DYNAMICS OF AN INTERIOR SPIKE IN THE GIERER-MEINHARDT SYSTEM

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Abstract. We study the dynamics of an interior spike of the Gierer-Meinhardt system. Under certain assumptions on the domain size, the diffusion coefficients, and the decay rates, we prove that the velocity of the center of the spike is proportional to the negative gradient of \(R(\xi,\xi)\), where \(R(x,\xi)\) is the regular part of the Green's function of the Laplacian with the Neumann boundary condition. Hence, an interior spike moves towards local minima of \(R(\xi,\xi)\) and therefore stays as an interior spike forever. This dynamics is fundamentally different from that of the shadow Gierer-Meinhardt system where an interior spike moves towards the closest point on the boundary.

1. Introduction

1.1. The Gierer-Meinhardt system. We consider the Gierer-Meinhardt system, for \(A = A(x,t)\) and \(H = H(x,t)\),

\[
\begin{align*}
A_t &= D_A \Delta A - k_A A + l_A A^2/H, \\
H_t &= D_H \Delta H - k_H H + l_H A^2, \\
\partial_n A &= 0 = \partial_n H, \\
A(x,0) &= A_0(x) > 0, \quad H(x,0) = H_0(x) > 0, \\
x &\in \delta \Omega = \{z | z \in \Omega\}, \quad t > 0,
\end{align*}
\]

Here \(\Omega \subset \mathbb{R}^N\), \(N = 2, 3\), is a bounded domain with \(C^3\) boundary and unit volume, \(\Delta\) is the Laplace operator, \(\partial_n\) is the exterior normal derivative, and \(\delta\) is the size of the physical domain. System (1.1) was proposed in [6] (see also [11]) as a model for biochemical reactions of activator-inhibitor type in which a short-range substance, the activator \(A\), promotes its own production as well as that of rapidly diffusing antagonist, the inhibitor \(H\).

In this paper, we assume that \(D_A, D_H, k_A,\) and \(k_H\), representing the diffusion coefficients and the decaying rates of species \(A\) and \(H\), are positive constants and satisfy

\[
\frac{k_A}{k_H} \ll 1, \quad \frac{D_A}{k_A} \ll \delta^2 \ll \frac{D_H}{k_H}.
\]

These conditions reflect the following scenario: (i) The half-life \((\ln 2/k_A)\) of the activator \(A\) is much longer than that of the inhibitor \(H\); (ii) With respect to the size \(\delta\) of the domain and the half-life of the species, the diffusion rate \(D_A/(k_A \delta^2)\) of \(A\) is small whereas that of \(H\) is large; namely, regional population differences of \(A\) are not easily evened out in the life time of the species \(A\), whereas regional population differences of \(H\) are almost instantaneously evened out by the diffusion. In such a scenario, a local increase in the concentration of the activator will be further amplified (due to the \(l_A A^2/H\) term), forming regions with high concentration of the activator surrounded by the “sea” of essentially uniformly distributed inhibitor. We speak of spikes if

\[
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\]

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the activator concentrates near a single point or a set of isolated points. Since in general spikes concentrated at more than one point are unstable, in this article we only study the dynamics of single spike solutions.

We introduce dimensionless constants

\[ \tau = \frac{k_A}{k_H}, \quad \varepsilon^2 = \frac{D_A}{\frac{k_A}{\bar{d}^2}}, \quad D = \frac{D_H}{k_H \bar{d}^2}, \]

and rescale the independent and dependent variables via

\[ t \mapsto \frac{t}{k_A}, \quad x \mapsto \delta x, \quad A \mapsto \frac{k_H}{l_\varepsilon} A, \quad H \mapsto \frac{k_A l_B}{k_H l_A} \varepsilon^N H. \]

Then the Gierer-Meinhardt system (1.1) takes the non-dimensional form

\[
\begin{align*}
A_t &= \varepsilon^2 \Delta A - A + f(A, H), & x \in \Omega, t > 0, \\
\tau H_t &= D \Delta H - H + \varepsilon^{-N} g(A), & x \in \Omega, t > 0, \\
\partial_n A &= 0 = \partial_n H, & x \in \partial \Omega, t > 0, \\
A(x, 0) &= A_0(x) > 0, \quad H(x, 0) = H_0(x) > 0, & x \in \Omega,
\end{align*}
\]

where

\[ f(A, H) = A^2 H^{-1}, \quad g(A) = A^2. \]

Formally, as \( D \to \infty \) one obtains the following shadow Gierer-Meinhardt system, for \( A = A(x, t) \) and \( H = H(t) \),

\[
\begin{align*}
A_t &= \varepsilon^2 \Delta A - A + f(A, H), & x \in \Omega, t > 0, \\
\tau H_t &= -H + \varepsilon^{-N} \int_{\Omega} g(A), & x \in \Omega, t > 0, \\
\partial_n A &= 0 = \partial_n H, & x \in \partial \Omega, t > 0, \\
A(x, 0) &= A_0(x) > 0, \quad H(0) = \int_{\Omega} H_0(x) dx > 0, & x \in \Omega.
\end{align*}
\]

Note that steady states to (1.4), after the change of variables \( y = x/\varepsilon \), are solutions to

\[ \Delta y A - A + f(A, \int_{\Omega_y} A dy) = 0, \quad y \in \Omega_\varepsilon := \varepsilon^{-1} \Omega. \]

In recent years there has been much interest in studying (1.2), (1.4) and especially the associated steady state problem (1.5).

In a series of papers \([13, 14, 15]\), Ni and Takagi (also with Lin [10]) established the existence of stationary spikes (solutions to (1.5) with homogeneous Neumann boundary condition) concentrating at points of maximal mean curvature of \( \partial \Omega \). We refer the readers to the recent review article by Ni [12] and the references therein for more details on this subject.

In \([3]\) we studied the evolution of single-spike solutions to (1.4) and showed that a single, interior spike located at

\[ \xi = \xi(t) \in \Omega \] moves toward the boundary \( \partial \Omega \) with velocity

\[ \dot{\xi} := \frac{d}{dt} \xi(t) \propto \nabla \xi e^{-2d(\xi)/\varepsilon} \]

where \( d(\xi) \) is the distance from \( \xi \) to \( \partial \Omega \). From this formula, one sees that single interior spikes for the shadow Gierer-Meinhardt system move towards their closest points on \( \partial \Omega \), possibly except those which have more than one closest points on the boundary. We would like to point out that the dynamics (1.6) for the shadow Gierer-Meinhardt system (1.4) was first derived by Iron and Ward in \([7]\), whereas in \([3]\) we provided a rigorous proof (see also related work \([17]\)).
In the present paper we assume that $D$ is sufficiently large, but $D \ll e^{1/\varepsilon}$. In such a case one does not expect the spike to move exponentially slowly. In fact we show that if $\varepsilon^{3-2N-\kappa} \ll D \ll e^{1/\varepsilon}$ for some $\kappa > 0$, then an interior spike moves with a velocity

$$
(1.7) \quad \dot{\xi} \propto -\varepsilon^2 D^{-1} D_{\xi} R(\xi, \xi)
$$

where $D_{\xi}$ is the total derivative with respect to $\xi$, and $R(x, \xi)$ is the regular part of the Green’s function for $\Delta$ with the Neumann boundary condition. Since $R(\xi, \xi) \to \infty$ as $x, \xi \to \partial \Omega$, one sees from the formula (1.7) that an interior spike moves towards local minima of $R(\xi, \xi)$ and hence stays in $\Omega$ forever.

Clearly, the dynamics (1.7) for the Gierer-Meinhardt system (1.2) is totally different from the dynamics (1.6) for the shadow Gierer-Meinhardt system (1.4).

The main purpose of this paper is to prove rigorously the asymptotic formula (1.7), following the so called invariant manifold approach developed by Alikakos and Fusco in [1, 2] to study motions of circular fronts (bubbles) in solutions to the Cahn-Hilliard equation.

1.2. Statement of the main result. In this paper we shall use the following notation:

$$
\langle \phi \rangle := \int_{\Omega} \phi(x) dx, \quad \langle \phi, \psi \rangle := \langle \phi \psi \rangle = \int \phi \psi, \quad \|u\|_{p} := \|u\|_{L^{p}(\Omega)}, \quad \Omega_{\varepsilon} := \varepsilon^{-1} \Omega.
$$

We assume that $\tau \ll 1$ and $D \gg 1$. Then we can argue from (1.2) that $H(\cdot, t)$ is almost a constant equal to $\varepsilon^{-N} \langle g(A) \rangle = \varepsilon^{-N} \langle g(A(\cdot, t)) \rangle$ (recall that the volume of $\Omega$ is 1). Hence it is convenient to decompose $H$ as

$$
H(x, t) = \varepsilon^{-N} \langle g(A) \rangle + h(x, t),
$$

$$
h(x, t) = h_{0}(t) + h_{1}(x, t),
$$

$$
h_{0}(t) := \langle H \rangle - \varepsilon^{-N} \langle g(A) \rangle,
$$

$$
h_{1}(x, t) := H(x, t) - \varepsilon^{-N} \langle g(A) \rangle - h_{0}(t) = H - \langle H \rangle.
$$

Then (1.2) can be written in terms of unknowns $A$, $h_{0}$, and $h_{1}$,

$$
(1.8) \quad \begin{cases}
A_{t} - \varepsilon^{2} \Delta A + A = f(A, \varepsilon^{-N} \langle g(A) \rangle + h_{0} + h_{1}), & x \in \Omega, t > 0, \\
\tau h_{0, t} + h_{0} = -\tau \varepsilon^{-N} \langle g'(A) A_{t} \rangle, & t > 0, \\
\tau h_{1, t} - D \Delta h_{1} + h_{1} = \varepsilon^{-N} [g(A) - \langle g(A) \rangle], & x \in \Omega, t > 0, \\
\partial_{n} A = 0 = \partial_{n} h_{1}, & x \in \partial \Omega, t > 0.
\end{cases}
$$

If we ignore $h_{0}$ and $h_{1}$ and use the stretched variable $y = x / \varepsilon$, (1.8a) becomes

$$
A_{t} - \Delta_{y} A + A = f(A, \int_{\Omega_{\varepsilon}} g(A) \, dy), \quad y \in \Omega_{\varepsilon}, t > 0,
$$

which is the limit, as $\tau \to 0$, of the shadow Gierer-Meinhardt system (1.4). Since $\varepsilon \ll 1$, a solution to the equation

$$
(1.9) \quad -\Delta_{y} A + A = f(A, \int_{R^{N}} g(A) \, dy) \quad \text{in } R^{N}
$$

will be almost a stationary solution.

With $f$ and $g$ given by (1.3), it is known that (1.9) has a unique, positive, radially symmetric solution, which we denote by $W(r, r = |y|)$. As $W(r) \to 0$ exponentially fast as $r \to \infty$, for every $\xi \in \Omega$, \{ $A = W(|x - \xi| / \varepsilon), h_{0} = 0, h_{1} = 0$ \} is almost an equilibrium of (1.8). In the sequel, a solution with $A(x, t) \approx W(|x - \xi(t)| / \varepsilon), h_{0} \approx 0$ and $h_{1} \approx 0$ will be called a spike solution located at $\xi(t)$ at time $t$. 

We consider only spikes that initially stay away from the boundary. To this end let \( \mu \) be a parameter in the range \( \frac{1}{2} \) diameter(\( \Omega \)) > \( \mu > 4 \varepsilon \log(D \varepsilon^{-2}) \). We define
\[
d(\xi) = \text{distance from } \xi \text{ to } \partial \Omega, \quad \Omega^\mu = \{ \xi \in \Omega \mid d(\xi) > \mu \}.
\]

It is convenient to work with approximate solutions to (1.8) which have compact support. Hence, we modify \( W(|y|) \) and \( W(|x - \xi|/\varepsilon) \) into compactly supported functions \( W^\varepsilon(y) \) and \( w^\varepsilon(x, \xi) \) as follows. Let \( \zeta(s) \) be a cut-off function such that \( \zeta = 1 \) if \( |s| < 1/2 \), \( \zeta = 0 \) if \( |s| > 1 \) and \( |D^n \zeta| \leq 2^{n+1}, n = 1, 2, 3 \). We define
\[
W^\varepsilon(y) = W(|y|) \zeta(|y|/\varepsilon/\mu), \quad y \in \mathbb{R}^N,
\]
\[
w^\varepsilon(x, \xi) = W^\varepsilon(|x - \xi|/\varepsilon) \zeta(|x - \xi|/\varepsilon/\mu), \quad x, \xi \in \Omega.
\]

We define the approximate invariant manifold \( \mathcal{M} \) by
\[
\mathcal{M} = \{ w^\varepsilon(\cdot, \xi) \mid \xi \in \bar{\Omega} \}.
\]

It is known (see Lemma 3.1 to follow) that there exists a positive constant \( c_0 > 0 \) (depending only on \( \Omega \)) such that if \( \text{dist}(A(\cdot, t), \mathcal{M}) \leq c_0 \varepsilon^{N/2} \), then we can uniquely decompose \( A(\cdot, t) \) as
\[
A(x, t) = w^\varepsilon(x, \xi(t)) + \phi(x, t), \quad \xi(t) \in \bar{\Omega}, \quad \| \phi(\cdot, t) \|_2 = \text{dist}(A(\cdot, t), \mathcal{M}) := \inf_{w \in \mathcal{M}} \| A - w \|_2.
\]

We define
\[
T^* := \sup \{ T > 0 \mid \text{dist}(A(\cdot, t), \mathcal{M}) \leq c_0 \varepsilon^{N/2}, \xi(t) \in \Omega^\mu \ \forall t \in [0, T] \}.
\]

**Theorem 1.1. (Quasi-Invariance of the Manifold \( \mathcal{M} \))** Let \( \kappa \in (0, 1/8) \) be any fixed constant. Let \( \varepsilon, \tau, D, \) and \( \mu \) be positive parameters such that there hold the relations
\[
0 < \varepsilon < 1, \quad 0 < \tau < 1, \quad \varepsilon^{2 - N - 2 \kappa} < D, \quad 4 \varepsilon \log(D \varepsilon^{-2}) < \mu < \frac{1}{2} \text{ diameter}(\Omega).
\]

Let \( (A, h_0, h_1) \) be solutions to (1.8) with initial values \( h_0(0), h_1(\cdot, 0) \) and \( A(\cdot, 0) = w^\varepsilon(\cdot, \xi_0) + \phi(\cdot, 0) \), where \( \langle h_1(\cdot, 0) \rangle = 0, \xi_0 \in \Omega^\mu \), and \( \| \phi(\cdot, 0) \|_2 = \text{dist}(A(\cdot, 0), \mathcal{M}) \). Assume that
\[
|h_0(0)| + \| h_1(\cdot, 0) \|_\infty + \varepsilon^{-N/2} \| \phi(\cdot, 0) \|_2 \leq D^{-1} \varepsilon^{2 - N - \kappa}.
\]

Decompose \( A \) as in (1.11) in the interval \([0, T^*]\) with \( T^* \) defined as in (1.12).

There exist small positive constants \( \varepsilon_0 \) and \( \tau_0 \) and a positive constant \( C_0 \), all of which depend only on \( \Omega \) and \( \kappa \), such that if \( \varepsilon \in (0, \varepsilon_0) \) and \( \tau \in (0, \tau_0) \), then
\[
|h_0(t)| + \| h_1(\cdot, t) \|_\infty + \varepsilon^{-N/2} \| \phi(\cdot, t) \|_2 \leq C_0 D^{-1} \varepsilon^{2 - N - \kappa} < c_0, \quad \forall t \in (0, T^*).
\]

In addition, either \( T^* = \infty \) or \( d(\xi(T^*)) = \mu \) (i.e., \( \xi(T^*) \in \partial \Omega^\mu \)).

To describe the dynamics of the spike (and therefore show that \( T^* = \infty \)) we introduce the Green’s function \( G(x, \xi) \) of \( \Delta \) with the Neumann boundary condition; i.e., for each \( \xi \in \Omega \), \( G(\cdot, \xi) \) solves
\[
\left\{
\begin{array}{ll}
-\Delta G(x, \xi) = \delta(x - \xi) - 1 & \text{in } \Omega, \\
\frac{\partial G}{\partial \nu} = 0 & \text{on } \partial \Omega, \\
\int_{\Omega} G(x, \xi) \, dx = 0.
\end{array}
\right.
\]

Let \( \Gamma(x) = -(2\pi)^{-1} \log |x| \) for \( N = 2 \) and \( -(4\pi |x|)^{-1} \) for \( N = 3 \) be the fundamental solution of \( \Delta \) and let \( R(x, \xi) := G(x, \xi) - \Gamma(x - \xi) \) be the regular part of the Green’s function.

**Theorem 1.2. (Dynamics of An Interior Spike)** In addition to the assumptions of the previous Theorem we assume that
\[
\| h_1(\cdot, 0) - (D \Delta)^{-1} [w^\varepsilon(\cdot, \xi_0)]^2 - \langle w^\varepsilon(\cdot, \xi_0)^2 \rangle \|_\infty \leq D^{-2} \varepsilon^{4 - 2N - 2\kappa}.
\]
The following formula holds true
\begin{equation}
\dot{\xi} = \alpha_0 D^{-1} \epsilon^2 \left( - D_\xi R(\xi, \xi) + O(\epsilon) d(\xi)^{-N} + O(\epsilon^{3-2N-2\kappa} D^{-1}) \right), \quad \forall t \in (0,T^*)
\end{equation}
where $\alpha_0$, given in (2.4) below, is a positive constant depending only on $N$.

Consequently if we further assume that $D \geq \epsilon^{2-N-3\kappa}$ and $\mu$ is sufficiently small, then $T^* = \infty$ and $\xi(t) \in \Omega^\mu$ for all $t > 0$.

**Remark 1.1.** Condition (1.13) implicitly imposes an upper bound on $D$:
\begin{equation}
D < \epsilon^2 e^{\mu/(4\epsilon)} < \epsilon^2 \exp(\text{diameter}([\Omega]/(8\epsilon))].
\end{equation}

If $D$ is too large, say, $\log(D) > 1/\epsilon$, then (1.2) should be considered as a small perturbation of the shadow Gierer-Meinhardt system, and therefore the dynamics (1.6) should prevail.

Intuitively, for any given $\xi \in \Omega$, making the magnitudes of the right-hand sides of (1.6) and (1.7) equal should give us the critical size of $D$ to determined which

dynamics dominates. We believe that when $D$ is exponentially large, i.e., $\log(D) = O(1/\epsilon)$, our analysis in [3] and the analysis presented in this paper can be combined to obtain the leading order expansion of the velocity of motion of single interior spikes, which somehow should be the sum of the right-hand sides of (1.7) and (1.6).

**Remark 1.2.** Our lower bound $\epsilon^{2-N-2\kappa}$ for the magnitude of $D$ for the quasi-invariance of the manifold $\mathcal{M}$ in Theorem 1.1 is possibly sharp. Indeed, it is proved in [18] that when $N = 2$ and $D = 1$, the stationary spike attains a maximum of the order $O(|\ln \epsilon|)$ as $\epsilon \to 0$, whereas in our case the spikes remain bounded by a constant independent on $\epsilon$.

**Remark 1.3.** One notices that taking smaller $\kappa$ in our theorems makes the results stronger. Nevertheless, we cannot take $\kappa = 0$. We expect terms involving $\ln \epsilon$ will come up if we set $\kappa = 0$.

In (1.15), the term $O(\epsilon^{3-2N-2\kappa} D^{-1})$ does not match with the combination $\mathcal{E}_\kappa := \epsilon^{2-N-\kappa} D^{-1}$. We believe the actual size of this term should be $O(\mathcal{E}_\kappa)$. To prove this, one needs an approximation better than approximating $H$ by a constant function.

**Remark 1.4.** We observe that since $w^\epsilon$ is bounded by a constant independent on $\epsilon$ therefore the assumptions on $h_1(\cdot, 0)$ in Theorem 1.1 and Theorem 1.2 can be satisfied simultaneously.

**Remark 1.5.** When $N = 2$ and $\Omega$ is a disk of radius $1/\sqrt{\pi}$ (so area of $\Omega$ is 1), we have an explicit formula for $R(x, \xi)$. Indeed, identifying points as complex numbers, the Green’s function is given by
\begin{equation}
G(z, \xi) = -\frac{1}{2\pi} \ln |z - \xi| + \ln |\xi - 1/\pi| + \frac{1}{4} (|z|^2 + |\xi|^2) + K_0, \quad K_0 = \frac{3}{4\pi} \ln \pi + \frac{5}{16\pi}.
\end{equation}
It then follows that
\begin{equation}
R(\xi, \xi) = -\frac{1}{2\pi} \ln |\xi|^2 - 1/\pi| + \frac{1}{2} |\xi|^2 + K_0, \quad D_\xi R(\xi, \xi) = \frac{2 - \pi |\xi|^2}{1 - \pi |\xi|^2}, \quad \forall \xi \in \Omega.
\end{equation}
Hence, a spike will move towards the origin in the radial direction.

For more explicit formulae of the regular part $R(x, \xi)$ of the Green’s function of certain other domains, see Fraenkel [5].

Later, in Lemma 3.5, we shall show that for any smooth domain $\Omega$, $|D_\xi R(\xi, \xi)| \propto d(\xi)^{1-N}$ as $\xi \to \partial \Omega$, so that $D_\xi R(\xi, \xi)$ is the leading order term in (1.15).
In the next section we shall formally derive the dynamics of $\xi(t)$. Then in the subsequent sections, we verify the dynamics rigorously.

In what follows, we shall always assume that $\varepsilon$ and $\tau$ are small positive constants and that $D, \mu$ satisfy (1.13).

2. Formal derivation of the dynamics

To explain better our idea of the proof, here we first provide a formal derivation of the dynamics (1.15). Let $W(r)$ be the solution to (1.9) and $W^\varepsilon, w^\varepsilon$ be functions defined in (1.10). We define

$$
\sigma^\varepsilon = \int_{\mathbb{R}^N} g(W^\varepsilon(|y|)) \, dy = \varepsilon^{-N} \int_{|x| < |\xi|} g(w^\varepsilon(x, \xi)) \, dx,
$$

$$
r^\varepsilon = r^\varepsilon(x, \xi) = \varepsilon^2 \Delta w^\varepsilon - w^\varepsilon + f(w^\varepsilon, \sigma^\varepsilon) = \Delta_y W^\varepsilon - W^\varepsilon + f(W^\varepsilon, \sigma^\varepsilon)|_{y=(x-\xi)/\varepsilon}.
$$

Since $W(y)$ decays to zero exponentially fast as $|y| \to \infty$, we can regard $r^\varepsilon$ as zero. (On the contrary, for the shadow Gierer-Meinhardt system, $r^\varepsilon$ is the main term forcing a spike to move towards the boundary.)

If we decompose $A(x, t)$ as in (1.11) then equation (1.8a) can be written as

$$
\sum_{j=1}^N w_{\xi_j}^\varepsilon \dot{\xi}_j + \phi_t = L^\varepsilon \phi + f_H(w^\varepsilon, \sigma^\varepsilon)(h_0 + h_1) + r^\varepsilon + N
$$

where

$$
L^\varepsilon \phi = \varepsilon^2 \Delta \phi - \phi + f_A(w^\varepsilon, \sigma^\varepsilon)\phi + f_H(w^\varepsilon, \sigma^\varepsilon)\varepsilon^{-N} \langle g'(w^\varepsilon) \phi \rangle,
$$

$$
N = f(w^\varepsilon + \phi, \varepsilon^{-N} (g(w^\varepsilon + \phi)) + h_0 + h_1) - f(w^\varepsilon, \sigma^\varepsilon) - f_A(w^\varepsilon, \sigma^\varepsilon)\phi - f_H(w^\varepsilon, \sigma^\varepsilon) \varepsilon^{-N} \langle g'(w^\varepsilon \phi) \rangle + h_0 + h_1.
$$

Multiplying (2.2) by $w_{\xi_i}^\varepsilon$, and using

$$
\int_\Omega w_{\xi_i}^\varepsilon w_{\xi_j}^\varepsilon = \delta_{ij} |w_{\xi_i}^\varepsilon|^2, \quad \int_\Omega \phi w_{\xi_i}^\varepsilon = \int_\Omega x w_{\xi_i}^\varepsilon = - \sum_{j=1}^N \int_\Omega \phi w_{\xi_j}^\varepsilon \dot{\xi}_j
$$

(since $\phi \perp TM$, $\langle \phi, w_{\xi_i} \rangle = 0$ for all $i$), we obtain

$$
\langle |w_{\xi_i}^\varepsilon|^2 \rangle \dot{\xi}_i - \sum_{j=1}^N \langle w_{\xi_i}^\varepsilon, \phi \rangle \dot{\xi}_j = \int_\Omega w_{\xi_i}^\varepsilon L^\varepsilon \phi + \int_\Omega f_H(w^\varepsilon, \sigma^\varepsilon)(h_0 + h_1) w_{\xi_i}^\varepsilon + \int_\Omega N w_{\xi_i}^\varepsilon.
$$

Since $w_{\xi_i}$ is almost in the kernel of $L^\varepsilon$ and therefore also its adjoint $L^\varepsilon^*$, we can ignore terms involving $\phi$ and $N$ to obtain

$$
\dot{\xi}_i \langle |w_{\xi_i}^\varepsilon|^2 \rangle \approx \int_\Omega f_H(w^\varepsilon, \sigma^\varepsilon)(h_0 + h_1) w_{\xi_i}^\varepsilon = \int_\Omega (h_0 + h_1) |Q(w^\varepsilon, \sigma^\varepsilon)|_{\xi_i},
$$

where

$$
Q(w, \sigma) = \int_0^w f_H(s, \sigma) \, ds = - \frac{w^3}{3\sigma^2}, \quad \forall w, \sigma \in (0, \infty).
$$

Since $Q(w^\varepsilon, \sigma^\varepsilon)_{\xi_i} = -Q(w^\varepsilon, \sigma^\varepsilon)_{\xi_i}$, and $Q \equiv 0$

on $\partial \Omega$, we have

$$
\dot{\xi}_i \langle |w_{\xi_i}^\varepsilon|^2 \rangle \approx \int_\Omega Q(w^\varepsilon, \sigma^\varepsilon)(h_0 + h_1)_{\xi_i} = \int_\Omega Q(w^\varepsilon, \sigma^\varepsilon) h_{1, \xi_i}.
$$

Because $\tau$ is small and $D$ is large, equation (1.8c) for $h_1$ gives

$$
h_1 \approx \varepsilon^{-N} (-D \Delta)^{-1} (g(w^\varepsilon) - \langle g(w^\varepsilon) \rangle).
$$
Writing the Green’s function for $\Delta$ as $G(x,x') = \Gamma(x-x') + R(x,x')$ and using the fact that $(G_,x)) = 0$, we then have
\[ h_1 \approx \varepsilon^{-N}D^{-1} \int_{\Omega} G(x,x')\mathcal{G}(w^\varepsilon) - (g(w^\varepsilon))\mathcal{G}(x') dx' = \varepsilon^{-N}D^{-1} \int_{\Omega} G(x,x')g(w^\varepsilon) dx' \]
\[ = \varepsilon^{-N}D^{-1} \int_{\Omega} \Gamma(x-x')g(w^\varepsilon)(x') dx' + \varepsilon^{-N}D^{-1} \int_{\Omega} R(x,x')g(w^\varepsilon)(x') dx' =: h_{11} + h_{12}. \]
Therefore
\[ \hat{\xi}_i (|w_{\xi}\xi|^2) \approx \int_{\Omega} [Q(w^\varepsilon,\sigma^\varepsilon)h_{11,x_i} + Q(w^\varepsilon,\sigma^\varepsilon)h_{12,x_i}] \]
Since $w^\varepsilon$ and $h_{11}$ are radially symmetric about $\xi$ and $w^\varepsilon$ has compact support,
\[ \int_{\Omega} Q(w^\varepsilon,\sigma^\varepsilon)h_{11,x_i} = \int_{\mathbb{R}^N} Q(w^\varepsilon,\sigma^\varepsilon)h_{11,x_i} = 0. \]
Hence
\[ \hat{\xi}_i (|w_{\xi}\xi|^2) \approx \varepsilon^{-N}D^{-1} \int_{\Omega} Q(w^\varepsilon,\sigma^\varepsilon)R_{x_i}(x,x')g(w^\varepsilon(x',\xi)) dx'dx \]
\[ = \varepsilon^{-N}D^{-1} \int_{\Omega} \left( \int_{\Omega} \varepsilon^{-N}Q(w^\varepsilon(x',\xi),\sigma^\varepsilon)R_{x_i}(x,x') dx \right) \varepsilon^{-N}g(w^\varepsilon(x',\xi)) dx'. \]
Observe that as $\varepsilon \to 0$,
\[ \varepsilon^{-N}g(w^\varepsilon(x',\xi)) \to c_1\delta(x'-\xi), \quad c_1 := \int_{\mathbb{R}^N} g(W(y)) dy, \]
\[ \varepsilon^{-N}Q(w^\varepsilon(x',\xi)) \to -c_2\delta(x'-\xi), \quad c_2 := \int_{\mathbb{R}^N} |Q(W(y))| dy, \]
\[ \varepsilon^{-2N} \int_{\Omega} |w_{\xi}\xi|^2 = \int_{\mathbb{R}^N} |W_{\xi y}|^2 \to c_3, \quad c_3 := \frac{1}{N} \int_{\mathbb{R}^N} |
abla W|^2. \]
We then have
\[ (2.3) \]
\[ \hat{\xi}_i \approx -\frac{c_1c_2}{c_3} \frac{\varepsilon^2}{D} \frac{N}{\int_{\mathbb{R}^N} W^3} \frac{W^3}{\int_{\mathbb{R}^N} W^2} = \frac{N}{6 \int_{\mathbb{R}^N} W^2} \]
\[ \frac{6 \int_{\mathbb{R}^N} |
abla W|^2}{6 \int_{\mathbb{R}^N} W^2} \]
by using the fact that $R(\xi,x) = R(x,\xi)$ so $\nabla x R(\xi,x) = \nabla_x R(\xi,\xi) = \frac{1}{2} D_{\xi} R(\xi,\xi)$. 

Finally, using the definition of $g$, $\sigma$ and $Q$ we have
\[ (2.4) \]
\[ \alpha_0 := \frac{c_1c_2}{2c_3} = \frac{N}{6 \int_{\mathbb{R}^N} W^3} \frac{W^3}{\int_{\mathbb{R}^N} W^2} \]
\[ \frac{N}{6 \int_{\mathbb{R}^N} W^2} \]
Here in the last equality, we have used the identity $\int_{\mathbb{R}^N} W^3 / \int_{\mathbb{R}^N} W^2 = \int_{\mathbb{R}^N} (|
abla W|^2 + W^2)$ obtained by integrating $0 = W(-\Delta W + W - W^2 / \int_{\mathbb{R}^N} W^2)$ over $\mathbb{R}^N$.

In the sequel we shall make the derivation of (2.3) rigorous.

3. Preliminaries

3.1. The ground state $W$. With $f(A,H) = A^2 / H$, equation (1.9) for $W$ reads
\[ -\Delta W + W - \frac{W^2}{\int_{\mathbb{R}^N} W^2} = 0 \quad \text{in } \mathbb{R}^N. \]
It is well-known that this equation possesses a unique, non-negative, radially symmetric solution $W$ (referred to as the ground state) with its unique maximum attained at the origin. In addition, there exists a positive constant $K$ such that as $r \to \infty$,
\[ (3.1) \]
\[ |D^\alpha_y W(y)| \leq K e^{-|y|}, \quad \alpha = 0,1,2. \]
For more details on the ground state solution $W$, see [4, 9, 8, 16, 19] and the references therein.
From (3.1) one sees that there exists a positive constant $C$, which is independent of $\varepsilon$ and $\mu$, such that $r^\varepsilon(x, \xi)$ defined in (2.1) satisfies
$$
|r^\varepsilon(x, \xi)| + |\nabla_\xi r^\varepsilon(x, \xi)| \leq Ce^{-\mu/(2\varepsilon^2)} \leq CD^{-2}\varepsilon^4, \quad \forall x, \xi \in \Omega,
$$
since $\mu \geq 4\varepsilon \log(D\varepsilon^{-2})$.

3.2. Local coordinates near $\mathcal{M}$. For convenience in what follows we shall often drop the superscript and write
$$
\sigma^\varepsilon = \sigma, \quad u^\varepsilon = u, \text{ etc.}
$$

**Lemma 3.1.** There exists a positive constant $c_0$ depending only on $\Omega$, such that if $\varepsilon \in (0, 1]$ and
$$
\text{dist}(u, \mathcal{M}) = \inf_{w \in \mathcal{M}} \|u - w\|_{L^2(\Omega)} < c_0\varepsilon^{N/2},
$$
then there exists a unique $\xi \in \overline{\Omega}$ such that
$$
u = w(\cdot, \xi) + \psi, \quad \|\psi\|_2 = \text{dist}(u, \mathcal{M}).$$

Consequently, if $\xi \in \Omega$, then $\psi \perp T\xi \mathcal{M}$, the tangent space of $\mathcal{M}$ at $w(\cdot, \xi)$; that is, $\langle \psi, w_i(\cdot, \xi) \rangle = 0$ for all $i = 1, \ldots, N$.

The standard proof of this result is left to the reader.

3.3. Eigenvalue estimates. Multiplying (2.2) by $\phi$ and integrating over $\Omega$ yields, after using $\phi \perp \mathcal{M}$,
$$
\frac{1}{2} \frac{d}{dt} \|\phi\|_2^2 = \langle L^\varepsilon \phi, \phi \rangle + h_0 \langle f, \phi \rangle + h_1 \langle f, \phi \rangle + \langle r^\varepsilon + \mathcal{N}, \phi \rangle,
$$
where the operator $L^\varepsilon$ can be written as
$$
L^\varepsilon \phi = \varepsilon^2 \Delta \phi - \phi + 2\sigma^{-1} w\phi - 2\varepsilon^{-N} \sigma^{-2} \langle w, \phi \rangle w^2, \quad w = w^\varepsilon(\cdot, \xi).
$$

**Lemma 3.2.** There exists a positive constant $\nu$ which is independent of $\varepsilon$ and $\mu$ such that for all sufficiently small positive $\varepsilon$,
$$
\langle L^\varepsilon \phi, \phi \rangle \leq -\nu \{\varepsilon^2 \|\nabla \phi\|_2^2 + \|\phi\|_2^2\} \quad \forall \phi \in H^1(\Omega), \xi \in \Omega^\mu.
$$

We refer the reader to [3] (Lemma 2.4) for the proof of the Lemma. It is worth mentioning here that this eigenvalue estimate lemma is the key to our whole analysis.

3.4. Some $L^\infty$ estimates for parabolic equations.

**Lemma 3.3.** There exists a positive constant $C(\Omega)$ such that for every constants $D \geq 1$, $\tau > 0$ and $T > 0$, and functions $u_0 : \Omega \rightarrow (0, \infty)$ and $F(\cdot, \cdot) : \Omega \times (0, T) \rightarrow (0, \infty)$, the solution $v$ to
$$
\begin{align*}
\tau v_t - D\Delta v + v &= F(x, t) \quad \text{in } \Omega \times (0, T), \\
\partial_\nu v &= 0 \quad \text{on } \partial \Omega \times (0, T), \\
v(\cdot, 0) &= u_0(\cdot) \quad \text{in } \Omega \times \{0\}
\end{align*}
$$
satisfies
Lemma 3.4. For each $p > N/2$, there exists a positive constant $C(\Omega, p)$ such that for every positive constants $T$, $\tau$ and $\eta$, and every functions $F \in L^\infty((0, T); L^p(\Omega))$ and $v_0 \in L^\infty(\Omega)$ with $\langle v_0 \rangle = \langle F(\cdot, t) \rangle = 0$ for all $t$, the solution $v$ to
\[
\begin{cases}
\tau v_t - \Delta v + \eta v = F(x, t) & \text{in } \Omega \times (0, T), \\
\partial_n v = 0 & \text{on } \partial \Omega \times (0, T), \\
v(x, 0) = v_0(x) & \text{in } \Omega \times \{0\}
\end{cases}
\]
satisfies for every $t_0 \in [0, T]$,
\[
\|v(\cdot, t_0)\|_{\infty} \leq \|v_0\|_{\infty} + C(\Omega, p) \sup_{0 \leq s \leq t_0} \|F(\cdot, s)\|_p.
\]

We leave the proof of the above two Lemmas to the last section.

3.5. The regular part of the Green's function. We assume that $\Omega$ is a bounded domain in $\mathbb{R}^N$ ($N = 2, 3$) with $C^3$ boundary and of unit volume. We denote by $G(x, \xi)$ the Green's function for $\Delta$ in $\Omega$ with homogeneous Neumann boundary condition, i.e., solution to (1.14). For each $\xi \in \Omega$ sufficiently close to the boundary $\partial \Omega$, the distance function $d(x)$ defined as the distance from $x$ to $\partial \Omega$ will be smooth near $\xi$, so that there
is a unique reflection point $\xi^* = \xi - 2d(\xi)\nabla_d d(\xi)$ of $\xi$ about $\partial \Omega$.

Lemma 3.5. For all $\xi$ sufficiently close to $\partial \Omega$,
\[
G(x, \xi) = \Gamma(x - \xi) + R(x, \xi), \quad R(x, \xi) = \Gamma(x - \xi^*) + J(x, \xi)
\]
where $\xi^* = \xi - 2d(\xi)\nabla_d d(\xi) \in \Omega^c$ is the unique reflection point of $\xi$ with respect to $\partial \Omega$ and the function $J(x, \xi)$ satisfies
\[
\|\nabla J(x, \xi)\| \leq C(\Omega) \ d(\xi)^{2-N}.
\]
Consequently,
\[
D_\xi R(\xi, \xi) = 2\nabla_\xi R(\xi, \xi) = 2^{2-N}(\omega_N)^{-1}d(\xi)^{1-N}[-\nabla_d d(\xi) + O(d(\xi))] \quad \text{as} \quad \xi \to \partial \Omega,
\]
where $\omega_N$ is the area of the unit sphere in $\mathbb{R}^N$.

We leave the proof to the last section of the paper.

4. $L^\infty$ Estimates

4.1. A lower bound on $H$.

Lemma 4.1. Assume that $D > 1$, $\tau > 0$, and that the initial value $H(\cdot, 0)$ of $H$ satisfy
\[
\|H(x, 0) - \sigma^\varepsilon\|_{\infty} \leq \frac{\sigma^\varepsilon}{4}.
\]
There exists a constant $C = C(\Omega) > 0$ such that for all $t \in (0, T^*)$,
\[
H(x, t) \geq \frac{1}{C(\Omega)}.
\]

Proof. First of all the constant $\sigma^\varepsilon := \int_{\mathbb{R}^N}(W^\varepsilon(y))^2dy = \varepsilon^{-N} \int_{|x| \leq \mu(\varepsilon)^2} \mu(\varepsilon^2)dx$ is bounded and positive, uniformly in $\varepsilon \in (0, 1]$.

When $t \in (0, T^*)$, $A = w + \phi$ with $\|\phi\|_{2} \leq c_0 \varepsilon^{N/2}$. It then follows that
\[
\bar{\sigma}(t) := \varepsilon^{-N} \int_\Omega g(A) = \varepsilon^{-N} \int_\Omega A^2 = \varepsilon^{-N} \int_\Omega (w + \phi)^2 \in (\sigma^\varepsilon/2, 3\sigma^\varepsilon/2)
\]
(taking smaller $c_0$ if necessary). The assertion of the lemma then follows directly from Lemma 3.3 with $F(x,t) = \varepsilon^{-N}g(A)$.

4.2. An upper bound for $A$.

**Lemma 4.2.** There exists a positive constant $C(\Omega)$ such that

$$\|A\|_{\infty, \Omega \times [0,T^*]} \leq C(\Omega), \quad \|\phi\|_{\infty, \Omega \times [0,T^*]} \leq C(\Omega).$$

**Proof.** Set $y = x/\varepsilon$. Then equation (1.2) can be written as

$$A_t - \Delta_y A + A = f(A,H), \text{ in } \Omega \times (0,T^*).$$

Fix $p \in (N/2, 2)$. Then the local and boundary parabolic estimates yield

$$\|A\|_{\infty, \Omega \times [0,T^*]} \leq \|A(\cdot,0)\|_{\infty} + C\sup_{0<t\leq T^*} \sup_{y \in \Omega} (\|A(t,\cdot)\|_{L^2(B_{1}(y) \cap \Omega_x)} + \|f(A,H)\|_{L^p(B_{1}(y) \cap \Omega_x)})$$

where $C$ depends only on the $C^{2+\alpha}$ norm of $\partial \Omega_x$, and hence is bounded independently on $\varepsilon \in (0,1]$. As $f(A,H) = A^2/H$,

$$\|f(A,H)\|_{L^p(B_{1}(y) \cap \Omega_x)} \leq \frac{\|A^2\|_{p,\Omega_x}}{\min H} \leq C \|A\|_{\infty}^{2-2/p} \|A\|_{2,\Omega_x}^{2/p} \leq \delta \|A\|_{\infty} + C(\delta) \|A\|_{2,\Omega_x}^{2/(2-p)} = \delta \|A\|_{\infty} + C(\delta) \left( \int_{\Omega} \varepsilon^{-N} A^2 dx \right)^{1/(2-p)}.$$ 

Taking small $\delta$ we then obtain from (4.2) that

$$\|A\|_{\infty, \Omega \times [0,T^*]} \leq \left\{ \left( \|A(\cdot,0)\|_{\infty} + C\sup_{(0,T^*)} \|ar{\sigma}(t) + C(\delta) \bar{\sigma}^{1/(2-p)}(t)\right)(1-C\delta)^{-1} \right\} \leq C(\Omega)$$

where $\bar{\sigma}(t)$ is as in (4.1). The proof of the lemma is complete. \qed

5. The flow in the normal space to $M$

5.1. Estimates for $\phi$. With $f$ and $g$ given by (1.3),

$$H = \varepsilon^{-N} \langle (w + \phi)^2 \rangle + h = \sigma + 2\varepsilon^{-N} \langle \phi w \rangle + \varepsilon^{-N} \|\phi\|^2_2 + h,$$

and equation (2.2) reads

$$\nabla \xi \cdot w - \dot{\xi} + L^\varepsilon \phi - \frac{w^2}{\sigma^2}h - N[w,\phi]$$

where $r^\varepsilon = \varepsilon^2 \Delta w - w + w^2/\sigma$ and

$$N[w,\phi] = \frac{\phi^2}{H} - \frac{\varepsilon^{-N} \|\phi\|^2_2 w^2}{\sigma^2} - \frac{(h + 2\varepsilon^{-N} \langle \phi w \rangle + \varepsilon^{-N} \|\phi\|^2_2)w\phi}{\sigma H} + \frac{w^2(H - \sigma)^2}{\sigma^2 H}.$$ 

**Lemma 5.1.** The following estimates hold for all $t \in [0,T^*)$:

$$|N[w,\phi]| \leq C \left( \phi^2 + \varepsilon^{-N} \|\phi\|^2_2 w^2 + h^2 w^2 \right),$$

(5.3)

$$\frac{1}{2\varepsilon^2} \frac{\|\phi\|^2_2}{\|\phi\|^2_2 + \|\phi\|^2_2} (e^{-N/2} \|\phi\|^2_2 - \frac{1}{2}) + C(\nu)\|\phi\|^2_2 + |r^\varepsilon|^2_{L^p(\Omega_x)} (5.4) \nu > 0 \text{ is the constant in Lemma 3.2.}$$
Proof. The estimate (5.3) follows from Lemma 4.1, the bounds

\[ ||\phi||_2 \leq c_0 \varepsilon^{N/2}, \quad ||w||_p^p = \varepsilon^N \int_{\mathbb{R}^N} (W^\varepsilon)^p dy = O(\varepsilon^N) \quad \forall p > 1, \]

and a straightforward calculation.

To prove (5.4), we multiply (5.1) by \( \phi \), integrate over \( \Omega \) and use \( \phi \perp M \), obtaining

\[ \frac{1}{2} \frac{d}{dt} ||\phi||_2^2 \leq (C^M \phi, \phi) + (r^\varepsilon, \phi) + ||w^2 h_\sigma^{-2}, \phi|| + ||(N[w, \phi], \phi)||. \]

Let \( \nu \) be the constant in Lemma 3.2. Using the bounds in (5.5), we can estimate

\[ |(r^\varepsilon, \phi)| \leq \frac{\nu}{2} ||\phi||_2^2 + C ||r^\varepsilon||_\infty^2, \]
\[ ||w^2 h_\sigma^{-2}, \phi|| \leq C \varepsilon^{N/2} ||\phi||_2 ||h||_\infty \leq \frac{\nu}{2} ||\phi||_2^2 + C \varepsilon^N ||h||_\infty^2, \]
\[ ||N[w, \phi], \phi|| \leq ||\phi||_3^3 \leq C \varepsilon^{-N/2} ||\phi||_2^3, \]
\[ ||w^2 \phi||_2 \leq C \varepsilon^{-N/2} ||h||_\infty^2 ||\phi||_2 \leq C \varepsilon^N ||h||_\infty^3. \]

Substituting these estimates into (5.6) and (5.3), and using Lemma 3.2, we then obtain (5.4). \( \square \)

5.2. Estimates for \( h \). We first estimate \( h_1 \). It is convenient to further decompose \( h_1 = h_{11} + h_{12} \), where

\[ \tau D^{-1}h_{11,t} - \Delta h_{11} + D^{-1}h_1 = \varepsilon^{-N} D^{-1} (w^2 - \langle w^2 \rangle), \]
\[ \tau D^{-1}h_{12,t} - \Delta h_{12} + D^{-1}h_{12} = \varepsilon^{-N} D^{-1} [(2\phi w + \phi^2) - \langle 2\phi w + \phi^2 \rangle]. \]

Both \( h_{11} \) and \( h_{12} \) satisfy the homogeneous Neumann boundary condition, \( h_{12}(\cdot, 0) \equiv 0 \), and \( h_{11}(x, 0) = H(x, 0) - \langle H(\cdot, 0) \rangle \).

**Lemma 5.2.** Assume that \( h_{12}(\cdot, 0) \equiv 0 \) and for some \( \kappa \in (0, 1/4) \), \( ||h_{11}(\cdot, 0)||_\infty \leq D^{-1} \varepsilon^{2-N-\kappa} \). Then there exists \( C = C(\kappa, \Omega) \) such that

\[ \lim_{t \to \infty} h_{11}(x, t) \leq CD^{-1} \varepsilon^{2-N-\kappa}, \]
\[ ||h_{12}||_\infty \leq CD^{-1} \varepsilon^{2-N-\kappa}(c_0 ||\phi||_2). \]

**Proof.** We set \( p = N/(2-\kappa) \in (N/2, 2) \). Using Lemma 3.4 we obtain

\[ ||h_{11}||_{\infty, \Omega \times [0,T^\ast]} \leq ||h_{11}(\cdot, 0)||_\infty + CD^{-1} \sup_{0 < s < T^\ast} ||\varepsilon^{-N} w^2||_p \]
\[ ||h_{12}||_{\infty, \Omega \times [0,T^\ast]} \leq CD^{-1} \sup_{0 < s < T^\ast} ||\varepsilon^{-N}(2\phi w + \phi^2)||_p. \]

Since \( \varepsilon^{-N} ||w^2||_p \leq C \varepsilon^{-N+N/p} = C \varepsilon^{2-N-\kappa} \), the estimate (5.7) follows from (5.9).

Also, we have

\[ ||\varepsilon^{-N} w\phi||_p \leq \varepsilon^{-N} ||\phi||_2 ||w||_{2p/(2-p)} \leq C \varepsilon^{-N-N(2-p)/(2p)} ||\phi||_2 = C \varepsilon^{-N-\kappa}(c_0 ||\phi||_2), \]
\[ ||\varepsilon^{-N} \phi^2||_p \leq \varepsilon^{-N} ||\phi||_2^{2p/(2-2p)} ||\phi||_{2p}^2 \leq C \varepsilon^{-N-\kappa}(c_0 ||\phi||_2) \]

since \( ||\phi||_\infty \leq C \) and \( \varepsilon^{-N/2} ||\phi||_2 \leq c_0 \). The inequality (5.8) then follows from (5.9) and the preceding estimates. \( \square \)

In the sequel we denote

\[ E_\kappa = E_\kappa(\varepsilon, D) = D^{-1} \varepsilon^{2-N-\kappa}, \quad \text{for } \kappa \in (0, 1/4). \]
We shall now estimate $h_0$, which solves
\begin{equation}
(5.10) \quad h_{0,t} + \tau^{-1} h_0 = -2\varepsilon^{-N} \int_{\Omega} AA_t.
\end{equation}

**Lemma 5.3.** The following estimate holds true for all $t \in [0, T^*)$,
\begin{equation}
(5.11) \quad \frac{1}{2} \frac{d}{dt} h_0^2 + \tau^{-1} h_0^2 \leq C(\kappa, \Omega) \left[ \|r^*\|^2_{\infty} + \|h\|^2_{\infty} + \varepsilon^{-N} \|\phi\|^2_{2} + \varepsilon^{-N} |h_0| (\varepsilon^2 \|\nabla \phi\|^2_{2} + \|\phi\|^2_{2}) \right].
\end{equation}

**Proof.** Substituting
\[ \int_{\Omega} AA_t = \int_{\Omega} (w + \phi) \{ r^* + C \xi \phi - \sigma^{-2} w^2 h + N(w, \phi) \} \]
into (5.10) and using a straightforward calculation similar to that in the proofs of Lemma 5.1 and Lemma 5.2, we obtain (5.11). We omit the details. \hfill \Box

5.3. **Proof of Theorem 1.1.** Adding estimates (5.4) and (5.11) we obtain
\begin{equation}
(5.12) \quad \frac{1}{2} \frac{d}{dt} (h_0^2 + \varepsilon^{-N} \|\phi\|^2_{2}) \quad \leq \quad (C|h_0| + C \varepsilon^{-N} \|\phi\|^2_{2}) - \nu/2 \varepsilon^{-N} (\varepsilon^2 \|\nabla \phi\|^2_{2} + \|\phi\|^2_{2}) + (C - \tau^{-1}) h_0^2 + C \|h\|^2_{\infty} + C \varepsilon^{-N} \|r^*\|^2_{\infty}.
\end{equation}

By Lemma 5.2,
\[ \|h\|^2_{\infty} = \|h_0 + h_{11} + h_{12}\|^2_{\infty} \leq 2h_0^2 + 2 \|h_{11}\|^2_{\infty} + 2 \|h_{12}\|^2_{\infty} \leq 2h_0^2 + C(\mathcal{E}_\kappa)^2 (1 + \varepsilon^{-N} \|\phi\|^2_{2}). \]

Since $\|r^*\|^2_{\infty} \leq C \varepsilon^{-\mu/(2\varepsilon)} < CD^{-2} \varepsilon^4$, taking $\tau_0 \leq \varepsilon_0 \leq c_0$ (in Lemma 3.1) sufficiently small we obtain from (5.12) that if $\varepsilon \in (0, \varepsilon_0), \tau \in (0, \tau_0)$, then
\begin{equation}
(5.13) \quad \frac{1}{2} \frac{d}{dt} (h_0^2 + \varepsilon^{-N} \|\phi\|^2_{2}) \quad \leq \quad -\frac{1}{2\tau} h_0^2 - (\nu/4 - C|h_0|) \varepsilon^{-N} (\varepsilon^2 \|\nabla \phi\|^2_{2} + \|\phi\|^2_{2}) + C_2(\mathcal{E}_\kappa)^2 \\
\leq \quad -\frac{1}{C_1} (h_0^2 + \varepsilon^{-N} \|\phi\|^2_{2}) + C_2(\mathcal{E}_\kappa)^2
\end{equation}
for all $t \in [0, \hat{T}]$ where $[0, \hat{T}]$ is the maximal interval in $[0, T^*)$ in which $|h_0| \leq \nu/(8C)$. Here $C_1$ and $C_2$ are constants depending only on $N, \kappa$, and $\Omega$.

Applying Gronwall’s inequality to (5.13) we conclude that there exists a constant $C = C(\Omega, \kappa, N)$ such that when $\varepsilon \in (0, \varepsilon_0]$ and $\tau \in (0, \tau_0]$,
\[ h_0^2 + \varepsilon^{-N} \|\phi\|^2_{2} \leq C(\mathcal{E}_\kappa)^2 = C(D^{-1} \varepsilon^{2-N-\kappa})^2 \]
for all $t \in (0, \hat{T}]$. Since we assume that $D > \varepsilon^{2-N-2\kappa}$, we see from the above estimate that, taking $\varepsilon_0$ smaller if necessary, $|h_0| \leq \nu/(16C)$ and $\varepsilon^{-N/2} \|\phi\|^2_{2} \leq c_0/2$. Thus, we must have $\hat{T} = T^*$ and, by the definition of $T^*$, either $T^* = \infty$ or $\xi(T^*) \in \partial \Omega^\mu$. This completes the proof of Theorem 1.1. \hfill \Box

6. **The flow in the tangent space of $\mathcal{M}$**

In the sequel, we assume that $\varepsilon \in (0, \varepsilon_0], \tau \in (0, \tau_0]$, and that $D$ and $\mu$ satisfy (1.13). Then the assertion of Theorem 1.1 holds true.
6.1. The velocity.

**Lemma 6.1.** For all $t \in [0, T^*)$,

$$
\dot{\xi} = \frac{\varepsilon^2}{3\sigma^2 c_3} \left( 1 + O(\varepsilon) \right) \left( -\varepsilon^{-N} \langle \nabla_x h_{111}, w^3 \rangle + O(\varepsilon^{-1} \varepsilon_\kappa^2) \right)
$$

where $I$ is the identity matrix and $c_3 = N^{-1} \int_{\mathbb{R}^N} |\nabla W(y)|^2 dy$.

**Proof.** Multiplying (5.1) by $w_{\xi_j}$ and integrating the resulting equation over $\Omega$ yields

$$
\sum_{i=1}^{N} \dot{\xi}_i \left( \langle w_{\xi_i}, w_{\xi_j} \rangle - \langle w_{\xi_{ij}}, \phi \rangle \right) = \langle r^\varepsilon, w_{\xi} \rangle + \langle \mathcal{L}^\xi \phi, w_{\xi} \rangle - \langle \sigma^{-2} w^2 h, w_{\xi} \rangle + \langle \mathcal{N}, w_{\xi} \rangle.
$$

Note that

$$
\langle w_{\xi_i}, w_{\xi_j} \rangle = \varepsilon^{N-2} \int_{\mathbb{R}^N} W_{\xi_i} W_{\xi_j} \, dy = \varepsilon^{N-2} c_3 \delta_{ij} (1 + O(e^{-\mu/\varepsilon})�
$$

$$
\langle w_{\xi_{ij}}, \phi \rangle \leq C e^{N-2} (e^{-N/2} \|\phi\|_2) \leq C e^{N-2} \varepsilon_\kappa.
$$

Hence, (6.2) can be written as

$$
c_3 \varepsilon^{-N-2} (1 + O(\varepsilon)) \dot{\xi} = \langle r^\varepsilon, w_{\xi} \rangle + \langle \mathcal{L}^\xi \phi, w_{\xi} \rangle - \langle \sigma^{-2} w^2 h, w_{\xi} \rangle + \langle \mathcal{N}, w_{\xi} \rangle.
$$

We shall now estimate each term on the right-hand side.

First of all,

$$
\|r^\varepsilon, w_{\xi}\| \leq \|r^\varepsilon\|_\infty \|w_{\xi}\| \leq C e^{-\mu/(2\varepsilon)} \varepsilon^{N-1} \leq C e^{N-1} \varepsilon_\kappa^2.
$$

Denote

$$
(\mathcal{L}^\xi)^* \psi = \varepsilon^2 \Delta \psi + 2\sigma^{-1} w^2 \psi - 2\varepsilon^{-N} \sigma^{-2} \langle w^2 \psi \rangle w = \mathcal{L}^\xi \psi - 2\sigma^{-2} \varepsilon^{-N} (\langle w^2 \psi \rangle w - \langle w \psi \rangle w^2).
$$

Then $(\mathcal{L}^\xi)^* w_{\xi} = \mathcal{L}^\xi w_{\xi} = r^\varepsilon$. It follows that

$$
\|\langle \mathcal{L}^\xi \phi, w_{\xi} \rangle \| = \|\langle \phi, (\mathcal{L}^\xi)^* w_{\xi} \rangle \| = \|\langle \phi, r^\varepsilon \rangle \| \leq C e^{-\mu/(2\varepsilon)} \varepsilon^{N/2} \varepsilon_\kappa \leq e^{N-1} \varepsilon_\kappa^2.
$$

Next, from (5.3) we obtain

$$
\|\langle \mathcal{N}, w_{\xi} \rangle \| \leq C \int_{\Omega} (\phi^2 + \varepsilon^{-N} \|\phi\|_2^2 w^2 + h^2 w^2) |w_{\xi}| \leq C e^{N-1} [\varepsilon^{-N} \|\phi\|_2^2 + \|h\|_\infty^2] \leq C e^{N-1} \varepsilon_\kappa^2.
$$

As $h = h_0(t) + h_1(x, t) = h_0 + h_{111} + h_{112}$,

$$
\langle \sigma^{-2} w^2 h, w_{\xi} \rangle = \langle \sigma^{-2} w^2 h_1, w_{\xi} \rangle = \langle \sigma^{-2} w^2 h_{111}, w_{\xi} \rangle + \langle \sigma^{-2} w^2 h_{112}, w_{\xi} \rangle
$$

and

$$
\|\langle \sigma^{-2} w^2 h_{112}, w_{\xi} \rangle \| \leq C e^{N-1} \|h_{112}\|_\infty \leq C e^{N-1} \varepsilon_\kappa^2
$$

by Lemma 5.2.

Finally, observe that

$$
\langle \sigma^{-2} w^2 h_{111}, w_{\xi} \rangle = -\frac{1}{3} \langle h_{111}, (w^3)_{x} \rangle = \frac{1}{3} \langle \nabla_x h_{111}, w^3 \rangle.
$$

Substituting all these estimates into (6.3) we then obtain (6.1) and complete the proof of the lemma.

**Lemma 6.2.** Under the assumptions of Theorem 1.2 formula (1.15) holds with $a_0$ given by (2.4).
Proof. From (6.1), it suffices to estimate the term $\langle \nabla_x h_{11}, w^3 \rangle$.

We write $h_{11} = h_{110} + h_{111}$ where

$$
-\Delta h_{110} = D^{-1} \varepsilon^{-N} [w^2 - \langle w^2 \rangle], \\
\tau D^{-1} h_{111,t} - \Delta h_{111} + D^{-1} h_{111} = -D^{-1} (\tau h_{110,t} + h_{110})
$$

with the homogeneous Neumann boundary condition for both $h_{110}$ and $h_{111}$. Note that

$$
h_{111}(\cdot, 0) = h_{11}(\cdot, 0) - h_{110}(\cdot, 0) = h_{1}(\cdot, 0) - (D \Delta)^{-1}(w(\cdot, \xi_0)^2 - \langle w^2(\cdot, \xi_0) \rangle).
$$

Using the Green’s function for the $\Delta$, we have

$$
h_{110}(x) = D^{-1} \varepsilon^{-N} \int_{\Omega} \Gamma(x - \eta) w^2(\eta, \xi) \, d\eta + D^{-1} \varepsilon^{-N} \int_{\Omega} R(x, \eta) w^2(\eta, \xi) \, d\eta
$$

$$
def \psi_1(\|x - \xi\|) + \psi_2(x, \xi).
$$

Note that $\langle w^3, \nabla_x \psi_1 \rangle = 0$, so that $\langle \nabla_x h_{110}, w^3 \rangle = \langle \nabla_x \psi_2, w^3 \rangle$.

As $|\nabla_x \nabla_x R(x, \xi)| \leq C(\Omega) d(\xi)^{-N}$ for all $x \in \Omega,$

$$
\left| \int_{\Omega} R_{x_{\eta}}(x, \eta) w^2(\eta, \xi) \, d\eta - R_{x\eta}(x, \xi) \int_{\Omega} w^2(\eta, \xi) \, d\eta \right|
$$

$$
\leq \int_{|\eta - \xi| \leq \mu} |R_{x_{\eta}}(x, \eta) - R_{x\eta}(x, \xi)| w^2(\eta, \xi) \, d\eta
$$

$$
\leq \int_{|\eta - \xi| < \mu/2} |\nabla_x \nabla_x R| |\eta - \xi| w^2 \, d\eta
$$

$$
+ \int_{\mu/2 < |\eta - \xi| \leq \mu} |R_{x_{\eta}}(x, \eta) - R_{x\eta}(x, \xi)| w^2(\eta, \xi) \, d\eta \leq C \varepsilon^{N+1} d(\xi)^{-N}.
$$

Similarly, since $|\nabla_x \nabla_x R(x, \xi)| \leq C d(\xi)^{-N},$

$$
\left| \int_{\Omega} R_{x_{\eta}}(x, \xi) w^3(\eta, \xi) \, dx - R_{x\eta}(x, \xi) \int_{\Omega} w^3(\xi, \eta) \, dx \right| \leq C \varepsilon^{N+1} d(\xi)^{-N}.
$$

It then follows that

$$
\langle \varepsilon^{-N} w^3, h_{110, \xi} \rangle = D^{-1} \int_{\Omega} \varepsilon^{-N} w^3(x, \xi) \int_{\Omega} \varepsilon^{-N} R_{x\eta}(x, \eta) w^2(\eta, \xi) \, d\eta \, dx
$$

$$
= D^{-1} R_{x\eta}(\xi, \xi) \int_{\mathbb{R}^N} W^2 \int_{\mathbb{R}^N} W^3 + O(\varepsilon) D^{-1} d(\xi)^{-N}.
$$

(6.5)

On the other hand by Lemma 3.4 we have

$$
\|h_{111}\|_{\infty} \leq \|h_{110}(\cdot, 0)\|_{\infty} + \sup_{0 < t < T^*} D^{-1} \tau \|h_{110,t}\|_{p} + \|h_{111}\|_{p}
$$

$$
\leq C \varepsilon^2 + D^{-2} \sup_{0 < t < T^*} (\tau \|\Delta^{-1} \varepsilon^{-N} \nabla \xi w \cdot \xi\|_{p} + \|\Delta^{-1} \varepsilon^{-N} w^2\|_{p})
$$

$$
\leq C \varepsilon^2 + CD^{-2} \sup_{0 < t < T^*} [(\tau |\xi| + 1) \varepsilon^{-N} \|w^2\|_{p}]
$$

$$
\leq C \varepsilon^2
$$

since from (5.7) and (6.1), $|\xi| = o(1)$ as long as $t < T^*$. Thus,

$$
\varepsilon^{-N} \|w^2 h_{111}, w_{\xi, \xi}\| \leq C \varepsilon^{-1} \|h_{111}\|_{\infty} \leq C \varepsilon^{-1} \varepsilon^{-2}.
$$

Hence, from (1.15), the preceding estimates, and the definition of $\alpha_0$, we obtain

$$
\xi = \alpha_0 \varepsilon^2 D^{-1} \left(1 + O(\varepsilon)\right) \left(-D \xi R(\xi, \xi) + O(\varepsilon) d(\xi)^{-N} + O(\varepsilon^{-2N-2 \kappa} D^{-1})\right).
$$

Finally notice that $|\xi, D \xi R(\xi, \xi)| \leq C d(\xi)^{-N} \varepsilon^{-2N-\kappa} D^{-1} = C \varepsilon^{-2N-\kappa} D^{-1} \varepsilon^{-N} \leq C \varepsilon^{-2N-2 \kappa} D^{-1}.$

Equation (1.15) thus follows. \(\square\)
6.2. **Proof of Theorem 1.2.** It remains to show that $T^* = \infty$ when $D > \varepsilon^{3-2N-3\kappa}$ and $\mu$ is small.

When $\xi(t)$ is near the boundary $\partial \Omega$, we have, from (1.15) and (3.4),

$$
\dot{\xi} = 2^{2-N} \alpha \varepsilon^2 (\omega N D)^{-1} \left( \nabla \xi d(\xi) + O(\varepsilon/d(\xi)) + O(\varepsilon^{3-2N-2\kappa} D^{-1}) \right)
$$

d

(\xi^{-1}) \text{Itthenfowllowsthat} \quad d(\xi(t)) \geq 0 \quad \text{whenever} \quad \xi(t) = \text{is close enough to the boundary}. \quad \text{Consequently,} \quad d(\xi(t)) > \mu \quad \text{for all} \quad t \in [0, T^*]

if we take $\mu$ small enough. Therefore, by Theorem 1.1, $T^* = \infty$. \hfill \Box

7. **Proofs of auxiliary lemmas**

**Proof of Lemma 3.3.** Integrating the differential equation over $\Omega$ yields

$$
\tau \frac{d}{dt} \int_{\Omega} v + \int_{\Omega} v = \int_{\Omega} F(x, t) \, dx \geq \min_{[0, T]} \int_{\Omega} F(x, t) \, dx.
$$

Gronwall's inequality then gives

\begin{equation}
\int_{\Omega} v(x, t) \, dx \geq \min \left\{ \int_{\Omega} v_0(x) \, dx, \min_{[0, T]} \int_{\Omega} f(x, t) \, dx \right\}.
\end{equation}

(7.1)

To prove (3.2), we consider two cases: (i) $t_0 \in [0, \tau]$; (ii) $t_0 \in [\tau, T]$.

- **Case (i):** $t_0 \in [0, \tau]$. Comparing $v$ with a subsolution $v = e^{-t/\tau} \min_{\Omega} v_0$ gives

\begin{equation}
v(x, t_0) \geq v \geq \frac{1}{\varepsilon} \min_{\Omega} v_0, \quad \forall x \in \Omega.
\end{equation}

(7.2)

- **Case (ii):** $t_0 \in [\tau, T]$. We define

$$
s = D(t - t_0)/\tau + 1, \quad \tilde{v} = e^{(t-t_0)/\tau} v = e^{(s-1)/D} v.
$$

Then $\tilde{v}_s - \Delta \tilde{v} \geq 0$ so that there exists $\bar{C}(\Omega)$ such that

\begin{equation}
\tilde{v}(\cdot, t_0 - \tau/D) \leq \bar{C}(\Omega)e^{-1/D} \int_{\Omega} v(\cdot, t_0 - \tau/D);
\end{equation}

namely,

\begin{equation}
\min_{\Omega} v(\cdot, t_0) \geq \bar{C}(\Omega)e^{-1/D} \int_{\Omega} v(\cdot, t_0 - \tau/D), \quad \forall t_0 \in [\tau, T].
\end{equation}

(7.3)

Combining (7.1)-(7.3) then yields the assertion (3.2) of the Lemma. \hfill \Box

**Proof of Lemma 3.4.** Since the equation is linear we can assume that $v_0 \equiv 0$. In addition, by the change of variables $t' = t/\tau$ we can also assume that $\tau = 1$. Furthermore, we can assume that $\eta = 0$ because, by defining $\tilde{v} = e^{nt} v$ and $\tilde{F} = e^{nt} F$, if (3.2) holds for $(\tilde{v}, \tilde{F})$ then it automatically holds for $(v, F)$ by the assumption that $\eta \geq 0$. Thus it suffices to establish (3.2) for the case when $\tau = 1, \eta = 0$ and $v_0 \equiv 0$.

Integrating the differential equation over $\Omega$ yields $\langle v \rangle = 0$ for all $t \in [0, T]$.

Multiplying the differential equation by $v$ and integrating the resulting equation over $\Omega$ gives

$$
\frac{1}{2} \frac{d}{dt} \int_{\Omega} v^2 + \int_{\Omega} |\nabla v|^2 = \int_{\Omega} F v \leq \|F\|_{2^*} \|v\|_{2^*} \leq C(\delta) \|F\|_{2^*}^2 + \delta \|v\|_{2^*}^2.
$$
where $2^* = 2N/(2-N)$ for $N \geq 3$ and any large number when $N = 2$. First applying the Poincare inequality $\|v\|_2 \leq C(\Omega)\|\nabla v\|_2$, then choosing $\delta = 1/(2C(\Omega))$, and finally applying the Gronwall’s inequality, we then obtain
\[
\|v(\cdot, t)\|_2 \leq C \sup_{0 < s < t} \|F(\cdot, s)\|_p
\]
for any $p > (2^*)' = 2N/(2 + N)$ when $N \geq 3$ and $p > 1$ when $N = 2$. The assertion of the Lemma then follows from the parabolic estimate
\[
\|v(\cdot, t)\|_\infty \leq C(\Omega,p) \sup_{0 < s < t} (\|v(\cdot, s)\|_2 + \|f(\cdot, s)\|_p) \quad \forall p \in (N/2, \infty),
\]
since $v(\cdot, 0) = 0$. The proof of the lemma is complete. \hfill \Box

Proof of Lemma 3.5. Since (3.4) follows directly from (3.3) and the definition of $\Gamma$ and $\xi^*$, we only need to show (3.3).

Note that $J(x, \xi) := G(x, \xi) - \Gamma(x - \xi) - \Gamma(x - \xi^*)$ satisfies $\Delta_x J(x, \xi) = 1$ in $\Omega$, $\partial_\nu J(x, \xi) = b(x, \xi)$ on $\partial \Omega$, and $\int_\Omega J(x, \xi)\,dx = c(\xi)$ where
\[
b(x, \xi) = -\partial_\nu \Gamma(x - \xi) - \partial_\nu \Gamma(x - \xi^*), \quad c(\xi) = -\int_\Omega (\Gamma(x - \xi) + \Gamma(x - \xi^*))\,dx.
\]
A geometric argument shows that for all $x \in \partial \Omega$ and $\xi \in \Omega$,
\[
|b(x, \xi)| = \frac{1}{\omega_N} \left| \frac{(x - \xi) \cdot n(x)}{|x - \xi|^N} + \frac{(x - \xi^*) \cdot n(x)}{|x - \xi^*|^N} \right| \leq C(\Omega)|\xi - x|^{2-N}.
\]

Using the Green’s formula and noting that $\langle G(\cdot, \xi) \rangle = 0$, we have
\[
J(x, \xi) = c(\xi) + \int_{\partial \Omega} b(x', \xi)G(x, x')\,dS_{x'},
\]
\[
\nabla_x J(x, \xi) = \int_{\partial \Omega} b(x', \xi)\nabla_x G(x, x')\,dS_{x'}.
\]
Using the known fact that $|\nabla_x G(x, x')| \leq C(\Omega)|x - x'|^{1-N}$ we then obtain from (7.4) and (7.5) that
\[
|\nabla_x J(\xi, \xi)| \leq C(\Omega) \int_{\partial \Omega} |\xi - x'|^{2-N}|x' - \xi|^{1-N}dS_{x'} \leq C(\Omega)d(\xi)^{2-N}.
\]
This completes the proof. \hfill \Box

References


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