Discrete Mathematics 21-228: Summer II 2000
Homework 2
Due Monday, 7/17

July 16, 2000

You may discuss the problems in groups of at most 3 people, but each person must write her own homework. Please list the members of your group. You may not use any sources other than the notes and texts listed on the course page.

1. (i) Let the points $1, 2, \ldots, 2n$ be equally spaced on a circle. Let h_n denote the number ways to join these points in n pairs so the resulting line segments do not intersect. Establish a recurrence relation for h_n.

   [Hint: In each configuration, the point 1 must be paired with another point. Count the number of solutions in which 1 is paired with i for appropriate values of i, and sum all of these together.]

   (ii) Show that number of solutions above is equal to the number of montone paths from $(0, 0)$ to (a, a) on an a by a grid which do not contain points (m, n) such that $m > n$ (paths that may touch but do not dip below the line $y = x$). Deduce that $h_n = \frac{1}{a+1}\binom{2a}{a}$ for an appropriate value of a.

   [Note: You may either establish a bijection or show that both satisfy "similar" recurrence relations.]

2. (i) Formulate and solve a recurrence for counting a_n, the number of binary strings of length n with no consecutive 1's. For $n = 3$ we have 000, 001, 010, 100, and 101.

   (ii) Prove the following neat equality:

$$\sum_{k=0}^{\lceil \frac{n}{2} \rceil} \binom{n-k+1}{k} = \frac{\phi^{n+2} - \left(\frac{-1}{\phi}\right)^{n+2}}{\sqrt{5}},$$

   where $\phi = \frac{1+\sqrt{5}}{2} \approx 1.618\ldots$ is the golden ration.

   [Hint: Use (i) and problem 2 from hw1. How does a_n relate to the RHS above?]

3. Our job is to count the number of ways of tiling a 2xn grid. We have at our disposal 1x1, 2x1, and 2x2 sized tiles. Formulate and solve a recurrence (don't forget initial conditions) for each of the following cases (assume $a_0 = 1$):

   (i) using only 2x1 tiles

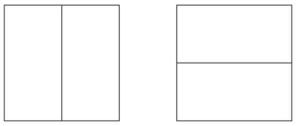
(ii) using 2x1 and 2x2 tiles

Extra Credit: (iii) using 1x1, 2x1, and 2x2 tiles

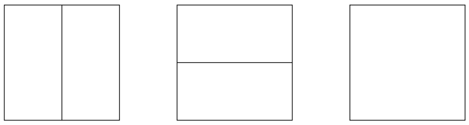
[Note: You should end up with second order linear recurrences for each case. You may use Mathematica or Maple if the algebra gets messy.]

For n=2:

Only 2x1 tiles:



2x1 and 2x2 tiles:



1x1, 2x1, and 2x2 tiles: