Selections and Binomial Coefficients

Permutations

Let X be any fixed set of size n e.g. $X = [n] = \{1, 2, \ldots, n\}$.
A *permutation* of X is a sequence of length n in which each element of X appears exactly once.
$p(n)$ denotes the number of permutations of X.
E.g. $n = 3$, $X = \{a, b, c\}$. $p(3) = 6$.

$$abc, acb, bac, bca, cab, cba,$$

$$\begin{aligned} p(1) &= 1 \\ p(n) &= np(n-1), \qquad n \geq 2 \end{aligned}$$

n choices for the first element x_1. For *each* choice of x_1 there are $p(n-1)$ ways of completing the sequence.

Thus

$$p(n) = n!.$$

Ordered selection without repetition

$p(n, r)$ is the number of sequences of length r in which each element of X appears at most once.
E.g. $n = 4$, $r = 2$, $X = \{a, b, c, d\}$, $p(4, 2) = 12$.

$$ab, ac, ad, ba, bc, bd, ca, cb, cd, da, db, dc.$$

$$\begin{aligned} p(n, 0) &= 1 \\ p(n, r) &= np(n-1, r-1), \qquad r \geq 1 \end{aligned}$$

n choices for the first element x_1. For *each* choice of x_1 there are $p(n-1, r-1)$ ways of completing the sequence.

$$\begin{aligned} p(n, r) &= n(n-1)\cdots(n-r+1) \\ &= \frac{n!}{(n-r)!} \end{aligned}$$

Ordered selection with repetition

$q(n,r)$ denotes the number of sequences of length r with elements from X.

E.g. $n = 3, r = 2$, $X = \{a, b, c\}$, $q(3,2) = 9$.

$$aa, ab, ac, ba, bb, bc, ca, cb, cc.$$

$$\begin{aligned} q(n,0) &= 1 \\ q(n,r) &= nq(n,r-1), \qquad r \geq 1 \end{aligned}$$

So

$$q(n,r) = n^r.$$

Unordered selection without repetition

What if the order of selection is immaterial?

$c(n, r)$ is the number of ways of choosing a set of r elements from $[n]$.

$$p(n, r) = r!c(n, r).$$

Each of the $c(n, r)$ unordered choices of a set can be ordered in $r!$ ways to make a sequence of length r from $[n]$.

$$c(n, r) = \binom{n}{r} = \frac{n!}{r!(n-r)!}$$

Here $n! = 0$, $n < 0$ and $0! = 1$

Unordered selection with repetition

$f(n,r)$ is the number of ways of choosing r elements from a set of size n where order does not matter and repetitions are allowed.

$$f(n,r) = \binom{n+r-1}{r}.$$

What if each element must be chosen at least once? If $n = 3, k = 4$:

$$\begin{array}{c} 2R + 1B + 1W \\ 1R + 2B + 1W \\ 1R + 1B + 2W \end{array}$$

Let

$$
\begin{aligned}
X &= \{(x \in \{1, 2, \ldots\}^n : x_1 + \cdots + x_n = r\} \\
X' &= \{x' \in \{0, 1, 2, \ldots\}^n : x'_1 + \cdots + x'_n = r - n\}.
\end{aligned}
$$

We claim that $|X| = |X'| = \binom{r-1}{n-1}$.

Consider $f : X \to X'$ where
$f(x_1, x_2, \ldots, x_n) = (x_1 - 1, x_2 - 1, \ldots, x_n - 1)$.
f is a bijection with inverse g
$g(x'_1, x'_2, \ldots, x'_n) = (x'_1 + 1, x'_2 + 1, \ldots, x'_n + 1)$.
This implies

$$|X| = |X'| = \binom{n + (r-n) - 1}{n - 1} = \binom{r - 1}{n - 1}$$

Summary of Selection

S is a set of n distinct objects and we must choose r objects:

Order matters	No repetion allowed:	$P(n,r)$
Order matters	Repetion allowed:	n^r
Order does not matter	No repetion allowed:	$\binom{n}{r}$
Order does not matter	Repetion allowed:	$\binom{n+r-1}{r}$