Exercises on Catalan and Related Numbers

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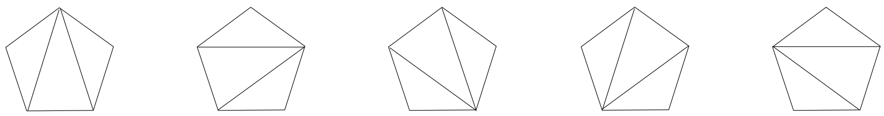
19. [1]–[3+] Show that the Catalan numbers $C_n = \frac{1}{n+1}\binom{2n}{n}$ count the number of elements of the 66 sets S_i, (a) $\leq i \leq$ (nnn) given below. We illustrate the elements of each S_i for $n = 3$, hoping that these illustrations will make any undefined terminology clear. (The terms used in (vv)–(yy) are defined in Chapter 7.) Ideally S_i and S_j should be proved to have the same cardinality by exhibiting a simple, elegant bijection $\phi_{ij} : S_i \to S_j$ (so 4290 bijections in all). In some cases the sets S_i and S_j will actually coincide, but their descriptions will differ.

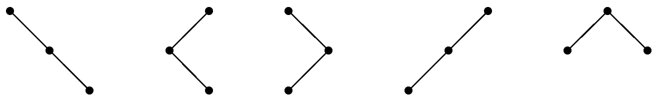
(a) triangulations of a convex $(n+2)$-gon into n triangles by $n-1$ diagonals that do not intersect in their interiors



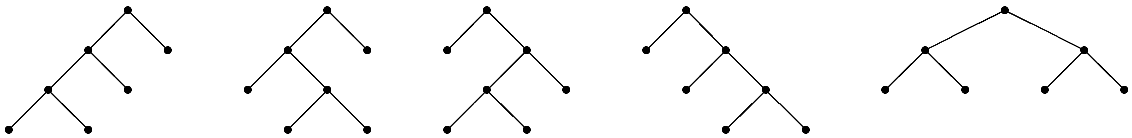
(b) binary parenthesizations of a string of $n+1$ letters

$$(xx \cdot x)x \quad x(xx \cdot x) \quad (x \cdot xx)x \quad x(x \cdot xx) \quad xx \cdot xx$$

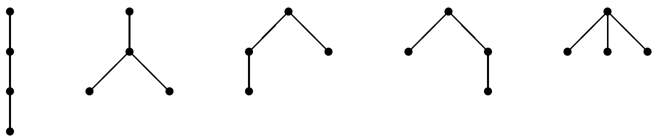
(c) binary trees with n vertices



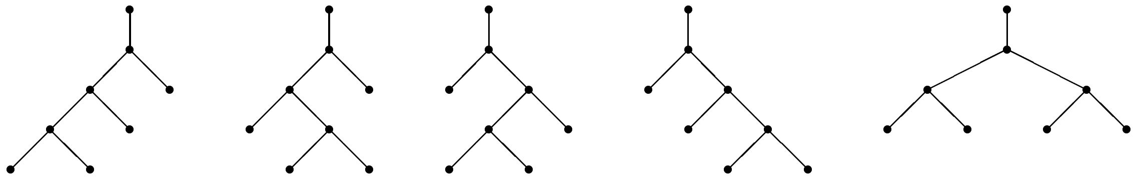
(d) plane binary trees with $2n+1$ vertices (or $n+1$ endpoints)



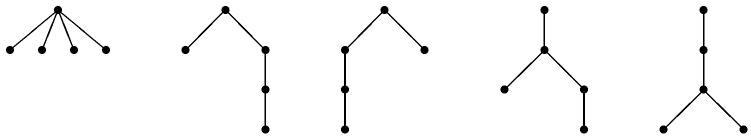
(e) plane trees with $n+1$ vertices



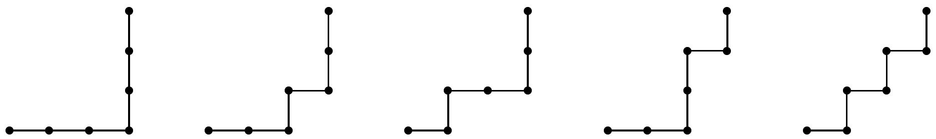
(f) planted (i.e., root has degree one) trivalent plane trees with $2n+2$ vertices



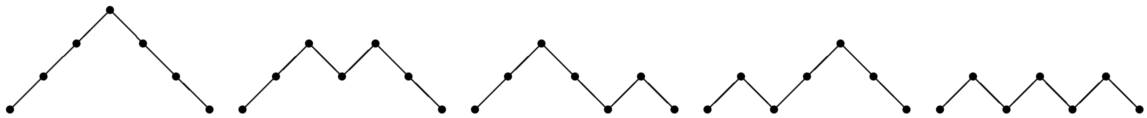
(g) plane trees with $n+2$ vertices such that the rightmost path of each subtree of the root has even length



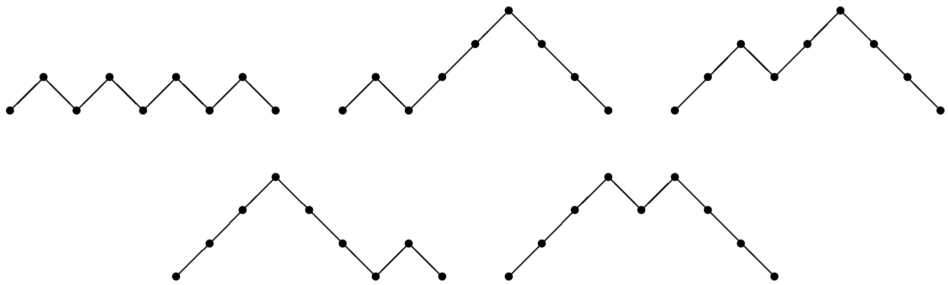
(h) lattice paths from $(0,0)$ to (n,n) with steps $(0,1)$ or $(1,0)$, never rising above the line $y=x$



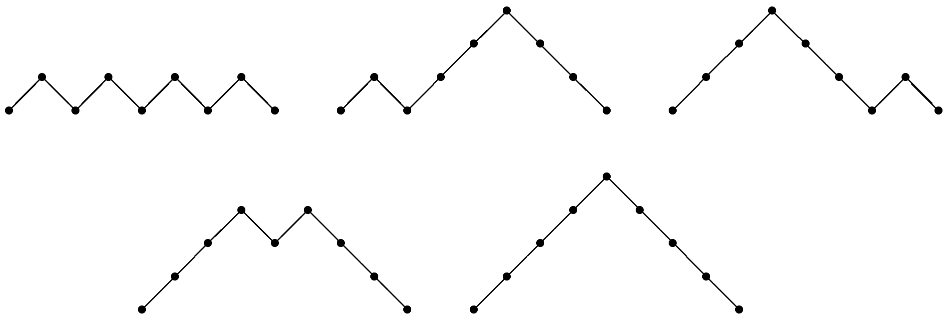
(i) Dyck paths from $(0,0)$ to $(2n,0)$, i.e., lattice paths with steps $(1,1)$ and $(1,-1)$, never falling below the x-axis



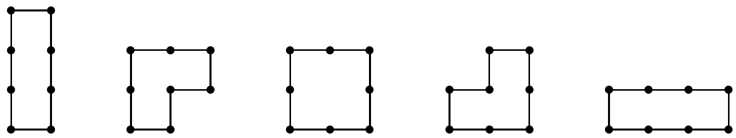
(j) Dyck paths (as defined in (i)) from $(0,0)$ to $(2n+2,0)$ such that any maximal sequence of consecutive steps $(1,-1)$ ending on the x-axis has odd length



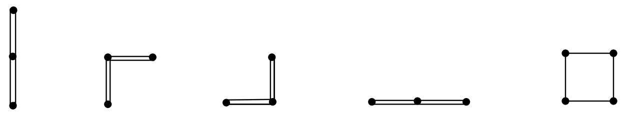
(k) Dyck paths (as defined in (i)) from $(0,0)$ to $(2n+2,0)$ with no peaks at height two.



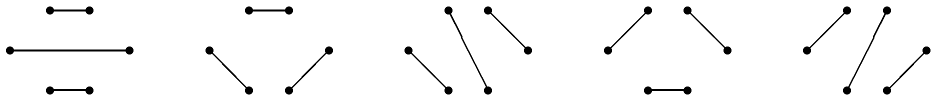
(l) (unordered) pairs of lattice paths with $n+1$ steps each, starting at $(0,0)$, using steps $(1,0)$ or $(0,1)$, ending at the same point, and only intersecting at the beginning and end



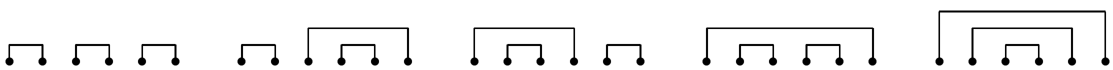
(m) (unordered) pairs of lattice paths with $n-1$ steps each, starting at $(0,0)$, using steps $(1,0)$ or $(0,1)$, ending at the same point, such that one path never arises above the other path



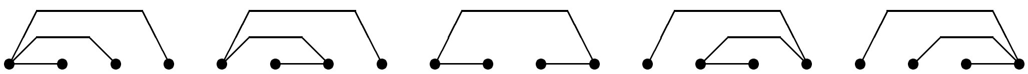
(n) n nonintersecting chords joining $2n$ points on the circumference of a circle



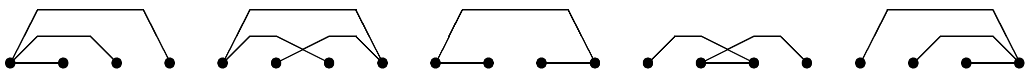
(o) ways of connecting $2n$ points in the plane lying on a horizontal line by n nonintersecting arcs, each arc connecting two of the points and lying above the points



(p) ways of drawing in the plane $n+1$ points lying on a horizontal line L and n arcs connecting them such that (α) the arcs do not pass below L, (β) the graph thus formed is a tree, (γ) no two arcs intersect in their interiors (i.e., the arcs are noncrossing), and (δ) at every vertex, all the arcs exit in the same direction (left or right)



(q) ways of drawing in the plane $n+1$ points lying on a horizontal line L and n arcs connecting them such that (α) the arcs do not pass below L, (β) the graph thus formed is a tree, (γ) no arc (including its endpoints) lies strictly below another arc, and (δ) at every vertex, all the arcs exit in the same direction (left or right)



(r) sequences of n 1's and n -1's such that every partial sum is nonnegative (with -1 denoted simply as $-$ below)

$$111{-}{-}{-} \qquad 11{-}1{-}{-} \qquad 11{-}{-}1{-} \qquad 1{-}11{-}{-} \qquad 1{-}1{-}1{-}$$

(s) sequences $1 \le a_1 \le \cdots \le a_n$ of integers with $a_i \le i$

111 112 113 122 123

(t) sequences $a_1 < a_2 < \cdots < a_{n-1}$ of integers satisfying $1 \le a_i \le 2i$

12 13 14 23 24

(u) sequences $a_1, a_2, \ldots, a_n$ of integers such that $a_1 = 0$ and $0 \le a_{i+1} \le a_i + 1$

000 001 010 011 012

(v) sequences $a_1, a_2, \ldots, a_{n-1}$ of integers such that $a_i \le 1$ and all partial sums are nonnegative

$$0,0 \qquad 0,1 \qquad 1,-1 \qquad 1,0 \qquad 1,1$$

(w) sequences $a_1, a_2, \ldots, a_n$ of integers such that $a_i \ge -1$, all partial sums are nonnegative, and $a_1 + a_2 + \cdots + a_n = 0$

$$0,0,0 \qquad 0,1,-1 \qquad 1,0,-1 \qquad 1,-1,0 \qquad 2,-1,-1$$

(x) sequences $a_1, a_2, \ldots, a_n$ of integers such that $0 \le a_i \le n-i$, and such that if $i < j$, $a_i > 0$, $a_j > 0$, and $a_{i+1} = a_{i+2} = \cdots = a_{j-1} = 0$, then $j - i > a_i - a_j$

000 010 100 200 110

(y) sequences $a_1, a_2, \ldots, a_n$ of integers such that $i \le a_i \le n$ and such that if $i \le j \le a_i$, then $a_j \le a_i$

123 133 223 323 333

(z) sequences $a_1, a_2, \ldots, a_n$ of integers such that $1 \le a_i \le i$ and such that if $a_i = j$, then $a_{i-r} \le j - r$ for $1 \le r \le j - 1$

111 112 113 121 123

(aa) equivalence classes B of words in the alphabet $[n-1]$ such that any three consecutive letters of any word in B are distinct, under the equivalence relation $uijv \sim ujiv$ for any words u, v and any $i, j \in [n-1]$ satisfying $|i-j| \geq 2$

$$\{\emptyset\} \quad \{1\} \quad \{2\} \quad \{12\} \quad \{21\}$$

(For $n = 4$ a representative of each class is given by $\emptyset$, 1, 2, 3, 12, 21, 13, 23, 32, 123, 132, 213, 321, 2132.)

(bb) partitions $\lambda = (\lambda_1, \ldots, \lambda_{n-1})$ with $\lambda_1 \leq n-1$ (so the diagram of λ is contained in an $(n-1) \times (n-1)$ square), such that if $\lambda' = (\lambda'_1, \lambda'_2, \ldots)$ denotes the conjugate partition to λ then $\lambda'_i \geq \lambda_i$ whenever $\lambda_i \geq i$

$$(0,0) \quad (1,0) \quad (1,1) \quad (2,1) \quad (2,2)$$

(cc) permutations $a_1 a_2 \cdots a_{2n}$ of the multiset $\{1^2, 2^2, \ldots, n^2\}$ such that: (i) the first occurrences of $1, 2, \ldots, n$ appear in increasing order, and (ii) there is no subsequence of the form $\alpha\beta\alpha\beta$

112233 112332 122331 123321 122133

(dd) permutations $a_1 a_2 \cdots a_{2n}$ of the set $[2n]$ such that: (i) $1, 3, \ldots, 2n-1$ appear in increasing order, (ii) $2, 4, \ldots, 2n$ appear in increasing order, and (iii) $2i-1$ appears before $2i$, $1 \leq i \leq n$

123456 123546 132456 132546 135246

(ee) permutations $a_1 a_2 \cdots a_n$ of $[n]$ with longest decreasing subsequence of length at most two (i.e., there does not exist $i < j < k$, $a_i > a_j > a_k$), called 321-*avoiding* permutations

123 213 132 312 231

(ff) permutations $a_1 a_2 \cdots a_n$ of $[n]$ for which there does not exist $i < j < k$ and $a_j < a_k < a_i$ (called 312-*avoiding* permutations)

123 132 213 231 321

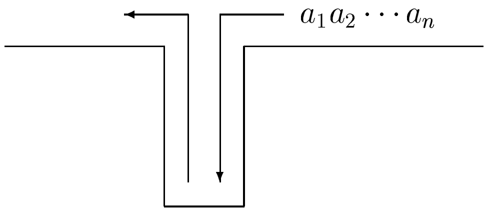
(gg) permutations w of $[2n]$ with n cycles of length two, such that the product $(1, 2, \ldots, 2n) \cdot w$ has $n+1$ cycles

$$\begin{aligned}
(1,2,3,4,5,6)(1,2)(3,4)(5,6) &= (1)(2,4,6)(3)(5) \\
(1,2,3,4,5,6)(1,2)(3,6)(4,5) &= (1)(2,6)(3,5)(4) \\
(1,2,3,4,5,6)(1,4)(2,3)(5,6) &= (1,3)(2)(4,6)(5) \\
(1,2,3,4,5,6)(1,6)(2,3)(4,5) &= (1,3,5)(2)(4)(6) \\
(1,2,3,4,5,6)(1,6)(2,5)(3,4) &= (1,5)(2,4)(3)(6)
\end{aligned}$$

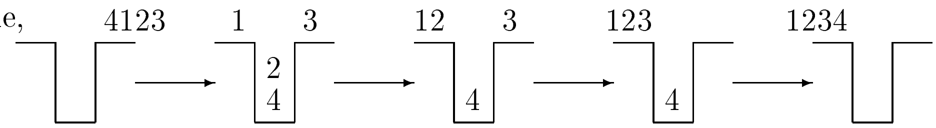
(hh) pairs (u, v) of permutations of $[n]$ such that u and v have a total of $n+1$ cycles, and $uv = (1, 2, \ldots, n)$

$$(1)(2)(3) \cdot (1,2,3) \quad (1,2,3) \cdot (1)(2)(3) \quad (1,2)(3) \cdot (1,3)(2)$$
$$(1,3)(2) \cdot (1)(2,3) \quad (1)(2,3) \cdot (1,2)(3)$$

(ii) permutations $a_1a_2 \cdots a_n$ of $[n]$ that can be put in increasing order on a single stack, defined recursively as follows: If $\emptyset$ is the empty sequence, then let $S(\emptyset) = \emptyset$. If $w = unv$ is a sequence of distinct integers with largest term n, then $S(w) = S(u)S(v)n$. A *stack-sortable* permutation w is one for which $S(w) = w$.

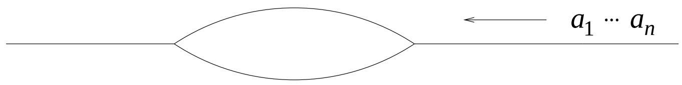


For example,



$$123 \quad 132 \quad 213 \quad 312 \quad 321$$

(jj) permutations $a_1a_2 \cdots a_n$ of $[n]$ that can be put in increasing order on two parallel queues. Now the picture is



$$123 \quad 132 \quad 213 \quad 231 \quad 312$$

(kk) fixed-point free involutions w of $[2n]$ such that if $i < j < k < l$ and $w(i) = k$, then $w(j) \neq l$ (in other words, 3412-avoiding fixed-point free involutions)

$$(12)(34)(56) \quad (14)(23)(56) \quad (12)(36)(45) \quad (16)(23)(45) \quad (16)(25)(34)$$

(ll) cycles of length $2n+1$ in $\mathfrak{S}_{2n+1}$ with descent set $\{n\}$

$$2371456 \quad 2571346 \quad 3471256 \quad 3671245 \quad 5671234$$

(mm) Baxter permutations (as defined in Exercise 55) of $[2n]$ or of $[2n+1]$ that are reverse alternating (as defined at the end of Section 3.16) and whose inverses are reverse alternating

$$132546 \quad 153426 \quad 354612 \quad 561324 \quad 563412$$

$$1325476 \quad 1327564 \quad 1534276 \quad 1735462 \quad 1756342$$

(nn) permutations w of $[n]$ such that if w has ℓ inversions then for all pairs of sequences $(a_1, a_2, \ldots, a_\ell), (b_1, b_2, \ldots, b_\ell) \in [n-1]^\ell$ satisfying

$$w = s_{a_1} s_{a_2} \cdots s_{a_\ell} = s_{b_1} s_{b_2} \cdots s_{b_\ell},$$

where s_j is the adjacent transposition $(j, j+1)$, we have that the ℓ-element *multisets* $\{a_1, a_2, \ldots, a_\ell\}$ and $\{b_1, b_2, \ldots, b_\ell\}$ are equal (thus, for example, $w = 321$ is not counted, since $w = s_1 s_2 s_1 = s_2 s_1 s_2$, and the multisets $\{1, 2, 1\}$ and $\{2, 1, 2\}$ are not equal)

123 132 213 231 312

(oo) permutations w of $[n]$ with the following property: Suppose that w has ℓ inversions, and let

$$R(w) = \{(a_1, \ldots, a_\ell) \in [n-1]^\ell : w = s_{a_1} s_{a_2} \cdots s_{a_\ell}\},$$

where s_j is as in (nn). Then

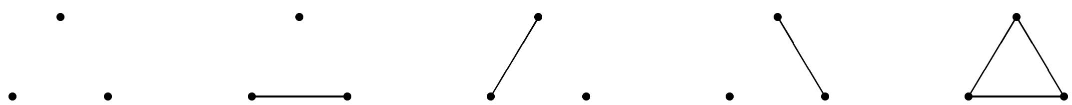
$$\sum_{(a_1,\ldots,a_\ell) \in R(w)} a_1 a_2 \cdots a_\ell = \ell!.$$

$$R(123) = \{\emptyset\}, \quad R(213) = \{(1)\}, \quad R(231) = \{(1, 2)\}$$
$$R(312) = \{(2, 1)\}, \quad R(321) = \{(1, 2, 1), (2, 1, 2)\}$$

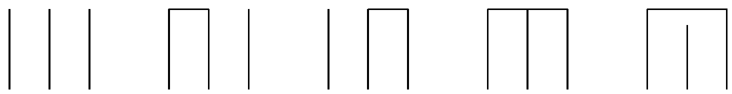
(pp) noncrossing partitions of $[n]$, i.e., partitions $\pi = \{B_1, \ldots, B_k\} \in \Pi_n$ such that if $a < b < c < d$ and $a, c \in B_i$ and $b, d \in B_j$, then $i = j$

123 12–3 13–2 23–1 1–2–3

(qq) partitions $\{B_1, \ldots, B_k\}$ of $[n]$ such that if the numbers $1, 2, \ldots, n$ are arranged in order around a circle, then the convex hulls of the blocks $B_1, \ldots, B_k$ are pairwise disjoint



(rr) noncrossing Murasaki diagrams with n vertical lines



(ss) noncrossing partitions of some set $[k]$ with $n+1$ blocks, such that any two elements of the same block differ by at least three

1–2–3–4 14–2–3–5 15–2–3–4 25–1–3–4 16–25–3–4

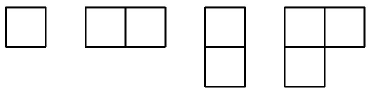
(tt) noncrossing partitions of $[2n+1]$ into $n+1$ blocks, such that no block contains two consecutive integers

137–46–2–5 1357–2–4–6 157–24–3–6 17–246–3–5 17–26–35–4 (uu) *nonnesting partitions* of $[n]$, i.e., partitions of $[n]$ such that if a, e appear in a block B and b, d appear in a *different* block B' where $a < b < d < e$, then there is a $c \in B$ satisfying $b < c < d$

123   12–3   13–2   23–1   1–2–3

(The unique partition of [4] that isn't nonnesting is 14–23.)

(vv) Young diagrams that fit in the shape $(n-1, n-2, \dots, 1)$

$\emptyset$ 

(ww) standard Young tableaux of shape (n, n) (or equivalently, of shape $(n, n-1)$)

$$\begin{array}{c} 123 \\ 456 \end{array} \quad \begin{array}{c} 124 \\ 356 \end{array} \quad \begin{array}{c} 125 \\ 346 \end{array} \quad \begin{array}{c} 134 \\ 256 \end{array} \quad \begin{array}{c} 135 \\ 246 \end{array}$$

or

$$\begin{array}{l} 123 \\ 45 \end{array} \quad \begin{array}{l} 124 \\ 35 \end{array} \quad \begin{array}{l} 125 \\ 34 \end{array} \quad \begin{array}{l} 134 \\ 25 \end{array} \quad \begin{array}{l} 135 \\ 24 \end{array}$$

(xx) pairs (P, Q) of standard Young tableaux of the same shape, each with n squares and at most two rows

$$(123, 123) \quad \left(\begin{array}{l} 12 \\ 3 \end{array}, \begin{array}{l} 12 \\ 3 \end{array} \right) \quad \left(\begin{array}{l} 12 \\ 3 \end{array}, \begin{array}{l} 13 \\ 2 \end{array} \right) \quad \left(\begin{array}{l} 13 \\ 2 \end{array}, \begin{array}{l} 12 \\ 3 \end{array} \right) \quad \left(\begin{array}{l} 13 \\ 2 \end{array}, \begin{array}{l} 13 \\ 2 \end{array} \right)$$

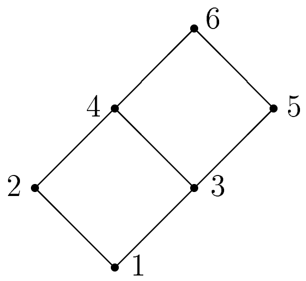
(yy) column-strict plane partitions of shape $(n-1, n-2, \dots, 1)$, such that each entry in the ith row is equal to $n-i$ or $n-i+1$

$$\begin{array}{l} 3\,3 \\ 2 \end{array} \quad \begin{array}{l} 3\,3 \\ 1 \end{array} \quad \begin{array}{l} 3\,2 \\ 2 \end{array} \quad \begin{array}{l} 3\,2 \\ 1 \end{array} \quad \begin{array}{l} 2\,2 \\ 1 \end{array}$$

(zz) convex subsets S of the poset $\mathbb{Z} \times \mathbb{Z}$, up to translation by a diagonal vector (m, m), such that if $(i, j) \in S$ then $0 < i - j < n$.

$\emptyset$ $\{(1,0)\}$ $\{(2,0)\}$ $\{(1,0),(2,0)\}$ $\{(2,0),(2,1)\}$

(aaa) linear extensions of the poset $\mathbf{2} \times \mathbf{n}$



123456
123546
132456
132546
135246

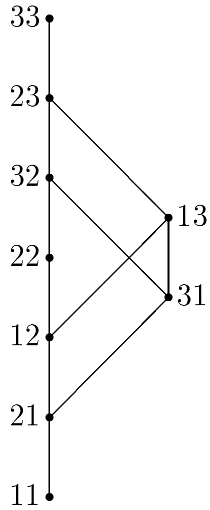
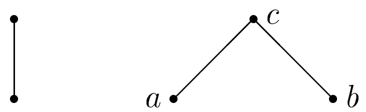


Figure 5: A poset with $C_4 = 14$ order ideals

(bbb) order ideals of $\mathrm{Int}(\boldsymbol{n-1})$, the poset of intervals of the chain $\mathbf{n}-\mathbf{1}$



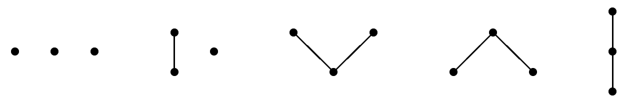
2 Int(**2**)

$\emptyset,\quad a,\quad b,\quad ab,\quad abc$

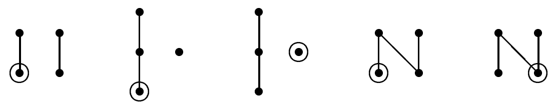
(ccc) order ideals of the poset A_n obtained from the poset $(\mathbf{n}-\mathbf{1})\times(\mathbf{n}-\mathbf{1})$ by adding the relations $(i,j)<(j,i)$ if $i>j$ (see Figure 5 for the Hasse diagram of A_4)

$\emptyset \qquad \{11\} \qquad \{11,21\} \qquad \{11,21,12\} \qquad \{11,21,12,22\}$

(ddd) nonisomorphic n-element posets with no induced subposet isomorphic to $\mathbf{2}+\mathbf{2}$ or $\mathbf{3}+\mathbf{1}$



(eee) nonisomorphic $(n+1)$-element posets that are a union of two chains and that are not a (nontrivial) ordinal sum, rooted at a minimal element

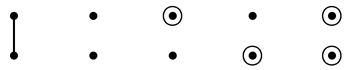


(fff) relations R on $[n]$ that are reflexive (iRi), symmetric ($iRj \Rightarrow jRi$), and such that if $1\le i<j<k\le n$ and iRk, then iRj and jRk (in the example below we write

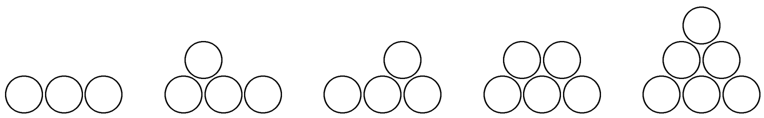
ij for the pair (i,j), and we omit the pairs ii)

$$\emptyset \quad \{12,21\} \quad \{23,32\} \quad \{12,21,23,32\} \quad \{12,21,13,31,23,32\}$$

(ggg) joining some of the vertices of a convex $(n-1)$-gon by disjoint line segments, and circling a subset of the remaining vertices



(hhh) ways to stack coins in the plane, the bottom row consisting of n consecutive coins



(iii) n-tuples $(a_1, a_2, \ldots, a_n)$ of integers $a_i \geq 2$ such that in the sequence $1a_1a_2\cdots a_n1$, each a_i divides the sum of its two neighbors

14321 13521 13231 12531 12341

(jjj) n-element multisets on $\mathbb{Z}/(n+1)\mathbb{Z}$ whose elements sum to 0

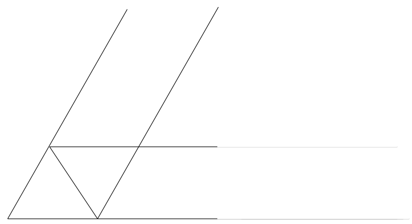
000 013 022 112 233

(kkk) n-element subsets S of $\mathbb{N} \times \mathbb{N}$ such that if $(i,j) \in S$ then $i \geq j$ and there is a lattice path from $(0,0)$ to (i,j) with steps $(0,1)$, $(1,0)$, and $(1,1)$ that lies entirely inside S

$$\{(0,0),(1,0),(2,0)\} \quad \{(0,0),(1,0),(1,1)\} \quad \{(0,0),(1,0),(2,1)\}$$

$$\{(0,0),(1,1),(2,1)\} \quad \{(0,0),(1,1),(2,2)\}$$

(lll) regions into which the cone $x_1 \geq x_2 \geq \cdots \geq x_n$ in $\mathbb{R}^n$ is divided by the hyperplanes $x_i - x_j = 1$, for $1 \leq i < j \leq n$ (the diagram below shows the situation for $n = 3$, intersected with the hyperplane $x_1 + x_2 + x_3 = 0$)



$$
\begin{array}{ccccccccccccccccccccccccccc}
1&&1&&1&&1&&1&&1&&1&&1&&1&&1&&1&&1&&1&&1\\
&1&&3&&2&&1&&5&&1&&2&&3&&1&&3&&2&&1&&5&\\
&&2&&5&&1&&4&&4&&1&&5&&2&&2&&5&&1&&4&&\\
&&&3&&2&&3&&3&&3&&2&&3&&3&&3&&2&&3&&&\\
&&&&1&&5&&2&&2&&5&&1&&4&&4&&1&&5&&&&\\
&&&&&2&&3&&1&&3&&2&&1&&5&&1&&2&&&&&\\
&&&&&&1&&1&&1&&1&&1&&1&&1&&1&&&&&&
\end{array}
$$

Figure 6: The frieze pattern corresponding to the sequence $(1,3,2,1,5,1,2,3)$

(mmm) positive integer sequences $a_1, a_2, \ldots, a_{n+2}$ for which there exists an integer array (necessarily with $n+1$ rows)

$$
\begin{array}{ccccccccccccccccc}
1 && 1 && 1 && \cdots && 1 && 1 && 1 && \cdots && 1 && 1\\
& a_1 && a_2 && a_3 && \cdots && a_{n+2} && a_1 && a_2 && \cdots && a_{n-1} \\
&& b_1 && b_2 && b_3 && \cdots && b_{n+2} && b_1 && \cdots && b_{n-2} \\
&&&&&&&& \vdots &&&&&&&& \\
&&& r_1 && r_2 && r_3 && \cdots && r_{n+2} && r_1 &&& \\
&&&& 1 && 1 && 1 && \cdots && 1 &&&&
\end{array}
\tag{54}
$$

such that any four neighboring entries in the configuration $\begin{smallmatrix} & r & \\ s & & t \\ & u & \end{smallmatrix}$ satisfy $st = ru+1$ (an example of such an array for $(a_1, \ldots, a_8) = (1,3,2,1,5,1,2,3)$ (necessarily unique) is given by Figure 6):

12213 22131 21312 13122 31221

(nnn) n-tuples $(a_1, \ldots a_n)$ of positive integers such that the tridiagonal matrix

$$
\begin{bmatrix}
a_1 & 1 & 0 & 0 & \cdots & 0 & 0 \\
1 & a_2 & 1 & 0 & \cdots & 0 & 0 \\
0 & 1 & a_3 & 1 & \cdots & 0 & 0 \\
 & & & & \vdots & & \\
0 & 0 & 0 & 0 & \cdots & a_{n-1} & 1 \\
0 & 0 & 0 & 0 & \cdots & 1 & a_n
\end{bmatrix}
$$

is positive definite with determinant one

131 122 221 213 312

20. (a) [2+] Let m, n be integers satisfying $1 \leq n < m$. Show by a simple bijection that the number of lattice paths from $(1,0)$ to (m,n) with steps $(0,1)$ and $(1,0)$ that intersect the line $y = x$ in at least one point is equal to the number of lattice paths from $(0,1)$ to (m,n) with steps $(0,1)$ and $(1,0)$.

(b) [2–] Deduce that the number of lattice paths from $(0,0)$ to (m,n) with steps $(1,0)$ and $(0,1)$ that intersect the line $y=x$ only at $(0,0)$ is given by $\frac{m-n}{m+n}\binom{m+n}{n}$.

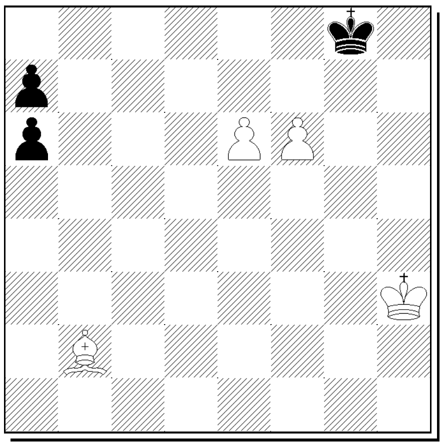
(c) [1+] Show from (b) that the number of lattice paths from $(0,0)$ to (n,n) with steps $(1,0)$ and $(0,1)$ that never rise above the line $y=x$ is given by the Catalan number $C_n = \frac{1}{n+1}\binom{2n}{n}$. (This gives a direct combinatorial proof of interpretation (h) of C_n in Exercise 19.)

21. (a) [2+] Let X_n be the set of all $\binom{2n}{n}$ lattice paths from $(0,0)$ to (n,n) with steps $(0,1)$ and $(1,0)$. Define the *excedance* (also spelled "exceedance") of a path $P \in X_n$ to be the number of i such that at least one point (i,i') of P lies above the line $y=x$ (i.e., $i' > i$). Show that the number of paths in X_n with excedance j is independent of j.

(b) [1] Deduce that the number of $P \in X_n$ that never rise above the line $y=x$ is given by the Catalan number $C_n = \frac{1}{n+1}\binom{2n}{n}$ (a direct proof of interpretation (h) of C_n in Exercise 19). Compare with Example 5.3.11, which also gives a direct combinatorial interpretation of C_n when written in the form $\frac{1}{n+1}\binom{2n}{n}$ (as well as in the form $\frac{1}{2n+1}\binom{2n+1}{n}$).

22. [2+] Show (bijectively if possible) that the number of lattice paths from $(0,0)$ to $(2n,2n)$ with steps $(1,0)$ and $(0,1)$ that avoid the points $(2i-1,2i-1)$, $1 \le i \le n$, is equal to the Catalan number C_{2n}.

23. [3–] Consider the following chess position.



Black is to make 19 consecutive moves, after which White checkmates Black in one move. Black may not move into check, and may not check White (except possibly on his last move). Black and White are *cooperating* to achieve the aim of checkmate. (In chess problem parlance, this problem is called a *serieshelpmate in 19*.) How many different solutions are there?

24. [?] Explain the significance of the following sequence:

un, dos, tres, quatre, cinc, sis, set, vuit, nou, deu, ...

25. [2]–[5] Show that the Catalan number $C_n = \frac{1}{n+1}\binom{2n}{n}$ has the algebraic interpretations given below.

(a) number of two-sided ideals of the algebra of all $(n-1) \times (n-1)$ upper triangular matrices over a field

(b) dimension of the space of invariants of $\mathrm{SL}(2, \mathbb{C})$ acting on the $2n$th tensor power $T^{2n}(V)$ of its "defining" two-dimensional representation V

(c) dimension of the irreducible representation of the symplectic group $\mathrm{Sp}(2(n-1), \mathbb{C})$ (or Lie algebra $\mathfrak{sp}(2(n-1), \mathbb{C})$) with highest weight λ_{n-1}, the $(n-1)$st fundamental weight

(d) dimension of the primitive intersection homology (say with real coefficients) of the toric variety associated with a (rationally embedded) n-dimensional cube

(e) the generic number of $\mathrm{PGL}(2, \mathbb{C})$ equivalence classes of degree n rational maps with a fixed branch set

(f) number of translation conjugacy classes of degree $n+1$ monic polynomials in one complex variable, all of whose critical points are fixed

(g) dimension of the algebra (over a field K) with generators $\epsilon_1, \ldots, \epsilon_{n-1}$ and relations

$$\begin{aligned} \epsilon_i^2 &= \epsilon_i \\ \beta\epsilon_i\epsilon_j\epsilon_i &= \epsilon_i, \quad \text{if } |i-j| = 1 \\ \epsilon_i\epsilon_j &= \epsilon_j\epsilon_i, \quad \text{if } |i-j| \geq 2, \end{aligned}$$

where β is a nonzero element of K

(h) number of $\oplus$-sign types indexed by A_{n-1}^+ (the set of positive roots of the root system A_{n-1})

(i) Let the symmetric group $\mathfrak{S}_n$ act on the polynomial ring $A = \mathbb{C}[x_1, \ldots, x_n, y_1, \ldots, y_n]$ by $w \cdot f(x_1, \ldots, x_n, y_1, \ldots, y_n) = f(x_{w(1)}, \ldots, x_{w(n)}, y_{w(1)}, \ldots, y_{w(n)})$ for all $w \in \mathfrak{S}_n$. Let I be the ideal generated by all invariants of positive degree, i.e.,

$$I = \langle f \in A : w \cdot f = f \text{ for all } w \in \mathfrak{S}_n, \text{ and } f(0) = 0 \rangle.$$

Then (conjecturally) C_n is the dimension of the subspace of A/I affording the sign representation, i.e.,

$$C_n = \dim\{f \in A/I : w \cdot f = (\operatorname{sgn} w)f \text{ for all } f \in \mathfrak{S}_n\}.$$

26. (a) [3–] Let D be a Young diagram of a partition λ, as defined in Section 1.3. Given a square s of D let t be the lowest square in the same column as s, and let u be the rightmost square in the same row as s. Let $f(s)$ be the number of paths from

t to u that stay within D, and such that each step is one square to the north or one square to the east. Insert the number $f(s)$ in square s, obtaining an array A. For instance, if $\lambda = (5,4,3,3)$ then A is given by

16	7	2	1	1
6	3	1	1	
3	2	1		
1	1	1		

Let M be the largest square subarray (using consecutive rows and columns) of A containing the upper left-hand corner. Regard M as a matrix. For the above example we have

$$M = \begin{bmatrix} 16 & 7 & 2 \\ 6 & 3 & 1 \\ 3 & 2 & 1 \end{bmatrix}.$$

Show that $\det M = 1$.

(b) [2] Find the unique sequence $a_0, a_1, \dots$ of real numbers such that for all $n \geq 0$ we have

$$\det \begin{bmatrix} a_0 & a_1 & \cdots & a_n \\ a_1 & a_2 & \cdots & a_{n+1} \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ a_n & a_{n+1} & \cdots & a_{2n} \end{bmatrix} = \det \begin{bmatrix} a_1 & a_2 & \cdots & a_n \\ a_2 & a_3 & \cdots & a_{n+1} \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ a_n & a_{n+1} & \cdots & a_{2n-1} \end{bmatrix} = 1.$$

(When $n = 0$ the second matrix is empty and by convention has determinant one.)

27. (a) [3–] Let V_n be a real vector space with basis $x_0, x_1, \dots, x_n$ and scalar product defined by $\langle x_i, x_j \rangle = C_{i+j}$, the $(i+j)$-th Catalan number. It follows from Exercise 26(b) that this scalar product is positive definite, and therefore V has an orthonormal basis. Is there an orthonormal basis for V_n whose elements are *integral* linear combinations of the x_i's ?

(b) [3–] Same as (a), except now $\langle x_i, x_j \rangle = C_{i+j+1}$.

(c) [5–] Investigate the same question for the matrices M of Exercise 26(a) (so $\langle x_i, x_j \rangle = M_{ij}$) when λ is self-conjugate (so M is symmetric).

28. (a) [3–] Suppose that real numbers $x_1, x_2, \dots, x_d$ are chosen uniformly and independently from the interval $[0,1]$. Show that the probability that the sequence $x_1, x_2, \dots, x_d$ is convex (i.e., $x_i \leq \frac{1}{2}(x_{i-1}+x_{i+1})$ for $2 \leq i \leq d-1$) is $C_{d-1}/(d-1)!^2$, where C_{d-1} denotes a Catalan number.

(b) [3–] Let $\mathcal{C}_d$ denote the set of all points $(x_1, x_2, \ldots, x_d) \in \mathbb{R}^d$ such that $0 \leq x_i \leq 1$ and the sequence $x_1, x_2, \ldots, x_d$ is convex. It is easy to see that $\mathcal{C}_d$ is a d-dimensional convex polytope, called the *convexotope*. Show that the vertices of $\mathcal{C}_d$ consist of the points

$$\left(1, \tfrac{j-1}{j}, \tfrac{j-2}{j}, \ldots, \tfrac{1}{j}, 0, 0, \ldots, 0, \tfrac{1}{k}, \tfrac{2}{k}, \ldots, 1\right) \tag{55}$$

(with at least one 0 coordinate), together with $(1, 1, \ldots, 1)$ (so $\binom{d+1}{2} + 1$ vertices in all). For instance, the vertices of $\mathcal{C}_3$ are $(0, 0, 0)$, $(0, 0, 1)$, $(0, \frac{1}{2}, 1)$, $(1, 0, 0)$, $(1, \frac{1}{2}, 0)$, $(1, 0, 1)$, $(1, 1, 1)$.

(c) [3] Show that the Ehrhart quasi-polynomial $i(\mathcal{C}_d, n)$ of $\mathcal{C}_d$ (as defined in Section 4.6) is given by

$$\begin{aligned} y_d &:= \sum_{n \geq 0} i(\mathcal{C}_d, n) x^n \\ &= \frac{1}{1-x} \left(\sum_{r=1}^{d} \frac{1}{[1][r-1]!} * \frac{1}{[1][d-r]!} - \sum_{r=1}^{d-1} \frac{1}{[1][r-1]!} * \frac{1}{[1][d-1-r]!} \right), \end{aligned} \tag{56}$$

where $[i] = 1 - x^i$, $[i]! = [1][2]\cdots[i]$, and $*$ denotes Hadamard product. For instance,

$$\begin{aligned} y_1 &= \frac{1}{(1-x)^2} \\ y_2 &= \frac{1+x}{(1-x)^3} \\ y_3 &= \frac{1+2x+3x^2}{(1-x)^3(1-x^2)} \\ y_4 &= \frac{1+3x+9x^2+12x^3+11x^4+3x^5+x^6}{(1-x)^2(1-x^2)^2(1-x^3)} \\ y_5 &= \frac{1+4x+14x^2+34x^3+63x^4+80x^5+87x^6+68x^7+42x^8+20x^9+7x^{10}}{(1-x)(1-x^2)^2(1-x^3)^2(1-x^4)}. \end{aligned}$$

Is there a simpler formula than (56) for $i(\mathcal{C}_d, n)$ or y_d?

29. [3] Suppose that $n+1$ points are chosen uniformly and independently from inside a square. Show that the probability that the points are in convex position (i.e., each point is a vertex of the convex hull of all the points) is $(C_n/n!)^2$.

30. [3–] Let f_n be the number of partial orderings of the set $[n]$ that contain no induced subposets isomorphic to $\mathbf{3}+\mathbf{1}$ or $\mathbf{2}+\mathbf{2}$. (This exercise is the labelled analogue of Exercise 19(ddd). As mentioned in the solution to this exercise, such posets are called *semiorders*.) Let $C(x) = 1 + x + 2x^2 + 5x^3 + \cdots$ be the generating function for Catalan numbers. Show that

$$\sum_{n \geq 0} f_n \frac{x^n}{n!} = C(1 - e^{-x}), \tag{57}$$

the composition of $C(x)$ with the series $1 - e^{-x} = x - \frac{1}{2}x^2 + \frac{1}{6}x^3 - \cdots$.

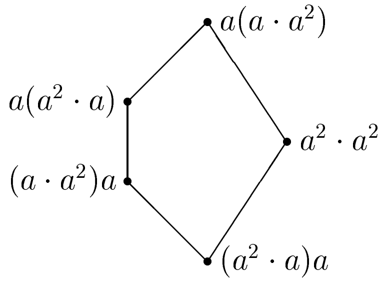


Figure 7: The Tamari lattice T_3

31. (a) [3–] Let $\mathcal{P}$ denote the convex hull in $\mathbb{R}^{d+1}$ of the origin together with all vectors $e_i - e_j$, where e_i is the ith unit coordinate vector and $i < j$. Thus $\mathcal{P}$ is a d-dimensional convex polytope. Show that the relative volume of $\mathcal{P}$ (as defined in Section 4.6) is equal to $C_d/d!$, where C_d denotes a Catalan number.

    (b) [3] Let $i(\mathcal{P}, n)$ denote the Ehrhart polynomial of $\mathcal{P}$. Find a combinatorial interpretation of the coefficients of the i-Eulerian polynomial (in the terminology of Section 4.3)

$$(1-x)^{d+1}\sum_{n\geq 0} i(\mathcal{P}, n)x^n.$$

32. (a) [3–] Define a partial order T_n on the set of all binary bracketings (parenthesizations) of a string of length $n+1$ as follows. We say that v covers u if u contains a subexpression $(xy)z$ (where x, y, z are bracketed strings) and v is obtained from u by replacing $(xy)z$ with $x(yz)$. For instance, $((a^2\cdot a)a^2)(a^2\cdot a^2)$ is covered by $((a\cdot a^2)a^2)(a^2\cdot a^2)$, $(a^2(a\cdot a^2))(a^2\cdot a^2)$, $((a^2\cdot a)a^2)(a(a\cdot a^2))$, and $(a^2\cdot a)(a^2(a^2\cdot a^2))$. Figures 7 and 8 show the Hasse diagrams of T_3 and T_4. (In Figure 8, we have encoded the binary bracketing by a string of four $+$'s and four $-$'s, where a $+$ stands for a left parenthesis and a $-$ for the letter a, with the last a omitted.) Let U_n be the poset of all integer vectors $(a_1, a_2, \ldots, a_n)$ such that $i \leq a_i \leq n$ and such that if $i \leq j \leq a_i$ then $a_j \leq a_i$, ordered coordinatewise. Show that T_n and U_n are isomorphic posets.

    (b) [2] Deduce from (a) that T_n is a lattice (called the *Tamari lattice*).

33. Let C be a convex n-gon. Let $\mathcal{S}$ be the set of all sets of diagonals of C that do not intersect in the interior of C. Partially order the element of $\mathcal{S}$ by inclusion, and add a $\hat{1}$. Call the resulting poset A_n.

    (a) [3–] Show that A_n is a simplicial Eulerian lattice of rank $n-2$, as defined in Section 3.14.

    (b) [3] Show in fact that A_n is the lattice of faces of an $(n-3)$-dimensional convex polytope $\mathcal{Q}_n$.

    (c) [3–] Find the number $W_i = W_i(n)$ of elements of A_n of rank i. Equivalently, W_i is the number of ways to draw i diagonals of C that do not intersect in their interiors. Note that by Proposition 2.1, $W_i(n)$ is also the number of plane trees with $n+i$ vertices and $n-1$ endpoints such that no vertex has exactly one successor.

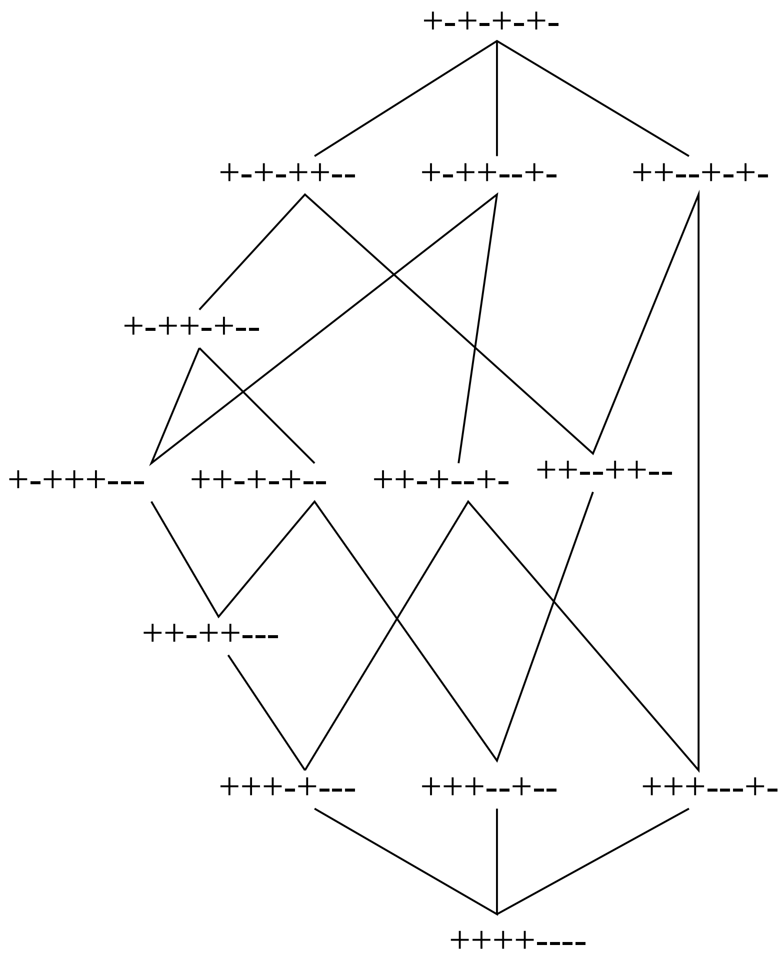


Figure 8: The Tamari lattice T_4

(d) [3–] Define

$$\sum_{i=0}^{n-3} W_i(x-1)^{n-i-3} = \sum_{i=0}^{n-3} h_i x^{n-3-i}, \tag{58}$$

as in equation (3.44). The vector $(h_0, \ldots, h_{n-3})$ is called the *h-vector* of A_n (or of the polytope $\mathcal{Q}_n$). Find an explicit formula for each h_i.

34. There are many possible q-analogues of Catalan numbers. In (a) we give what is perhaps the most natural "combinatorial" q-analogue, while in (b) we give the most natural "explicit formula" q-analogue. In (c) we give an interesting extension of (b), while (d) and (e) are concerned with another special case of (c).

(a) [2+] Let

$$C_n(q) = \sum_P q^{A(P)},$$

where the sum is over all lattice paths P from $(0,0)$ to (n,n) with steps $(1,0)$ and $(0,1)$, such that P never rises above the line $y = x$, and where $A(P)$ is the area under the path (and above the x-axis). Note that by Exercise 19(h), we have $C_n(1) = C_n$. (It is interesting to see what statistic corresponds to $A(P)$ for many of the other combinatorial interpretations of C_n given in Exercise 19.) For instance, $C_0(q) = C_1(q) = 1$, $C_2(q) = 1+q$, $C_3(q) = 1+q+2q^2+q^3$, $C_4(q) = 1+q+2q^2+3q^3+3q^4+3q^5+q^6$. Show that

$$C_{n+1}(q) = \sum_{i=0}^{n} C_i(q) C_{n-i}(q) q^{(i+1)(n-i)}.$$

Deduce that if $\tilde{C}_n(q) = q^{\binom{n}{2}} C_n(1/q)$, then the generating function

$$F(x) = \sum_{n \geq 0} \tilde{C}_n(q) x^n$$

satisfies

$$xF(x)F(qx) - F(x) + 1 = 0.$$

From this we get the continued fraction expansion

$$F(x) = \cfrac{1}{1 - \cfrac{x}{1 - \cfrac{qx}{1 - \cfrac{q^2x}{1 - \cdots}}}}. \tag{59}$$

(b) [2+] Define

$$c_n(q) = \frac{1}{(\boldsymbol{n+1})} \binom{\boldsymbol{2n}}{\boldsymbol{n}}.$$

For instance, $c_0(q) = c_1(q) = 1$, $c_2(q) = 1 + q^2$, $c_3(q) = 1 + q^2 + q^3 + q^4 + q^6$, $c_4(q) = 1 + q^2 + q^3 + 2q^4 + q^5 + 2q^6 + q^7 + 2q^8 + q^9 + q^{10} + q^{12}$. Show that

$$c_n(q) = \sum_w q^{\mathrm{maj}(w)},$$

where w ranges over all sequences $a_1a_2 \cdots a_{2n}$ of n 1's and n -1's such that each partial sum is nonnegative, and where

$$\mathrm{maj}(w) = \sum_{\{i\,:\,a_i > a_{i+1}\}} i,$$

the major index of w.

(c) [3–] Let t be a parameter, and define

$$c_n(t;q) = \frac{1}{(\boldsymbol{n+1})} \sum_{i=0}^{n} \binom{\boldsymbol{n}}{\boldsymbol{i}} \binom{\boldsymbol{n}}{\boldsymbol{i+1}} q^{i^2+it}.$$

Show that

$$c_n(t;q) = \sum_w q^{\mathrm{maj}(w)+(t-1)\mathrm{des}(w)},$$

where w ranges over the same set as in (b), and where

$$\mathrm{des}(w) = \#\{i : a_i > a_{i+1}\},$$

the number of descents of w. (Hence $c_n(1;q) = c_n(q)$.)

(d) [3–] Show that

$$c_n(0;q) = \frac{1+q}{1+q^n} c_n(q).$$

For instance, $c_0(0;q) = c_1(0;q) = 1$, $c_2(0;q) = 1+q$, $c_3(0;q) = 1+q+q^2+q^3+q^4$, $c_4(0;q) = 1 + q + q^2 + 2q^3 + 2q^4 + 2q^5 + 2q^6 + q^7 + q^8 + q^9$.

(e) [3+] Show that the coefficients of $c_n(0;q)$ are *unimodal*, i.e., if $c_n(0;q) = \sum b_i q^i$, then for some j we have $b_0 \le b_1 \le \cdots \le b_j \ge b_{j+1} \ge b_{j+2} \ge \cdots$. (In fact, we can take $j = \lfloor \frac{1}{2} \deg c_n(0;q) \rfloor = \lfloor \frac{1}{2}(n-1)^2 \rfloor$.)

35. Let Q_n be the poset of direct-sum decompositions of an n-dimensional vector space V_n over the field $\mathbb{F}_q$, as defined in Example 5.5.2(b). Let $\bar{Q}_n$ denote Q_n with a $\hat{0}$ adjoined, and let $\mu_n(q) = \mu_{\bar{Q}_n}(\hat{0}, \hat{1})$. Hence by (5.74) we have

$$-\sum_{n\ge 1} \mu_n(q) \frac{x^n}{q^{\binom{n}{2}} (\boldsymbol{n})!} = \log \sum_{n \ge 0} \frac{x^n}{q^{\binom{n}{2}} (\boldsymbol{n})!}.$$

(a) [3–] Show that

$$\mu_n(q) = \frac{1}{n}(-1)^n (q-1)(q^2-1)\cdots(q^{n-1}-1) P_n(q),$$

where $P_n(q)$ is a polynomial in q of degree $\binom{n}{2}$ with nonnegative integral coefficients, satisfying $P_n(1) = \binom{2n-1}{n}$. For instance,

$$\begin{aligned} P_1(q) &= 1 \\ P_2(q) &= 2+q \\ P_3(q) &= 3+3q+3q^2+q^3 \\ P_4(q) &= (2+2q^2+q^3)(2+2q+2q^2+q^3). \end{aligned}$$

(b) Show that

$$\exp \sum_{n\geq 1} q^{\binom{n}{2}} P_n(1/q) \frac{x^n}{n} = \sum_{n\geq 1} q^{\binom{n}{2}} C_n(1/q) x^n,$$

where $C_n(q)$ is the q-Catalan polynomial defined in Exercise 34(a).

36. (a) [2+] The *Narayana numbers* $N(n,k)$ are defined by

$$N(n,k) = \frac{1}{n}\binom{n}{k}\binom{n}{k-1}.$$

Let X_{nk} be the set of all sequences $w = w_1 w_2 \cdots w_{2n}$ of n 1's and n -1's with all partial sums nonnegative, such that

$$k = \#\{j : w_j = 1, w_{j+1} = -1\}.$$

Give a combinatorial proof that $N(n,k) = \#X_{nk}$. Hence by Exercise 19(r), there follows

$$\sum_{k=1}^{n} N(n,k) = C_n.$$

(It is interesting to find for each of the combinatorial interpretations of C_n given by Exercise 19 a corresponding decomposition into subsets counted by Narayana numbers.)

(b) [2+] Let $F(x,t) = \sum_{n\geq 1}\sum_{k\geq 1} N(n,k) x^n t^k$. Using the combinatorial interpretation of $N(n,k)$ given in (a), show that

$$xF^2 + (xt + x - 1)F + xt = 0, \tag{60}$$

so

$$F(x,t) = \frac{1 - x - xt - \sqrt{(1-x-xt)^2 - 4x^2 t}}{2x}.$$

37. [2+] The *Motzkin numbers* M_n are defined by

$$\begin{aligned} \sum_{n\geq 0} M_n x^n &= \frac{1 - x - \sqrt{1 - 2x - 3x^2}}{2x^2} \\ &= 1 + x + 2x^2 + 4x^3 + 9x^4 + 21x^5 + 51x^6 + 127x^7 + 323x^8 \\ &\quad + 835x^9 + 2188x^{10} + \cdots. \end{aligned}$$

Show that $M_n = \Delta^n C_1$ and $C_n = \Delta^{2n} M_0$, where C_n denotes a Catalan number.

38. [3–] Show that the Motzkin number M_n has the following combinatorial interpretations. (See Exercise 46(b) for an additional interpretation.)

(a) Number of ways of drawing any number of nonintersecting chords among n points on a circle.

(b) Number of walks on $\mathbb{N}$ with n steps, with steps -1, 0, or 1, starting and ending at 0.

(c) Number of lattice paths from $(0,0)$ to (n,n), with steps $(0,2)$, $(2,0)$, and $(1,1)$, never rising above the line $y=x$.

(d) Number of paths from $(0,0)$ to (n,n) with steps $(1,0)$, $(1,1)$, and $(1,-1)$, never going below the x-axis. Such paths are called *Motzkin paths*.

(e) Number of pairs $1 \leq a_1 < \cdots < a_k \leq n$ and $1 \leq b_1 < \cdots < b_k \leq n$ of integer sequences such that $a_i \leq b_i$ and every integer in the set $[n]$ appears at least once among the a_i's and b_i's.

(f) Number of ballot sequences (as defined in Corollary 2.3(ii)) $(a_1,\ldots,a_{2n+2})$ such that we never have $(a_{i-1},a_i,a_{i+1}) = (1,-1,1)$.

(g) Number of plane trees with $n/2$ edges, allowing "half edges" that have no successors and count as half an edge.

(h) Number of plane trees with $n+1$ edges in which no vertex, the root excepted, has exactly one successor.

(i) Number of plane trees with n edges in which every vertex has at most two successors.

(j) Number of binary trees with $n-1$ edges such that no two consecutive edges slant to the right.

(k) Number of plane trees with $n+1$ vertices such that every vertex of odd height (with the root having height 0) has at most one successor.

(l) Number of noncrossing partitions $\pi = \{B_1,\ldots,B_k\}$ of $[n]$ (as defined in Exercise 3.68) such that if $B_i = \{b\}$ and $a < b < c$, then a and c appear in different blocks of π.

(m) Number of noncrossing partitions π of $[n+1]$ such that no block of π contains two consecutive integers.

39. [3–] The Schröder numbers r_n and s_n were defined in Section 2. Show that they have the following combinatorial interpretations.

(a) s_{n-1} is the total number of bracketings (parenthesizations) of a string of n letters.

(b) s_{n-1} is the number of plane trees with no vertex of degree one and with n endpoints.

(c) r_{n-1} is the number of plane trees with n vertices and with a subset of the endpoints circled.

(d) s_n is the number of binary trees with n vertices and with each right edge colored either red or blue.

(e) s_n is the number of lattice paths in the (x, y) plane from $(0, 0)$ to the x-axis using steps $(1, k)$, where $k \in \mathbb{P}$ or $k = -1$, never passing below the x-axis, and with n steps of the form $(1, -1)$.

(f) s_n is the number of lattice paths in the (x, y) plane from $(0, 0)$ to (n, n) using steps $(k, 0)$ or $(0, 1)$ with $k \in \mathbb{P}$, and never passing above the line $y = x$.

(g) r_{n-1} is the number of parallelogram polynominoes (defined in the solution to Exercise 19(l)) of perimeter $2n$ with each column colored either black or white.

(h) s_n is the number of ways to draw any number of diagonals of a convex $(n+2)$-gon that do not intersect in their interiors

(i) s_n is the number of sequences $i_1 i_2 \cdots i_k$, where $i_j \in \mathbb{P}$ or $i_j = -1$ (and k can be arbitrary), such that $n = \#\{j : i_j = -1\}$, $i_1 + i_2 + \cdots + i_j \geq 0$ for all j, and $i_1 + i_2 + \cdots + i_k = 0$.

(j) r_n is the number of lattice paths from $(0, 0)$ to (n, n), with steps $(1, 0)$, $(0, 1)$, and $(1, 1)$, that never rise above the line $y = x$.

(k) r_{n-1} is the number of $n \times n$ permutation matrices P with the following property: We can eventually reach the all 1's matrix by starting with P and continually replacing a 0 by a 1 if that 0 has at least two adjacent 1's, where an entry a_{ij} is defined to be adjacent to $a_{i\pm1,j}$ and $a_{i,j\pm1}$.

(l) Let $u = u_1 \cdots u_k \in \mathfrak{S}_k$. We say that a permutation $w = w_1 \cdots w_n \in \mathfrak{S}_n$ is u-*avoiding* if no subsequence $w_{a_1}, \ldots, w_{a_k}$ (with $a_1 < \cdots < a_k$) is in the same relative order as u, i.e., $u_i < u_j$ if and only if $w_{a_i} < w_{a_j}$. Let $\mathfrak{S}_n(u, v)$ denote the set of permutations $w \in \mathfrak{S}_n$ avoiding both the permutations $u, v \in \mathfrak{S}_4$. There is a group G of order 16 that acts on the set of pairs (u, v) of unequal elements of $\mathfrak{S}_4$ such that if (u, v) and (u', v') are in the same G-orbit (in which case we say that they are *equivalent*), then there is a simple bijection between $\mathfrak{S}_n(u, v)$ and $\mathfrak{S}_n(u', v')$ (for all n). Namely, identifying a permutation with the corresponding permutation matrix, the orbit of (u, v) is obtained by possibly interchanging u and v, and then doing a simultaneous dihedral symmetry of the square matrices u and v. There are then ten inequivalent pairs $(u, v) \in \mathfrak{S}_4 \times \mathfrak{S}_4$ for which $\#\mathfrak{S}_n(u, v) = r_{n-1}$, namely, $(1234, 1243)$, $(1243, 1324)$, $(1243, 1342)$, $(1243, 2143)$, $(1324, 1342)$, $(1342, 1423)$, $(1342, 1432)$, $(1342, 2341)$, $(1342, 3142)$, and $(2413, 3142)$.

(m) r_{n-1} is the number of permutations $w = w_1 w_2 \cdots w_n$ of $[n]$ with the following property: It is possible to insert the numbers $w_1, \ldots, w_n$ in order into a string, and to remove the numbers from the string in the order $1, 2, \ldots, n$. Each insertion must be at the beginning or end of the string. At any time we may remove the first (leftmost) element of the string. (*Example:* $w = 2413$. Insert 2, insert 4 at the right, insert 1 at the left, remove 1, remove 2, insert 3 at the left, remove 3, remove 4.)

(n) r_n is the number of sequences of length $2n$ from the alphabet A, B, C such that: (i) for every $1 \leq i < 2n$, the number of A's and B's among the first i terms is not

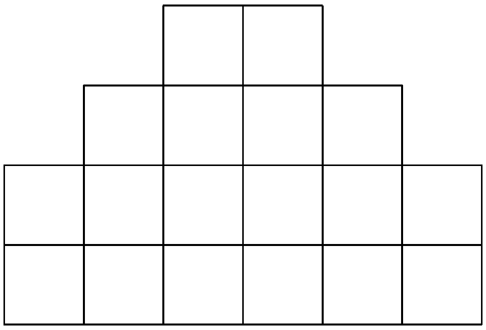


Figure 9: A board with $r_3 = 22$ domino tilings

less than the number of C's, (ii) the total number of A's and B's is n (and hence the also the total number of C's), and (iii) no two consecutive terms are of the form CB.

(o) r_{n-1} is the number of noncrossing partitions (as defined in Exercise 3.68) of some set $[k]$ into n blocks, such that no block contains two consecutive integers.

(p) s_n is the number of graphs G (without loops and multiple edges) on the vertex set $[n+2]$ with the following two properties: (α) All of the edges $\{1, n+2\}$ and $\{i, i+1\}$ are edges of G, and (β) G is *noncrossing*, i.e., there are not both edges $\{a, c\}$ and $\{b, d\}$ with $a < b < c < d$. Note that an arbitrary noncrossing graph on $[n+2]$ can be obtained from those satisfying (α)–(β) by deleting any subset of the required edges in (α). Hence the total number of noncrossing graphs on $[n+2]$ is $2^{n+2}s_n$.

(q) r_{n-1} is the number of reflexive and symmetric relations R on the set $[n]$ such that if iRj with $i < j$, then we never have uRv for $i \leq u < j < v$.

(r) r_{n-1} is the number of reflexive and symmetric relations R on the set $[n]$ such that if iRj with $i < j$, then we never have uRv for $i < u \leq j < v$.

(s) r_{n-1} is the number of ways to cover with disjoint dominos (or dimers) the set of squares consisting of $2i$ squares in the ith row for $1 \leq i \leq n-1$, and with $2(n-1)$ squares in the nth row, such that the row centers lie on a vertical line. See Figure 9 for the case $n = 4$.

40. [3–] Let a_n be the number of permutations $w = w_1 w_2 \cdots w_n \in \mathfrak{S}_n$ such that we never have $w_{i+1} = w_i \pm 1$, e.g., $a_4 = 2$, corresponding to 2413 and 3142. Equivalently, a_n is the number of ways to place n nonattacking kings on an $n \times n$ chessboard with one king in every row and column. Let

$$\begin{aligned} A(x) &= \sum_{n \geq 0} a_n x^n \\ &= 1 + x + 2x^4 + 14x^5 + 90x^6 + 646x^7 + 5242x^8 + \cdots. \end{aligned}$$

Show that $A(xR(x)) = \sum_{n \geq 0} n! x^n := E(x)$, where

$$R(x) = \sum_{n \geq 0} r_n x^n = \frac{1}{2x}\left(1 - x - \sqrt{1 - 6x + x^2}\right),$$

the generating function for Schröder numbers. Deduce that

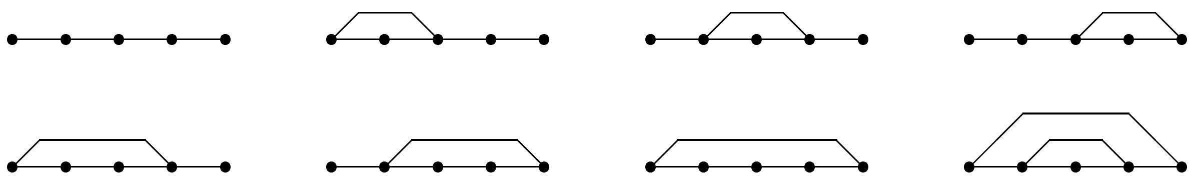
$$A(x) = E\left(\frac{x(1-x)}{1+x}\right).$$

41. [3] A permutation $w \in \mathfrak{S}_n$ is called *2-stack sortable* if $S^2(w) = w$, where S is the operator of Exercise 19(ii). Show that the number $S_2(n)$ of 2-stack sortable permutations in $\mathfrak{S}_n$ is given by

$$S_2(n) = \frac{2(3n)!}{(n+1)!\,(2n+1)!}.$$

42. [2] A king moves on the vertices of the infinite chessboard $\mathbb{Z} \times \mathbb{Z}$ by stepping from (i,j) to any of the eight surrounding vertices. Let $f(n)$ be the number of ways in which a king can walk from $(0,0)$ to $(n,0)$ in n steps. Find $F(x) = \sum_{n\geq 0} f(n)x^n$, and find a linear recurrence with polynomial coefficients satisfied by $f(n)$.

43. (a) [2+] A *secondary structure* is a graph (without loops or multiple edges) on the vertex set $[n]$ such that (a) $\{i, i+1\}$ is an edge for all $1 \leq i \leq n-1$, (b) for all i, there is at most one j such that $\{i,j\}$ is an edge and $|j-i| \neq 1$, and (c) if $\{i,j\}$ and $\{k,l\}$ are edges with $i < k < j$, then $i \leq l \leq j$. (Equivalently, a secondary structure may be regarded as a 3412-avoiding involution (as in Exercise 19(kk)) such that no orbit consists of two consecutive integers.) Let $s(n)$ be the number of secondary structures with n vertices. For instance, $s(5) = 8$, given by



Let $S(x) = \sum_{n\geq 0} s(n)x^n = 1 + x + x^2 + 2x^3 + 4x^4 + 8x^5 + 17x^6 + 37x^7 + 82x^8 + 185x^9 + 423x^{10} + \cdots$. Show that

$$S(x) = \frac{x^2 - x + 1 - \sqrt{1 - 2x - x^2 - 2x^3 + x^4}}{2x^2}.$$

(b) [3–] Show that $s(n)$ is the number of walks in n steps from $(0,0)$ to the x-axis, with steps $(1,0)$, $(0,1)$, and $(0,-1)$, never passing below the x-axis, such that $(0,1)$ is never followed directly by $(0,-1)$.

44. Define a *Catalan triangulation* of the Möbius band to be an abstract simplicial complex triangulating the Möbius band that uses no interior vertices, and has vertices labelled $1, 2, \ldots, n$ in order as one traverses the boundary. (If we replace the Möbius band by a disk, then we get the triangulations of Corollary 2.3(vi) or Exercise 19(a).) Figure 10 shows the smallest such triangulation, with five vertices (where we identify the vertical edges of the rectangle in opposite directions). Let MB(n) be the number of Catalan

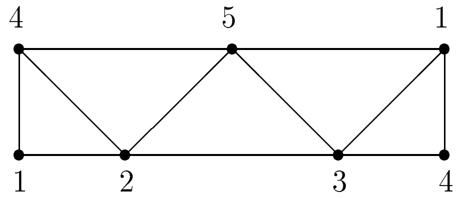
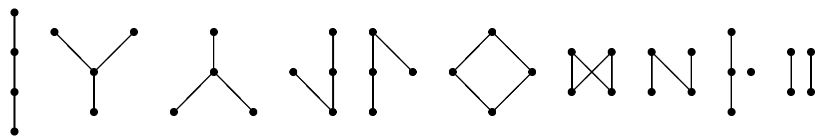


Figure 10: A Catalan triangulation of the Möbius band

triangulations of the Möbius band with n vertices. Show that

$$\begin{aligned}\sum_{n\geq 0}\mathrm{MB}(n)x^n &= \frac{x^2\left((2-5x-4x^2)+(-2+x+2x^2)\sqrt{1-4x}\right)}{(1-4x)\left(1-4x+2x^2+(1-2x)\sqrt{1-4x}\right)}\\ &= x^5+14x^6+113x^7+720x^8+4033x^9+20864x^{10}+\cdots.\end{aligned}$$

45. [3–] Let $f(n)$ be the number of nonisomorphic n-element posets with no 3-element antichain. For instance, $f(4)=10$, corresponding to



Let $F(x)=\sum_{n\geq 0} f(n)x^n = 1+x+2x^2+4x^3+10x^4+26x^5+75x^6+225x^7+711x^8+2311x^9+7725x^{10}+\cdots$. Show that

$$F(x)=\frac{4}{2-2x+\sqrt{1-4x}+\sqrt{1-4x^2}}.$$

46. (a) [3+] Let $f(n)$ denote the number of subsets S of $\mathbb{N}\times\mathbb{N}$ of cardinality n with the following property: If $p\in S$ then there is a lattice path from $(0,0)$ to p with steps $(0,1)$ and $(1,0)$, all of whose vertices lie in S. Show that

$$\begin{aligned}\sum_{n\geq 1} f(n)x^n &= \frac{1}{2}\left(\sqrt{\frac{1+x}{1-3x}}-1\right)\\ &= x+2x^2+5x^3+13x^4+35x^5+96x^6+267x^7\\ &\quad +750x^8+2123x^9+6046x^{10}+\cdots.\end{aligned}$$

(b) [3+] Show that the number of such subsets contained in the first octant $0\leq x\leq y$ is the Motzkin number M_{n-1} (defined in Exercise 37).

47. (a) [3] Let P_n be the Bruhat order on the symmetric group $\mathfrak{S}_n$ as defined in Exercise 3.75(a). Show that the following two conditions on a permutation $w\in\mathfrak{S}_n$ are equivalent:

i. The interval $[\hat{0},w]$ of P_n is rank-symmetric, i.e., if ρ is the rank function of P_n (so $\rho(w)$ is the number of inversions of w), then

$$\#\{u\in[\hat{0},w]:\rho(u)=i\}=\#\{u\in[\hat{0},w]:\rho(w)-\rho(u)=i\},$$

for all $0\leq i\leq\rho(w)$.

ii. The permutation $w = w_1w_2\cdots w_n$ is 4231 and 3412-avoiding, i.e., there do not exist $a < b < c < d$ such that $w_d < w_b < w_c < w_a$ or $w_c < w_d < w_a < w_b$.

(b) [3–] Call a permutation $w \in \mathfrak{S}_n$ *smooth* if it satisfies (i) (or (ii)) above. Let $f(n)$ be the number of smooth $w \in \mathfrak{S}_n$. Show that

$$\begin{aligned}\sum_{n\geq 0} f(n)x^n &= \frac{1}{1-x-\frac{x^2}{1-x}\left(\frac{2x}{1+x-(1-x)C(x)}-1\right)} \\ &= 1+x+2x^2+6x^3+22x^4+88x^5+366x^6 \\ &\quad +1552x^7+6652x^8+28696x^9+\cdots,\end{aligned}$$

where $C(x) = (1-\sqrt{1-4x})/2x$ is the generating function for the Catalan numbers.

48. [3] Let $f(n)$ be the number of 1342-avoiding permutations $w = w_1w_2\cdots w_n$ in $\mathfrak{S}_n$, i.e., there do not exist $a < b < c < d$ such that $w_a < w_d < w_b < w_c$. Show that

$$\begin{aligned}\sum_{n\geq 0} f(n)x^n &= \frac{32x}{1+20x-8x^2-(1-8x)^{3/2}} \\ &= 1+x+2x^2+6x^3+23x^4+103x^5+512x^6+2740x^7+15485x^8+\cdots.\end{aligned}$$

49. (a) [3–] Let B_n denote the board consisting of the following number of squares in each row (read top to bottom), with the centers of the rows lying on a vertical line: 2, 4, 6, …, $2(n-1)$, $2n$ (three times), $2(n-1)$, …, 6, 4, 2. Figure 11 shows the board B_3. Let $f(n)$ be the number of ways to cover B_n with disjoint dominos (or dimers). (A domino consists of two squares with an edge in common.) Show that $f(n)$ is equal to the central Delannoy number $D(n,n)$ (as defined in Section 3).

(b) [3–] What happens when there are only two rows of length $2n$?

50. [3] Let B denote the "chessboard" $\mathbb{N}\times\mathbb{N}$. A *position* consists of a finite subset S of B, whose elements we regard as pebbles. A *move* consists of replacing some pebble, say at cell (i,j), with two pebbles at cells $(i+1,j)$ and $(i,j+1)$, provided that each of these cells is not already occupied. A position S is *reachable* if there is some sequence of moves, beginning with a single pebble at the cell $(0,0)$, that terminates in the position S. A subset T of B is *unavoidable* if every reachable set intersects T. A subset T of B is *minimally unavoidable* if T is unavoidable, but no proper subset of T is unavoidable. Let $u(n)$ be the number of n-element minimally unavoidable subsets of B. Show that

$$\begin{aligned}\sum_{n\geq 0} u(n)x^n &= x^3\frac{(1-3x+x^2)\sqrt{1-4x}-1+5x-x^2-6x^3}{1-7x+14x^2-9x^3} \\ &= 4x^5+22x^6+98x^7+412x^8+1700x^9+6974x^{10}+28576x^{11}+\cdots.\end{aligned}$$

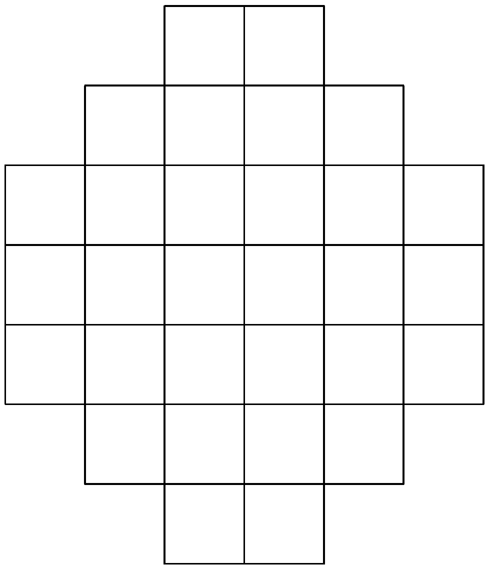


Figure 11: A board with $D(3,3) = 63$ domino tilings