem in harmonic equation. Let $-\triangle u = 0$ for $u \in C^2(\Omega)$ for any Ω open, then we have

$$u(x) = \fint_{\partial B(x,r)} u(y)dy = \fint_{B(x,r)} u(y)dy, \tag{1}$$

for any $B(x,r) \subset \Omega$.
Moreover, the condition $-\triangle u = 0$ and $u \in C^2(\Omega)$ can be weaken to $u \in C^1(\Omega)$ and

$$\int_{\partial B} \frac{\partial}{\partial \nu} u(y)dy = 0,$$

for all ball $B \subset \Omega$.
Now for the converse, suppose $u \in C(\Omega)$ and satisfies MVT, then u is smooth and also harmonic.
Comment: when we say u satisfies MVT, we actually only require that for each $x \in \Omega$, there exists $R(x) > 0$ such that for all $r < R(x)$ we have u satisfies (1). The point is that $R(x)$ is not necessary to be dist$(x, \partial\Omega)$, it could be smaller then it. However, R could be it as showed in HW.

Theorem 8.6. [The Maximal Principle]
Take Ω be open bounded, then the harmonic function u in Ω will only take the Max and Min on the boundary of Ω.

Theorem 8.7. [The Smoothness of u]
If u is harmonic in open Ω, then u is smooth in Ω.
Moreover, if $u \in C^0(\Omega)$ and MVT, then u is smooth as well. That is, the condition of u can be weaken then $C^2(\Omega)$

Theorem 8.8. [The Estimation of Derivative]
Suppose u satisfies MVT in Ω open, then u is smooth and in addition,

$$|D^\alpha u(x)| \le \frac{C(n,k)}{\alpha(n)r^n \cdot r^k}\int_{B(x,r)}|u(y)|\,dy = \frac{C(n,k)}{r^k}⨍_{B(x,r)}|u(y)|\,dy \le \frac{C(n,k)}{r^k}\|u\|_{L^\infty(B(x,r))}$$

Theorem 8.9. *If u is harmonic, then it is analytic. Moreover, if $-\Delta u = f$ and f is analytic, so is u.*

Corollary 8.10. *The directly application of u being analytic is that if Ω is open connected, then $u \equiv 0$ on Ω if $u \equiv 0$ on an neighborhood in Ω. Moreover, $u \equiv$ constant if $u \equiv$ constant in an neighborhood.*

Theorem 8.11. [Green's Reconstruction Formula]
Let Ω be open bounded with C^1 boundary. If $u \in C^2(\Omega) \cap C^0(\bar{\Omega})$ and Green's function exists on Ω, then

$$u(x) = \int_\Omega -\Delta u(y)\, G(x.y)dy - \int_{\partial\Omega} u(y)\frac{\partial_y G(x,y)}{\partial \nu}d\sigma(y).$$

??The formula is true if $u \in C^2(\bar{\Omega})$, but what kind of approximation should we use here??

Theorem 8.12. [The Properties of Green Function] *We list all properties of Green function*

(1) $G(x,y) = G(y,x)$
(2) ~~*??G(x, y) is strict positive.??*~~ *Yes, and we actually have $\phi(x,y) > G(x,y) > 0$.*
(3) *Take*

$$u(x) = \int_\Omega G(x,y)f(y)dy,$$

then u solves $-\Delta u(x) = f(x)$ for $x \in \Omega$ and $u = 0$ on $\partial\Omega$. ??what f?? ??$u(x) \to 0$ as $x \to \partial\Omega$??

Theorem 8.13 (The Poisson Reconstruction Formula in $B(0,R)$)**.** *We could uniquely solve Dirichlet problem*

$$\begin{cases} -\Delta u = 0 & \text{in } B(0,R) \\ u = \phi & \text{on } \partial B(0,R), \end{cases}$$

where ϕ is continuous on $\partial B(0,R)$, by defining

$$u(x) = \int_{\partial B(0,R)} \mathcal{K}(x,y)\,\phi(y)\,d\sigma(y)$$

where $\mathcal{K}$ is the Poisson Kernel defined as

$$\mathcal{K}(x,y) = \frac{R^2 - |x|^2}{n\alpha(n)R}\frac{1}{|y-x|^n}, \text{ for } x \in B(0,R),\ y \in \partial B(0,R),$$

then $u \in C^2(B(0,R)) \cap C^0(\overline{B(0,R)})$.

Theorem 8.14 (The Poisson Reconstruction Formula in $\mathbb{R}^N_+$)**.** *We could solve Dirichlet problem, however, may not uniquely, by*

$$\begin{cases} -\Delta u = 0 & \text{in } \mathbb{R}^N_+ \\ u = \phi & \text{on } \partial\mathbb{R}^N_+, \end{cases}$$

where ϕ is continuous on $\partial\mathbb{R}^N_+$, by defining

$$u(x) = \int_{\partial\mathbb{R}^N_+} \mathcal{K}(x,y)\,\phi(y)\,d\sigma(y)$$

where $\mathcal{K}$ is the Poisson Kernel defined as

$$\mathcal{K}(x,y) = \frac{2x_N}{n\alpha(n)}\frac{1}{|y-x|^n}, \text{ for } x \in \mathbb{R}^N_+,\ y \in \partial\mathbb{R}^N_+,$$

then $u \in C^2(\mathbb{R}^N_+) \cap C^0(\partial\mathbb{R}^N_+)$.

Comment: If g is continuous and bounded, then so is u and hence we have unique solution. Also, if g vanish at infinity, so is u.

Theorem 8.15 (The general existence theorem of Laplace equation for Dirichlet problem)**.** *We present two methods for general existence theorem of Dirichlet problem for Ω open bounded with regular boundary such that the problem*

$$\begin{cases} -\triangle u = 0 & \text{in } \Omega \\ u = g & \text{on } \partial\Omega, \end{cases}$$

where $g \in C^0(\Omega)$, has a unique solution $u \in C^2(\Omega) \cap C^0(\bar{\Omega})$.

(1) *The Perron method:*
We define

$$\mathcal{P} := \left\{ u \in C^0(\bar{\Omega}) \text{ is Sub-harmonic and } u \leq g \text{ on } \partial\Omega \right\}$$

and

$$\mathcal{P}g(x) := \sup_{u \in \mathcal{P}} u(x), \text{ for } x \in \Omega.$$

Then $\mathcal{P}g$ is the solution we want.

(2) *The Poincaré sweeping out:*
We use countably ball $\{B_n\}_{n=1}^{\infty}$ cover Ω and re-index them as

$$B_1, B_2, B_1, B_2, B_3, B_1, B_2, B_3, B_4, \ldots$$

Then, we start with a sub-harmonic function $u_0 \in C^0(\bar{\Omega})$ such that $u = g$ on $\partial\Omega$. We obtain u_1 by doing Possion re-construction on B_1. Likewise, we obtain $\{u_n\}_{n=1}^{\infty}$ and the limiting function u is the solution we are looking at.

9. General Elliptic PDEs - Strong Solutions

In this section we define the non-divergence form L on open bounded Ω such that

$$L := -a_{ij}\partial_i\partial_j + b_i\partial_i + c,$$

where a_{ij}, b_i and c are all $L^\infty(\Omega)$.

Lemma 9.1. *This is an simple observation but it is really nice. Suppose $u = 0$ on $\partial\Omega$, then $\nabla u(x_0) = \partial_\nu u(x_0)$, up to a sign. That is, the gradient is parallel to the direction derivative along the outer normal vector of $\partial\Omega$.*

Theorem 9.2. [The Weak Maximal Principle for $c \equiv 0$] *Let Ω be open bounded, $u \in C^2(\Omega) \cap C^0(\bar{\Omega})$. Then*

(1) *if $Lu \leq 0$, sub-solution, we have*

$$\sup_{\Omega} u = \sup_{\partial\Omega} u$$

(2) *if $Lu \geq 0$, super-solution, we have*

$$\inf_{\Omega} u = \inf_{\partial\Omega} u$$

(3) *if $Lu = 0$, we have*

$$\sup_{\Omega} u = \sup_{\partial\Omega} u \quad \text{and} \quad \inf_{\Omega} u = \inf_{\partial\Omega} u$$

Theorem 9.3 (A Priori Estimation)**.** *Let Ω be open bounded, i.e., $\Omega \subset \left\{ x \in \mathbb{R}^N,\ 0 \leq x \cdot \xi \leq d \right\}$, and $u \in C^2(\Omega) \cap C^0(\bar{\Omega})$ solves*

$$\begin{cases} Lu = f & \text{in } \Omega \\ u = g & \text{on } \partial\Omega, \end{cases}$$

then we have

$$\sup_{\Omega} u \leq \sup_{\partial\Omega} g^+ + C \sup_{\Omega} f^+,$$

and

$$C = \frac{1}{\theta}\left(e^{dL} - 1\right) \text{ where } L := 1 + \frac{\|b\|}{\theta}.$$

Theorem 9.4 (Uniqueness)**.** *Let Ω be open bounded and u_1 u_2 solves*

$$\begin{cases} Lu = f & \text{in } \Omega \\ u = g & \text{on } \partial\Omega, \end{cases}$$

then $u_1 = u_2$ in Ω. ~~??what kind of f and g??~~ We don't really worry about what kind of f and g here since we are only talk about Priori estimation.

Theorem 9.5. [The Weak Maximal Principle for $c \geq 0$] *Let Ω be open bounded, $u \in C^2(\Omega) \cap C^0(\bar{\Omega})$. Then*

(1) *if $Lu \leq 0$, sub-solution, we have*

$$\sup_{\Omega} u \leq \sup_{\partial\Omega} u^+$$

(2) *if $Lu \geq 0$, super-solution, we have*

$$\inf_{\Omega} u \geq \inf_{\partial\Omega} -u^-$$

(3) *if $Lu = 0$, we have*

$$\sup_{\Omega} |u| = \sup_{\partial\Omega} |u|$$

Theorem 9.6. [The Hope's Lemma for case $c \equiv 0$ and $c \geq 0$] *Let Ω be open bounded and and $u \in C^2(\Omega) \cap C^1(\bar{\Omega})$. Then assume there exists $x_0 \in \partial\Omega$ such that $u(x_0) > u(x)$ for all $x \in \Omega$ and Ω satisfies the interior ball condition at x_0. Then*

(1) *if $c \equiv 0$, we have*

$$\frac{\partial u}{\partial \nu} u(x_0) > 0;$$

(2) *if $c > 0$ and $u(x_0) \geq 0$, we have*

$$\frac{\partial u}{\partial \nu} u(x_0) > 0,$$

where ν is outer normal of Ω at x_0.

Theorem 9.7 (The Strong Maximal Principle)**.** *Let Ω be open bounded and $u \in C^2(\Omega) \cap C^0(\bar{\Omega})$ with $Lu \leq 0$. Then u can not assume Maximal inside interior if $c \equiv 0$ or can not assume non-negative maximal inside interior if $c > 0$.*

Theorem 9.8 (The Generally Case for c)**.** *Suppose there exists a $w \in C^2(\Omega) \cap C^1(\bar{\Omega})$ satisfying $w > 0$ in $\bar{\Omega}$ and $Lw \geq 0$. If $u \in C^2(\Omega) \cap C^0(\bar{\Omega})$ satisfies $Lu \leq 0$ in Ω, then u/w has Strong-Max-Principle for the case $c \geq 0$.*

Comment: If the domain is small enough we could have w for free.

Corollary 9.9 (Uniqueness)**.** *The operator L satisfies above will have uniqueness for Dirchlet problem.*

10. General Elliptic PDEs - Weak Solutions

In this section we take

$$L := -\partial_j(a_{ij}\partial_i) + b_i\partial i + c = -\text{div}(A \cdot \nabla) + b \cdot \nabla + c$$

as the divergence form.

Definition 10.1 (The Weak Solution)**.** We define, for Ω open bounded,

$$B[u, v] := \int_{\Omega} A\nabla u \nabla v \, dx + \int_{\Omega} b \cdot \nabla uv \, dx + \int_{\Omega} cuv \, dx = \int_{\Omega} a_{ij}\partial_i u \partial_j v \, dx + \int_{\Omega} b\partial_i uv \, dx + \int_{\Omega} cuv \, dx$$

We say $u \in H_0^1(\Omega)$ is a weak solution of

$$\begin{cases} Lu = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{2}$$

if

$$B[u, v] = \int_{\Omega} f v \, dx$$

for all $v \in H_0^1(\Omega)$.
Here we assume $f \in L^2$, but it could be weaken into H^{-1}.

Theorem 10.2. [The Weak Maximal Principle for $c \geq 0$] *Let Ω be open bounded, $u \in H^1(\Omega)$ is a Weak-solution. Then*

(1) *if $B[u,v] \leq 0$ for all $v \geq 0$ in $H_0^1(\Omega)$, Weak-sub-solution, we have*

$$ess\ sup_{\Omega} u \leq ess\ sup_{\partial\Omega} u^+$$

Theorem 10.3 (A Priori Estimation)**.** *Let $A \in C^1(\bar{\Omega})$ and Ω be open bounded, i.e., $\Omega \subset \left\{x \in \mathbb{R}^N,\ 0 \leq x \cdot \xi \leq d\right\}$, and $u \in H^1(\Omega)$ is a weak solution solves*

$$\begin{cases} Lu = f & in\ \Omega \\ u = g & on\ \partial\Omega, \end{cases}$$

then we have

$$ess\ sup_{\Omega} u \leq ess\ sup_{\partial\Omega} g^+ + C ess\ sup_{\Omega} f^+,$$

and

$$C = \frac{1}{\theta}\left(e^{dL} - 1\right) \text{ where } L := 1 + \frac{\|b\|_{L^\infty} + \|\nabla A\|_{L^\infty}}{\theta}.$$

Theorem 10.4 (The Three Existence Theorems)**.** *Now we present those three existence theorem in weak solutions.*

(1) *The 1st existence theorem.*
As the most fundamental existence theorem, it highly depends on the result in Lax-Mailgram Theorem. Let γ is large enough, which can be determined by A, b, and c, that for every $f \in L^2(\Omega)$ we have the equation

$$\begin{cases} Lu + \gamma u = f & in\ \Omega \\ u = 0 & on\ \partial\Omega, \end{cases}$$

has a unique solution.

(2) *The second existence theorem is the variation of Fredholm alternative. We simply notice that, by define $L_\gamma := L + \gamma$, that $T := (L_\gamma)^{-1}$ is a compact operator from $L^2 \to L^2$ and hence Fredholm applied. The result is:*
   (a) *There exactly one and only one will hold:*
      (i) *The equation*

$$\begin{cases} Lu = f & in\ \Omega \\ u = 0 & on\ \partial\Omega, \end{cases}$$

will have a unique solution for every $f \in L^2(\Omega)$

      (ii) *The equation*

$$\begin{cases} Lu = 0 & in\ \Omega \\ u = 0 & on\ \partial\Omega, \end{cases}$$

will have a non-trivial solution.

(3) *The 3rd existence theorem is the variation of the spectrum theorem in Hilbert space. The theorem states that the equation*

$$\begin{cases} Lu = \lambda u + f & in\ \Omega \\ u = 0 & on\ \partial\Omega, \end{cases}$$

has a unique solution for all f if and only if that $\lambda \notin \Sigma_L$, where Σ_L is the spectrum of L, i.e., the equation

$$\begin{cases} Lu = \lambda u & in\ \Omega \\ u = 0 & on\ \partial\Omega, \end{cases}$$

has non-trivial solution if $\lambda \in \Sigma_L$.
Moreover, if Σ_L is infinite, we have for $\{\alpha_n\}_{n=1}^\infty \subset \Sigma_L$, $\alpha_n \to +\infty$.
Comment: 3rd existence is a consequence of 2nd existence. It has nothing to do with 1st existence, so to say...

Remark 10.5. Hence, by 2nd existence theorem and W-M-P, we know that if $c \geq 0$ then equation (2) will always have unique weak solution.

Theorem 10.6. *Suppose $\lambda \notin \Sigma_L$, we have the equation*

$$\begin{cases} Lu = \lambda u + f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

has a unique solution $u \in H_0^1(\Omega)$, and moreover, we have

$$\|u\|_{H_0^1(\Omega)} \le C \|f\|_{L^2(\Omega)}$$

Theorem 10.7 (Continuity Depends on Dataum)**.** *Basically, for operators L_1 and L_2, if $\|A_1 - A_2\|_{L^\infty} + \|b_1 - b_2\|_{L^\infty} + \|c_1 - c_2\|_{L^\infty} < \varepsilon$ for arbitrary $\varepsilon > 0$, then if $\lambda \notin \Sigma_{L_1}$, $\lambda \notin \Sigma_{L_2}$ as well.*

Theorem 10.8 (The interior and exterior regularity)**.** *The interior estimation is the result of Characterization of $H_0^1(\Omega)$ and exterior estimation if the result of flat boundary condition. We all just consider about the equation*

$$\begin{cases} Lu = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

(1) *The H^2 regularity of interior and exterior.*
   (a) *As the interior regularity, we never care about the boundary behavior of solution u. What we will do is fix a compact subset $\Omega' \subset\subset \Omega$ and focus inside Ω'.*
   Assume now $f \in L^2(\Omega)$ and $A \in C^1(\Omega)$, b, $c \in L^\infty(\Omega)$, then we have $u \in H^2_{loc}(\Omega)$ and
   $$\|u\|_{H^2(\Omega')} \le C(\Omega', \Omega)(\|f\|_{L^2(\Omega)} + \|u\|_{L^2}).$$
   Moreover, we have $Lu = f$ a.e.
   (b) *As the exterior regularity, we now have to ask the boundary of Ω to be nice so that we could do a "flatten" argument at $\partial\Omega$.*
   Assume now $f \in L^2(\Omega)$ and $A \in C^1(\bar{\Omega})$ and b, $c \in L^\infty(\Omega)$. (Here we explicitly require that $A \in C^1(\bar{\Omega})$, as in interior case, A is implicitly assumed to be $C^1(\bar{\Omega}')$). Then if Ω has C^2 boundary, we have $u \in H^2(\Omega)$ and we have
   $$\|u\|_{H^2(\Omega)} \le C(\Omega)(\|f\|_{L^2(\Omega)} + \|u\|_{L^2}).$$

(2) *The H^m regularity of interior and exterior.*
   (a) *Assume now $f \in H^m(\Omega)$ and $A \in C^m(\Omega)$, b, $c \in C^{m-1}(\Omega)$, then we have $u \in H^{m+2}_{loc}(\Omega)$ and*
   $$\|u\|_{H^{m+2}(\Omega')} \le C(\Omega', \Omega)(\|f\|_{H^m(\Omega)} + \|u\|_{L^2}).$$
   (b) *Assume now $f \in H^m(\Omega)$ and $A \in C^{m+1}(\bar{\Omega})$ and b, $c \in C^m(\bar{\Omega})$. Then if Ω has C^{m+2} boundary, we have $u \in H^{m+2}(\Omega)$ and we have*
   $$\|u\|_{H^{m+2}(\Omega)} \le C(\Omega)(\|f\|_{H^m(\Omega)} + \|u\|_{L^2}).$$

   Also, we could have $f = g$ on $\partial\Omega$ where $g \in H^{m+2}$.

Comment: the continuity assumption can be weaken to Lipschitz continuous. The whole point is we can have a classical derivative and serve in integral sense, i.e., a.e. differentiable is good enough.
Comment: Since the interior regularity does not depends on the behavior on the boundary, hence we could work on the estimation of the entire space, locally, i.e., if $Lu = f$ on $\mathbb{R}^N$ and f is C^∞, then u is C^∞ as well, maybe even analytic.

Theorem 10.9 (The Eigenvalue and Eigenvector)**.** *Here we take*

$$L := -\partial_j(a_{ij}\partial_i) + c$$

for $c > 0$. Hence the Lax-Mailgram will always work and we sure have unique solutions for each $f \in L^2(\Omega)$

Theorem 10.10 (The Completion of Eigenvalue)**.** *In view that L is one-to-one and onto, so $T := L^{-1}$ $L^2(\Omega) \to L^2(\Omega)$ is self-adjoint operator and hence we have there exists sequence $\{\lambda_n\}_{n=1}^\infty$ such that $0 < \lambda_1 \le \lambda_2 \le \cdots$ and w_n such that*

$$\begin{cases} Lw_n = \lambda_n w_n & \text{in } \Omega \\ w_n = 0 & \text{on } \partial\Omega, \end{cases}$$

where $\{w_n\}_{n=1}^\infty$ forms an orthonormal basis in $L^2(\Omega)$ but each $w_n \in C^\infty(\bar{\Omega})$

Theorem 10.11 (The Rayleigh Quotient Theorem and Courant Min-Max Theorem)**.** *Those two theorems basically tell the same story so we put them together.*

(1) *The Rayleigh Quotient Theorem: we could identify* λ_k *by*

$$\lambda_k = \min_{\|u\|_{L^2(\Omega)}=1} \{B[u,u],\ (u, w_n) = 0 \text{ for all } 1 \le n \le k-1\}$$

(2) *The Courant Min-Max Theorem states that*

$$\lambda_k = \max_{S \subset H_0^1(\Omega)} \min_{\substack{u \in S^\perp \\ \|u\|_{L^2(\Omega)}=1}} B[u,u]$$

where S *is a* $(k-1)$ *dimensional subspace in* $H_0^1(\Omega)$

Theorem 10.12 (The Interpolation of H_0^m norm)**.** *Assume* Ω *open bounded and smooth boundary,* A*,* c *are all* $C^\infty(\bar{\Omega})$*. From the prove of Rayleigh Quotient Theorem, we deduce that for all* $u \in H_0^1(\Omega)$ *we have*

$$C_1 \|u\|_{H_0^1(\Omega)} \le \sum_{k=1}^{\infty} \lambda_k (u, w_k)^2 \le C_2 \|u\|_{H_0^1(\Omega)}.$$

Moreover, by induction, we could write for any $m \ge 1$ *be integer, we have*

$$C_1 \|u\|_{H_0^m(\Omega)} \le \sum_{k=1}^{\infty} \lambda_k^m (u, w_k)^2 \le C_2 \|u\|_{H_0^m(\Omega)}.$$

Even more, we could have for any $s \in [0, \infty)$*, that*

$$C_1 \|u\|_{H_0^s(\Omega)} \le \sum_{k=1}^{\infty} \lambda_k^s (u, w_k)^2 \le C_2 \|u\|_{H_0^s(\Omega)}.$$

where all C_1 *and* C_2 *are depends on* Ω*,* A*, and* c*.*

Theorem 10.13 (More Properties of λ_1)**.** *The first eigenvalue is a bit special, so to say...*

(1) λ_1 *has multiplicity* 1*, i,e., the dimension of the eigenspace is* 1*, as such* $0 < \lambda_1 < \lambda_2 \le \cdots$

(2) *If* w_1 *is the corresponding eigenfunction, then we may take* $w_1 > 0$ *in* Ω*.*

Part 5. **Micellaneous**

11. Measure Theory & Functional Analysis

Definition 11.1 (Measure & Signed Measure)**.** We say a map $\mu\colon \mathcal{B}(X) \to \mathbb{R}^+ \cup \{0\}$ is a measure if

(1) $\mu(\varnothing) = 0$;
(2) $\mu(A) \leq \mu(B)$ if $A \subset B$;
(3) $\mu(A \cup B) = \mu(A) + \mu(B)$ if $A \cap B = \varnothing$

A measure μ on X is *regular* if for each set $A \subset X$ there exists a μ-measurable set B such that $A \subset B$ and $\mu(A) = \mu(B)$.
A measure μ on $\mathbb{R}^N$ is called *Borel* if every Borel set is μ-measurable.
A measure μ on $\mathbb{R}^N$ is *Borel regular* if μ is Borel and for each $A \subset \mathbb{R}^N$ there exists a Borel set B such that $A \subset B$ and $\mu(A) = \mu(B)$.
A measure μ on $\mathbb{R}^N$ is a *Radon* measure if μ is Borel regular and $\mu(K) < \infty$ for each compact set $K \subset \mathbb{R}^N$.
A measure ν is said to be Signed if there exists $f \in L^1_{loc}(\mu)$ and

$$\nu(A) = \int_A f\, d\mu$$

Theorem 11.2. [Approximation by Open and Compact Sets]
Let μ be a Radon measure on $\mathbb{R}^N$. Then

(1) *for each set $A \subset \mathbb{R}^N$,*

$$\mu(A) := \inf\{\mu(U),\ A \subset U,\ U \text{ is open}\}$$

(2) *for each set $A \subset \mathbb{R}^N$ and A is measurable,*

$$\mu(A) := \sup\{\mu(K),\ K \subset A,\ K \text{ is compact}\}$$

Theorem 11.3. (The Convergence Theorem)

(1) *The Monotone Convergence: Let $u_n\colon X \to [0, +\infty]$ and $0 \leq u_1(x) \leq u_2(x) \leq \cdots \leq u_n(x) \leq \cdots$ for (almost) every $x \in X$, we have*

$$\lim_{n\to\infty} \int_X u_n\, dx = \int_X \lim_{n\to\infty} u_n(x)\, dx$$

And from this we conclude that for u_n non-negative,

$$\int_X \sum_{n=1}^{\infty} u_n(x)\, dx = \sum_{n=1}^{\infty} \int_X u_n(x)\, dx,$$

and

$$\sum_{n=1}^{\infty} \sum_{m=1}^{\infty} a_{n,m} = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} a_{n,m},$$

if $a_{n,m} \geq 0$.

(2) *The Fatou's Lemma: Let u_n be non-negative, then*

$$\liminf_{n\to\infty} \int_X u_n\, dx \geq \int_X \liminf_{n\to\infty} u_n(x)\, dx.$$

Also if u_n is uniformly bounded below and $\mu(X) < \infty$, the same will work.

(3) *The Dominate Convergence Theorem: Suppose $|u_n| \leq g$ a.e. $x \in X$ and $g \in L^1(X)$, then*

$$\lim_{n\to\infty} \int_X u_n\, dx = \int_X \lim_{n\to\infty} u_n(x)\, dx.$$

Thus, if $\left|\sum_{n=1}^{\infty} u_n\right| \leq g$ and $g \in L^1(X)$, we have

$$\int_X \sum_{n=1}^{\infty} u_n(x)\, dx = \sum_{n=1}^{\infty} \int_X u_n(x)\, dx.$$

Theorem 11.4. [Lusin - The Continuity Property of Measurable Function]
Let u be a measurable function on (X, μ), and a measurable set A with $\mu(A) < \infty$. Then for any $\varepsilon > 0$ there exists $g \in C_c(X)$ such that

$$\mu(\{f(x) \neq g(x)\}) < \varepsilon.$$

Moreover, g can be arranged that

$$\sup_{x\in X} |g| \leq \sup_{x\in X} |u|.$$

In other word, u is continuous on a compact set K such that $\mu(A - K) < \varepsilon$.

Check Remark 13.1 for the limitation of Lusin.

Theorem 11.5. [Egoroff]
Let $u_n \to u$ pointwise and a measurable set A with $\mu(A) < \infty$. Given any $\varepsilon > 0$ there exists a measurable set $B \subset A$ such that $\mu(A - B) < \varepsilon$ and $u_n \to u$ uniformly on B.

Check Remark 13.1 for the limitation of Lusin

L^p **Space Theorem**: The next couple of Theorems concern about L^p Theorem.

Theorem 11.6. [L^p Space Interpolation]
Let $u \in L^{p_1}(X) \cap L^{p_2}(x)$ where $p \leq q$ and $\mu(X)$ is not necessary finite, we have $u \in L^q(X)$ for any $p_1 \leq q \leq p_2$ and moreover,

$$\|u\|_{L^q(X)} \leq \|u\|^{\theta}_{L^{p_1}(X)} \|u\|^{1-\theta}_{L^{p_2}(X)},$$

where θ is defined as

$$\frac{1}{q} := \frac{1-\theta}{p_1} + \frac{\theta}{p_1}$$

Theorem 11.7. [L^p Space Approximation]
For any $1 \leq p < \infty$ and $u \in L^p(X)$, there exists $\{u_n\}_{n=1}^{\infty} \subset C_c^{\infty}(X) \cap L^p(X)$ such that

$$\lim_{n\to\infty} \|u_n - u\|_{L^p(X)} = 0.$$

Here we do not require X is finite or not.
For L^{∞} we can NOT because uniformly convergence will always result in continuous function.

Theorem 11.8. [Weierstrass Approximation Theorem]
Suppose f is a continuous real-valued function defined on the real interval $[a, b]$. For every $\varepsilon > 0$, there exists a polynomial $p(x)$ such that for all $x \in [a, b]$, we have $|f(x) - p(x)| < \varepsilon$, or equivalently, the supremum norm $\|f - p\|_{L^{\infty}} < \varepsilon$.

Corollary 11.9. *From Weierstrass Approximation, we could also deduce that the polynomial $p(x)$ is dense in $L^p([0, 1])$, $1 \leq p < \infty$ because of the density of continuous function in L^p space above.*

Theorem 11.10. [Vitali Convergence Theorem]
Let $\{u_n\}_{n=1}^{\infty}$ and $u \in L^1(X)$, then $u_n \to u$ in $L^1(X)$ if and only if

(1) *$u_n \to u$ in measure;*
(2) *(Tightness) For any $\varepsilon > 0$, there exists a set V such that $\mu(V) < \infty$ and*

$$\sup_{n\in\mathbb{N}} \int_{X-V} |u_n| \, d\mu < \varepsilon$$

(3) *(Uniform Integrable) For any $\varepsilon > 0$, there exists $\delta > 0$ such that for any set E, $\mu(E) < \delta$, we have*

$$\sup_{n\in\mathbb{N}} \int_{E} |u_n| \, d\mu < \varepsilon$$

??What is the analogous results in L^p, $p > 1$, in L^{∞} ??

Theorem 11.11. [The Weak Convergence in L^p for $p = 1$, $1 < p < \infty$, and $p = \infty$]

(1) *We assume* $1 < p < \infty$ *and* $\sup_{n\in\mathbb{N}} \|u_n\|_{L^p(X)} \leq M < \infty$, *then up to a subsequence, that* $u_n \rightharpoonup u$ *for some* $u \in L^p(X)$, *i.e., for every* $\varphi \in L^{p'}(X)$

$$\int_X u_n\varphi\,dx \to \int_X u\varphi\,dx.$$

Also, by Banach-Algoglu, *we have* $\|u\|_{L^p} \leq M$ *as well.* ~~??can we have l.s.c of norm??~~ *Yes, this is weak convergence theorem*

~~??Do we have for any $\Omega \subset X$ the above convergence still hold??~~ *No the testing space is getting larger.*

~~??If $\mu(X) < \infty$, do we have L^p weak implies L^q weak for $q < p$. Of course this is true in Strong sense??~~ *Yes, this is try by* L^p *embedding.*

(2) *For* $p = \infty$ *,we write* $u_n \overset{*}{\rightharpoonup} u$ *for some* $u \in L^\infty$. *Again, by* Banach-Algoglu, *we have* $\|u\|_{L^p} \leq M$ *as well.* ~~??can we have l.s.c of norm??~~ *Yes*

(3) *For* $p = 1$, *we certainly lose the weak completeness. We will have details in next theorem.*

Theorem 11.12. [Dunford-Pettis]
Suppose $\sup_{n\in\mathbb{N}} \|u_n\|_{L^1(X)} < \infty$ *and* $\{u_n\}_{n=1}^\infty$ *satisfies Tightness and Uniform Integrable. Then, up to a subsequence, that* $u_n \rightharpoonup u$ *for some* $u \in L^1(X)$. *Usually, we test against* $\varphi \in L^\infty(X)$, *or ??could we test against* $C_c(X)$ *is enough. I don't think so b/c* C_c *is not dense in* L^∞ *??.*

Theorem 11.13. [Characterization of $L^p(X)$ Weak Convergence]
We will do the case $p = 1$ *and* $1 < p \leq \infty$ *separately.*

(1) *For the case* $1 < p \leq \infty$, *we have* $u_n \rightharpoonup u$ *in* $L^p(\Omega)$ *(*$\overset{*}{\rightharpoonup}$ *if* $p = \infty$*) if and only if*

$$\lim_{n\to\infty} \int_{\Omega\cap Q} u_n\,dx = \int_{\Omega\cap Q} u\,dx,$$

for any cube and

$$\sup_{n\in\mathbb{N}} \|u_n\|_{L^p(\Omega)} < \infty$$

(2) *For the case* $p = 1$, *this is again the* Dunford-Pettis. *We have a sequence* $u_n \rightharpoonup u$ *in* $L^1(\Omega)$ *if and only if*

$$\lim_{n\to\infty} \int_{\Omega\cap Q} u_n\,dx = \int_{\Omega\cap Q} u\,dx,$$

and $\{u_n\}_{n=1}^\infty$ *satisfies uniformly integrable and tightness. (Notice that UI and Tightness implies* $\sup_{n\in\mathbb{N}} \|u_n\|_{L^1(\Omega)} < \infty$*)*

Theorem 11.14. [Weak Compactness of Radon Measure]
Let ν_n *be a sequence of Radon measure on* X *and we assume* $\sup_{n\in\mathbb{N}} \nu_n(K) < \infty$ *for each* K *compact, then, up to a subsequence, that*

$$\nu_n \overset{*}{\rightharpoonup} \nu, \text{ for some Radon measure } \nu,$$

i.e., for every $\varphi \in C_c(X)$,

$$\lim_{n\to\infty} \int_X \varphi\,d\nu_n = \int_X \varphi\,d\nu$$

Theorem 11.15. [Consequence of Weak Compactness]
Assume $\mu_n \overset{*}{\rightharpoonup} \mu$

(1) *For any* A *open, we have*

$$\liminf_{n\to\infty} \mu_n(A) \geq \mu(A);$$

(2) *For any* K *compact, we have*

$$\limsup_{n\to\infty} \mu_n(K) \geq \mu(K);$$

For any bounded Borel set B *and* $\mu(\partial B) = 0$, *we have*

$$\lim_{n\to\infty} \mu_n(B) = \mu(B)$$

Theorem 11.16. [Weak Compactness of Signed Measure]
Let σ_n be a sequence of Signed measure on X and we assume $\sup_{n\in\mathbb{N}} \|\sigma_n\| < \infty$, then, up to a subsequence, that

$$\sigma_n \overset{*}{\rightharpoonup} \sigma, \text{ for some Signed measure } \sigma,$$

i.e., for every $\varphi \in C_c(X)$,

$$\lim_{n\to\infty} \int_X \varphi \, d\sigma_n = \int_X \varphi \, d\sigma$$

~~*??Can we have l.s.c. of the norm of σ_n??*~~ *Yes, the l.s.c always hold for weak convergence, this is a consequence of general functional analysis conclusion.*

The derivative in measure, AC, and FTC

Theorem 11.17. [The Radon-Nikogy] *Let ν be a signed measure, then there exists two signed measure ν_{ac} and ν_s such that*

$$\nu_{ac} \ll \mathcal{L}^N, \ \nu_s \perp \mathcal{L}^N.$$

Moreover, there exists a unique $u \in L^1_{loc}(\mathbb{R}^N)$ such that

$$\nu_{ac}(E) = \int_E u \, d\mathcal{L}^N,$$

for every set E measurable. Also if ν is finite, u will be in $L^1(\mathbb{R}^N)$.

Definition 11.18. [The Maximal Function]
Given a measure μ, we define the Maximal Function of μ to be

$$M(\mu)(x) := \sup_{r>0} \frac{\mu(B(x,r))}{\mathcal{L}^N(B(x,r))},$$

and the Maximal Function for measurable function u to be

$$M(u)(x) := \sup_{r>0} \frac{1}{\mathcal{L}^N(B(x,r))} \int_{B(x,r)} |u| \, d\mathcal{L}^N,$$

Theorem 11.19. [The Vitali's Covering Theorem]
Let $\mathcal{F}$ be any collection of non degenerate closed balls in $\mathbb{R}^N$ with

$$\sup\{diam\, B \ \ B \in \mathcal{F}\} < \infty.$$

Then there exists a countable family $\mathcal{G}$ of disjoint balls in $\mathcal{F}$ such that

$$\bigcup_{B\in\mathcal{F}} B \subset \cup_{B\in\mathcal{G}} \hat{B}.$$

Corollary 11.20. [The Actual Usage of Vitali's Covering Theorem]
If $\mathcal{B}$ is the union of a finite collection of balls $B(x_i, r_i)$, $1 \le i \le N$, then there is a set $S \subset \{1, \dots, N\}$ so that

1. *the balls $B(x_i, r_i)$ with $i \in S$ are disjoint,*
2. *$\mathcal{B} \subset \bigcup_{i\in S}(B(x_i, 3r_i)$ and*
3. *$\mathcal{L}^N(\mathcal{B}) \le 3^k \sum_{i\in S} \mathcal{L}^N(B(s_i, r_i))$*

Theorem 11.21. [The Estimation of Maximal Function]
Suppose $u \in L^1(\mathbb{R}^N)$, then for any $\lambda > 0$ we have

$$\mathcal{L}^N\{M(u) > \lambda\} \le 3^N \lambda^{-1} \|u\|_{L^1}$$

Theorem 11.22. [The Lebesgue Points]
Let $u \in L^1(X)$, then a.e. $x \in \Omega$ we have

$$\lim_{r\to 0} \frac{1}{\mathcal{L}^N(B(x,r))} \int_{B(x,r)} u(y) \, dy = u(x),$$

or

$$\lim_{r\to 0} \frac{1}{\mathcal{L}^N(B(x,r))} \int_{B(x,r)} |u(y) - u(x)| \, dx = 0.$$

Theorem 11.23 (The Weak and Weak* Convergence)**.** *Notice the l.s.c part*

(1) *For weak convergence, suppose* $x_n \rightharpoonup x$ *and* $\{x_n\}_{n=1}^{\infty} \subset Y$ *where* Y *is closed and convex, then we have*
   (a) $x \in Y$
   (b) $\sup \|x_n\| < \infty$
   (c) $\liminf_{n\to\infty} \|x_n\| \geq \|x\|$
(2) *For weak* convergence, suppose* $x_n \overset{*}{\rightharpoonup} x$*, then we have*
   (a) $\sup \|x_n\| < \infty$
   (b) $\liminf_{n\to\infty} \|x_n\| \geq \|x\|$

Theorem 11.24 (Every Hilbert space has an orthonormal basis)**.** *The most efficient way to prove a orthonormal set is actually an orthonormal basis is to show this set is complete, i.e., if* $\{x_n\}_{n=1}^{\infty} \subset H$ *is an orthonormal set, then if* $(x_n, y) = 0$ *for all* x_n *implies that* $y = 0$*, we have* $\{x_n\}_{n=1}^{\infty}$ *is an ONB in* H*.*

Theorem 11.25. *[The Stampacchia and Lax-Milgram] Here are two most important theorems in Hilbert Space* H *which relevant to Variational Problem*

(1) *The Stampacchia: Assume that* $a(u, v)$ *is a continuous coercive bilinear form on* H*. Let* $K \subset H$ *be a nonempty closed and convex subset. Then, given any* $\varphi \in H^*$*, there exists a unique element* $u \in K$ *such that*
$$a(u, v-u) \geq \langle \varphi, v-u \rangle_{\langle H^*, H\rangle} \quad \forall v \in K.$$
Moreover, if a *is symmetric, then* u *is characterized by the property*
$$u \in K \text{ and } \frac{1}{2}a(u,u) - \langle \varphi, u\rangle = \min_{v\in K}\left\{\frac{1}{2}a(v,v) - \langle \varphi, v\rangle\right\}$$
Comment: the Stampacchia theorem is corresponding to the Unilateral constraints problem in Calculus of Variation. In application, we usual take $v - u = \pm\varepsilon u$ *to finally have equality. Also, this version of Stampacchia also gives the formula of minimizing problem.*
(2) *The Lax-Milgram: Assume that* $a(u, v)$ *is a continuous coercive bilinear form on* H*. Then, given any* $\varphi \in H^*$*, there exists a unique element* $u \in H$ *such that*
$$a(u, v) = \langle \varphi, v \rangle_{\langle H^*, H\rangle} \quad \forall v \in K.$$
Moreover, if a *is symmetric, then* u *is characterized by the property*
$$u \in H \text{ and } \frac{1}{2}a(u,u) - \langle \varphi, u\rangle = \min_{v\in H}\left\{\frac{1}{2}a(v,v) - \langle \varphi, v\rangle\right\}$$
Comment: The Lax-Mailgram is corresponding to the general variation problem. Notice that this version of Lax-Milgram also gives the form of minimizing problem.

Theorem 11.26. *Let* X *be a Banach space,* S *be a total subset of* X^**, i.e., the span of* S *is dense in* X^**,* $\{x_n\}_{n=1}^{\infty}$ *be a sequence in* X *and* $x \in X$ *is given. Then* $x_n \rightharpoonup x$ *as* $n \to \infty$ *if and only if* $\{x_n\}_{n=1}^{\infty}$ *is bounded and*
$$\langle y^*, x_n\rangle \to \langle y^*, x\rangle \text{ for all } y^* \in S.$$
Comment: This Theorem is extremely useful, as the boundedness is already proven, when we try to prove (verify) certain weak convergence sequence, such that ∇u_n *weakly goes to some* ∇u *in* L^p*, as* ∇u *is already determined. Hence, instead of testing against* $L^{p'}$ *function, we only need to test against* C_c^∞ *function and hence the IBP will come into play and we reduce the problem into the weak convergence of* u *itself. Also, consult Theorem 5.19.*
Comment: The same result hold for Weak Star convergence.

Theorem 11.27 (The Riesz Representation Theorem)**.** *Let* $L\colon C_c(\mathbb{R}^N; \mathbb{R}) \to \mathbb{R}$ *be a liner functional satisfying*
$$\sup\left\{L(f),\ f \in C_c(\mathbb{R}^N, \mathbb{R}),\ |f| \leq 1,\ spt(f) \subset K\right\} < \infty$$
for each compact set $K \subset \mathbb{R}^N$*. Then there exists a Radon measure* μ *on* $\mathbb{R}^N$ *and a* μ*-measurable function* $\sigma\colon \mathbb{R}^N \to \mathbb{R}^N$ *such that*

(1) $|\sigma(x)| = 1$ μ *a.e.,* x

(2)
$$L(f) = \int_{\mathbb{R}^N} \langle f, \sigma \rangle \, d\mu$$
for all $f \in C_c(\mathbb{R}^N, \mathbb{R})$

(3) *??* $\|L\| = \mu(\mathbb{R}^N)$ *if??*

Theorem 11.28 (Riemann-Lebesgue Lemma). *Let* $u \in L^p_{loc}(\mathbb{R}^N)$, $1 \le p \le \infty$, *be* KQ*-periodic. For every* $\varepsilon > 0$ *and* $x \in \mathbb{R}^N$ *set*
$$u_\varepsilon(x) := u\left(\frac{x}{\varepsilon}\right).$$
Then $u_\varepsilon \rightharpoonup \bar{u}$ *in* $L^p(E)$ ($\overset{*}{\rightharpoonup}$ *if* $p = \infty$) *for every bounded measurable set* $E \subset \mathbb{R}^N$, *where*
$$\bar{u}(x) \equiv const := \frac{1}{K^N} \int_{Q(0,K)} u(y) dy.$$

12. Other Calculation Facts

We collect some useful but easy to forget facts.

(1) Any function can be written as the summation of odd and even function.
$$u_{\text{even}} = \frac{u(x) + u(-x)}{2} \text{ and } u_{\text{odd}} = \frac{u(x) - u(-x)}{2}$$

(2) The triangle inequality for p power
$$|a - b|^p \le ||a| + |b||^p \le 2^{p-1}(|a|^p + |b|^p)$$

(3) Generally, for $\mathbb{R}^1$ uniformly continuous function u and v, the product uv is not necessary uniformly continuous unless both of them are bounded. (One bounded is still not enough. For example: $u(x) = x \cdot \frac{\sin(x^2)}{\sqrt{x}}$). The similar result happens in $W^{1,p}$ and ~~?? BV ??~~.
The space $BV(\mathbb{R})$ is better since any $u \in BV(\mathbb{R})$ is necessary $L^\infty(\mathbb{R})$. Actually, $BV(I)$ is an algebra for I bounded. ~~??what if I is unbounded??~~ The same result hold. The key is $BV(I) \subset L^\infty$. Moreover, since $W^{1,p}(I) \subset BV(I)$, the same result hold.

13. Counterexamples and Some Useful Explanation

In this section we collect all must-known counterexamples and some useful explanations of theorems

Remark 13.1. By the first glance, I thought Lusin and Egoroff all so powerful, it almost turns a barely measurable function into a continuous function and a pointwise convergence function into a uniformly convergence function. What we need to do is just push $\varepsilon \to 0$. But, of course, we can not. We point out, as in Egoroff, we actually have for each $\varepsilon > 0$, there exists $N_\varepsilon > 0$ such that for all $x \in B_\varepsilon$, $|u_n(x) - u(x)| < \varepsilon$. However, as $\varepsilon \to 0$, $N_\varepsilon \to \infty$, thus we have nothing in the end. Plus, all usage of Egoroff can be easily replaced by LDC.
Also, for Lusin, we mark that $\varepsilon > 0$ is essential, we can not obtain that u is a.e. continuous by pushing $\varepsilon \to 0$, as example 13.2 shows.

Example 13.2. We provide a function $u\colon \mathbb{R} \to \mathbb{R}$ such that u is no where continuous but u is continuous after we remove a measure 0 set.
We define u as

$$u(x) := \begin{cases} 0 & x \in \mathbb{Q} \\ 1 & x \notin \mathbb{Q} \end{cases}$$

Of course u is no where continuous but u is continuous without $\mathbb{Q}$ which is a measure 0 set.

Example 13.3. We define u_n, u such that $u_n, u \in W^{1,1}(0,1)$ such that $u_n \to u$ in $L^1(0,1)$ but

$$\lim_{n\to\infty} \left\|u_n'\right\|_{L^1(0,1)} > \|u'\|_{L^1(0,1)}.$$

We define u_n as the copy of n triangles with length $1/n$ and hight $1/2n$. Hence $\|u_n\|_{L^1(0,1)} = 1/4n$ and hence $u_n \to 0$ in L^1. However, $\left\|u_n'\right\|(0,1) \equiv 1$. Clearly we have $u \equiv 0$. Thus, we have strict inequality.

Example 13.4. There exists a function $u \in L^p(0,1)$ for all $p \geq 1$, but $u \notin L^\infty(0,1)$

Example 13.5. We show that $BV(0,1)$ is not separable.
Take $u_\varepsilon = \chi_{(1/2-\varepsilon, 1/2+\varepsilon)}$. Then we have $u_\varepsilon \in BV(0,1)$ and $\left\|u_{\varepsilon_1}' - u_{\varepsilon_2}'\right\|(0,1) = 2$ for every $\varepsilon_1 \neq \varepsilon_2$, and hence $\left\|u_{\varepsilon_1} - u_{\varepsilon_2}\right\|_{BV(0,1)} \geq 2$ for each $\varepsilon_1 \neq \varepsilon_2$. Hence we can build a uncountablely many ball centered at u_ε with radius $1/2$ and hence those balls are mutually disjoint.

Example 13.6. We discuss the relationship between $AC(I)$, $BV(I)$, and the uniformly continuous function on $[0,1]$, where I would be $[0,1]$. $(0,1)$ and $(0,\infty)$.

Example 13.7. Let $\Omega = B(0,1) \subset \mathbb{R}^N$, and $1 \leq p < N$, and consider the sequence of functions $u_n\colon \Omega \to \mathbb{R}$ defined by

$$u_n(x) := \begin{cases} n^{\frac{N-p}{p}}(1 - n|x|) & \text{if } |x| < \frac{1}{n} \\ 0 & \text{if } |x| \geq \frac{1}{n} \end{cases}$$

Then we have $\{u_n\}_{n=1}^\infty$ is bounded in $W^{1,p}(\Omega)$ but that it does not admit any subsequence strongly convergent in $L^{p^*}(\Omega)$

References

[1] Lawrece C. Evans. *Partial Differential Equations*, Graduate Studies in Mathematics, Volume 19, AMS.