

STABILITY OF THE STEINER SYMMETRIZATION OF CONVEX SETS

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Abstract. The isoperimetric inequality for Steiner symmetrization of any codimension is investigated and the equality cases are characterized. Moreover, a quantitative version of this inequality is proven for convex sets.

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1. INTRODUCTION

The present paper analyses the Steiner symmetrization of any codimension. Let $n \geq 2$ and $1 \leq k \leq n - 1$. For every set $E \subset \mathbb{R}^n$ and for every $x' \in \mathbb{R}^{n-k}$ we define

$$E_{x'} := \{y \in \mathbb{R}^k : (x', y) \in E\} \quad \text{and} \quad L(x') := \mathcal{L}^k(E_{x'}),$$

where \mathcal{L}^k stands for the *outer Lebesgue measure* in \mathbb{R}^k . We denote by $r(x')$ the radius of a k -dimensional ball in \mathbb{R}^k having \mathcal{L}^k -measure equal to $L(x')$, and we set

$$\pi(E)^+ := \{x' \in \mathbb{R}^{n-k} : L(x') > 0\}.$$

Then, the *Steiner symmetral* (of codimension k) of E is defined as

$$E^S := \{(x', y) \in \mathbb{R}^n : x' \in \pi(E)^+, |y| \leq r(x')\}.$$

The importance of Steiner symmetrization has been assessed by several authors, and relies upon the fact that it acts monotonically on many geometric and analytic quantities associated with subsets of \mathbb{R}^n , e.g. the perimeter. A characterization of the sets whose perimeter is preserved under symmetrization of codimension 1 was given by Chlebík, Cianchi and Fusco [3].

In this paper we tackle the general case $1 \leq k \leq n - 1$ using a new approach, based on the regularity properties of the barycenter of the sections $E_{x'}$ as x' varies in $\pi^+(E)$. The advantage of this approach is twofold. Firstly, we may recover and extend the result proved in [3] for $k = 1$ to any codimension, with a new and simpler proof. Secondly, we are able to obtain a quantitative isoperimetric estimate for convex sets which, to the best of our knowledge, is the first result of this kind in the framework of Steiner symmetrization.

We shall now proceed by providing a detailed account of the content of the paper. Our first result reads as follows.

Theorem 1.1. *Let E be a set of finite perimeter in \mathbb{R}^n . Then*

$$P(E^S; B \times \mathbb{R}^k) \leq P(E; B \times \mathbb{R}^k) \tag{1.1}$$

for every Borel set $B \subset \mathbb{R}^{n-k}$. Moreover, if $P(E^S) = P(E)$, then E is either equivalent to \mathbb{R}^n , or $\mathcal{L}^n(E) < \infty$ and for \mathcal{L}^{n-k} -a.e. $x' \in \pi(E)^+$

- (a) $E_{x'}$ is equivalent to a k -dimensional ball and $\mathcal{H}^{k-1}(\partial^* E_{x'} \Delta (\partial^* E)_{x'}) = 0$;
- (b) the functions $\nu_{x'}^E(x', \cdot)$ and $|\nu_y^E|(x', \cdot)$ are constant \mathcal{H}^{k-1} -a.e. in $\partial^* E_{x'}$.

Here, $P(E; B \times \mathbb{R}^k)$ denotes the perimeter of E in $B \times \mathbb{R}^k$, and $P(E)$ stands for the perimeter of E in the whole \mathbb{R}^n . The reduced boundary of E is indicated with $\partial^* E$, and its generalized inner normal with ν^E (see Section 2). Moreover, $\nu_{x'}^E$ and ν_y^E denote the first $n - k$ and the last k components of ν^E respectively. Finally, \mathcal{H}^d is the *Hausdorff measure* of dimension d .

Inequality (1.1) in high codimension is a result often taken for granted by the experts, in analogy with what is known in the case $k = 1$. However, as far as we know, it has never been explicitly stated and proven in the general framework of sets of finite perimeter (see e.g. [13], for the case $n = 3, k = 2, E$ of class C^2). Indeed, when $k > 1$, it turns out that the proof of this property is more delicate than in the case $k = 1$. The reason of this extra difficulty lies in the fact that the Radon measure

$$B \subset \mathbb{R}^{n-k} \mapsto \mu(B) := \int_{\partial^* E \cap (B \times \mathbb{R}^k) \cap \{\nu_y^E = 0\}} \nu_{x'}^E(x', y) d\mathcal{H}^{n-1}(x', y)$$

has a different behavior depending on whether $k = 1$ or $k > 1$. In fact, when $k = 1$, μ is *purely singular* with respect to the Lebesgue measure \mathcal{L}^{n-1} , while, if $k > 1$, it may contain a non-trivial absolutely continuous part. A somewhat surprising example by Almgren and Lieb (see Remark 3.2) shows that when $k > 1$ it may even happen that μ is absolutely continuous with respect to \mathcal{L}^{n-k} . In other words, when $k = 1$ the projection of $\{x \in \partial^* E : \nu_y^E(x) = 0\}$ (the “vertical part” of the boundary) on \mathbb{R}^{n-k} is a set of zero Lebesgue measure, while if $k > 1$ this projection may be smeared out on a set of positive \mathcal{L}^{n-k} measure.

As observed in [3], the equality $P(E) = P(E^S)$ does not imply that E and E^S are equivalent. In fact, if the boundary of E^S contains vertical parts, one can easily find a set E having the same perimeter of E^S and not equivalent to (any translated of) E^S (see Figure 1).

Therefore, in order to characterize the equality cases, at least in a local form, we shall assume that, given an open set $\Omega \subset \mathbb{R}^{n-k}$, E^S satisfies the following condition

$$\mathcal{H}^{n-1}(\{x \in \partial^* E^S : \nu_y^{E^S}(x) = 0\} \cap (\Omega \times \mathbb{R}^k)) = 0. \quad (1.2)$$

Assumption (1.2) is equivalent to the requirement $L \in W^{1,1}(\Omega)$ (see Proposition 3.5). Note that, for any set E of finite perimeter, $L \in BV(\mathbb{R}^{n-k})$ (see Lemma 3.1). Therefore, the presence of singular parts in the measure DL is equivalent to having vertical parts in the boundary of E^S . Furthermore, (1.2) is weaker than the correspondent condition on the set E

$$\mathcal{H}^{n-1}(\{x \in \partial^* E : \nu_y^E(x) = 0\} \cap (\Omega \times \mathbb{R}^k)) = 0. \quad (1.3)$$

However, (1.2) and (1.3) are equivalent when $P(E; \Omega \times \mathbb{R}^k) = P(E^S; \Omega \times \mathbb{R}^k)$ (see Proposition 3.6).

Condition (1.2), together with the equality of the perimeters, is not yet sufficient to conclude that E and E^S are equivalent. As shown even through simple examples (see Figure 1), this is due to the fact that the set E^S may not be connected in a proper sense. That is, the Lebesgue representative L^* of L may vanish in a set of positive \mathcal{H}^{n-k-1} -measure. Therefore, we are led to assume

$$L^*(x') > 0, \quad \text{for } \mathcal{H}^{n-k-1}\text{-a.e. } x' \in \Omega. \quad (1.4)$$

Then, the analogous of [3, Theorem 1.3] can be established, for every $1 \leq k \leq n - 1$.

Theorem 1.2. *Let $\Omega \subset \mathbb{R}^{n-k}$ be a connected open set, and let E be a set of finite perimeter such that $P(E^S; \Omega \times \mathbb{R}^k) = P(E; \Omega \times \mathbb{R}^k)$. If (1.2) and (1.4) are satisfied, then $E \cap (\Omega \times \mathbb{R}^k)$ is equivalent to (a translate along \mathbb{R}^k of) $E^S \cap (\Omega \times \mathbb{R}^k)$.*

Our proof of this result significantly simplifies the one given in [3] for the case of codimension 1, which was based on a delicate density argument and on the heavy use of the notion of

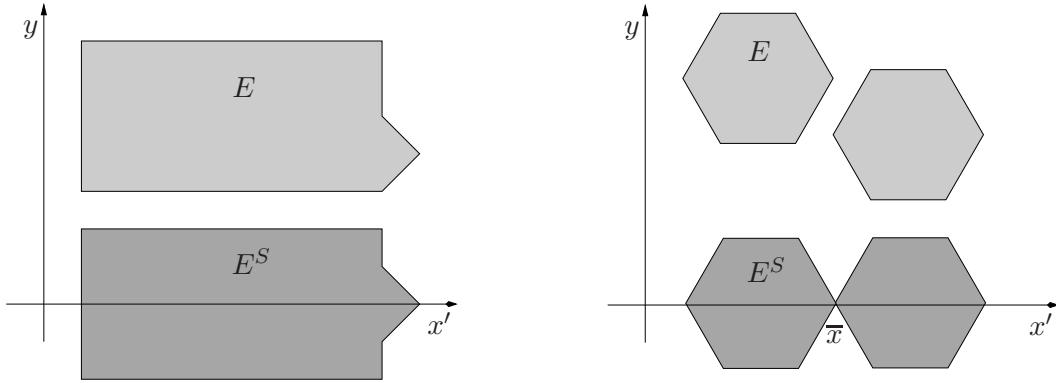


FIGURE 1. Here $n = 2$, $k = 1$. On the left, the boundary of set E^S contains vertical parts, thus violating condition (1.2). Instead, for the set E^S on the right $L^*(\bar{x}) = 0$, thus violating condition (1.4).

polarization. We have devised a different approach, based on the regularity properties of the barycenter $b : \mathbb{R}^{n-k} \rightarrow \mathbb{R}^k$ of the k -dimensional sections of E (see Definition 4.1). We start by observing that assumptions (1.2) and (1.4) yield that the barycenter b is an absolutely continuous function on almost every 1-dimensional section of the set Ω (see Theorem 4.3). Then, using the explicit expression of the derivatives of b (see (4.1)) we show that, if E and E^S have the same perimeter, these derivatives are all zero, thus proving that b is constant in Ω . This gives that E is equivalent to a translation of E^S .

The role played by the barycenter can be easily understood, observing that the sets enjoying property (a) of Theorem 1.1 are completely characterized by the functions L and b . Note, however, that even for these sets no regularity for the barycenter may be expected, if (1.3) and (1.4) are not satisfied (see Example 4.2).

The regularity of the barycenter is an essential tool also in dealing with the second issue addressed in the present paper, namely a quantitative version of the inequality $P(E^S) \leq P(E)$. In recent years quantitative isoperimetric and related functional inequalities have attracted the interest of several authors (witness [5, 8, 9, 10, 11, 12], just to name a few). Let us just recall here the result proved in [10]. To this aim, given a set of finite perimeter E , we define the *asymmetry index* $A(E)$ and the *isoperimetric deficit* $\delta(E)$ of E as

$$A(E) := \min \left\{ \frac{\mathcal{L}^n(E \Delta B_r(x))}{\mathcal{L}^n(E)} : x \in \mathbb{R}^n \right\}, \quad \delta(E) := \frac{P(E) - P(B_r)}{P(B_r)},$$

where $B_r(x)$ is the ball of radius r centered at x , with $\mathcal{L}^n(E) = \mathcal{L}^n(B_r(x))$, and B_r stands for $B_r(0)$.

Theorem 1.3. *There exists a constant $\gamma_n > 0$ such that for every set of finite perimeter $E \subset \mathbb{R}^n$*

$$A(E) \leq \gamma_n \sqrt{\delta(E)}. \quad (1.5)$$

In other words, this inequality states that, if the perimeter of E is close to the one of the ball with the same volume, then there exists an optimal ball which overlaps with E with the exception of a set whose measure is controlled by the square root of the difference $P(E) - P(B_r)$.

In this paper we prove an analogous of Theorem 1.3 for Steiner symmetrization. To state our result precisely, let us recall that the *eccentricity* of a bounded convex set C is defined as the ratio between the outer and inner radii of C , i.e., the radius of the smallest ball containing C and of the largest ball contained in C , respectively.

Let us also introduce the *asymmetry index* and the *isoperimetric deficit* of E relative to its Steiner symmetral E^S :

$$\lambda(E) := \inf \left\{ \frac{\mathcal{L}^n(E^S \Delta (E + (0, y)))}{\mathcal{L}^n(E^S)} : y \in \mathbb{R}^k \right\}, \quad D(E) := \frac{P(E) - P(E^S)}{P(E^S)}.$$

We start with the case $k = 1$.

Theorem 1.4. *Let $k = 1$. If E^S is a Steiner symmetric bounded convex set with eccentricity \mathcal{E} , then*

$$\lambda(E) \leq n 4^{n+1} \mathcal{E}^{2n} \sqrt{D(E)} \quad (1.6)$$

for every set $E \subset \mathbb{R}^n$ of finite perimeter satisfying (1.3), whose 1-dimensional sections $E_{x'}$ are segments, and whose Steiner symmetral is E^S .

An interesting feature of inequality (1.6) is that it applies with the same constant to a large class of sets E^S . On the other hand, in light of the characterization of the equality cases, it is clear that we cannot have a quantitative estimate of this kind if we allow the boundary of E^S to have even “almost” vertical parts (see figure 2). Similarly, it is also clear that Theorem 1.4 cannot hold, should the vertical sections shrink in an arbitrary way (see Figure 3). Therefore, the convexity assumption on E^S seems a natural geometric compromise to avoid both these phenomena, even though Theorem 1.4 may be proven under weaker assumptions (see Remark 5.6). Anyway, the set E is not required to be convex. Note also that inequality (1.6) cannot hold with a constant not depending on the eccentricity (see Example 5.4). Finally, considering that the exponent $1/2$ in (1.5) is optimal, it is hardly surprising that also in our inequality we cannot replace the exponent on the right-hand side with any number larger than $1/2$ (see Example 5.5).

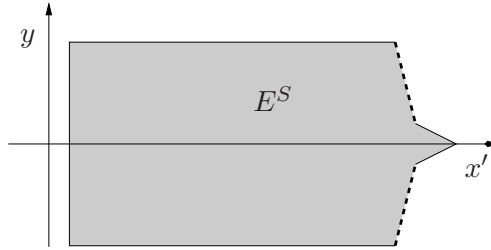


FIGURE 2. When E^S is as in figure, the quantitative inequality $\lambda(E) \leq c\sqrt{D(E)}$ holds with a constant c which goes to infinity as the slope of the two dotted segments becomes larger and larger.

In higher codimension our result is slightly weaker, since we have to assume that the set E is convex.

Theorem 1.5. *Let $2 \leq k \leq n - 1$. There exists a constant $c \equiv c(n, k)$ such that, if E^S is a Steiner symmetric bounded convex set, then*

$$\lambda(E) \leq c \mathcal{E}^{2n} \sqrt{D(E)} \quad (1.7)$$

for every convex set $E \subset \mathbb{R}^n$ with eccentricity equal to \mathcal{E} whose Steiner symmetral is E^S .

Also in this case one can give an explicit bound to the constant appearing in the quantitative isoperimetric inequality (see (5.8)).

By way of conclusion, a short overview of the proof of Theorems 1.4 and 1.5 is provided. Concerning the case $k = 1$, note that the convexity assumption on E^S implies that L is locally

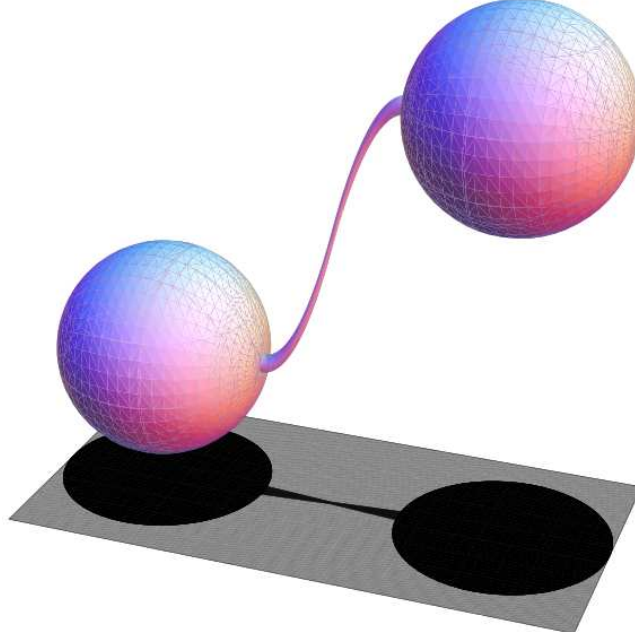


FIGURE 3. Here $n = 3$, $k = 1$, and x' varies in the horizontal plane. Let E_ε^S be given by the union of two balls, connected by a narrow neck whose surface measures ε . By lifting one of the two balls, we obtain a set E_ε as in figure, such that $\lambda(E_\varepsilon) \approx 1/2$, while $D(E_\varepsilon) \rightarrow 0$ as $\varepsilon \rightarrow 0$. Therefore, for every $\varepsilon > 0$ a quantitative isoperimetric inequality $\lambda(E) \leq c_\varepsilon \sqrt{D(E)}$ may hold only with a constant $c_\varepsilon \rightarrow \infty$.

Lipschitz in the projection Ω of the interior of E . More precisely (see Lemma 5.1), one can estimate the gradient of L at a point x' near the boundary by inverse of the distance $\text{dist}(x', \partial\Omega)$ from x' to $\partial\Omega$. Using this estimate, the key point of the proof is to derive the inequality

$$\int_{\Omega} |\nabla b(x')| \text{dist}(x', \partial\Omega) dx' \leq c \sqrt{D(E)},$$

from which (1.6) follows, thanks to a weighted Poincaré inequality (see Proposition 5.2) applied to the barycenter b . The case $k \geq 2$ is then obtained via a symmetrization argument, by applying estimate (1.6) k times.

2. PRELIMINARY RESULTS

In this section we recall some classical results of geometric measure theory, which will be instrumental in the forthcoming arguments. We refer the reader to the monograph [1] for a comprehensive illustration of the subject.

We first give some notation. Let $n \geq 2$ and $1 \leq k \leq n - 1$. For $x \in \mathbb{R}^n$, we write $x = (x', y)$, with $x' \in \mathbb{R}^{n-k}$ and $y \in \mathbb{R}^k$. Similarly, when $v = (v_1, \dots, v_n)$ is a vector in \mathbb{R}^n , we set

$$v_{x'} = (v_1, \dots, v_{n-k}) \in \mathbb{R}^{n-k}, \quad v_y = (v_{n-k+1}, \dots, v_n) \in \mathbb{R}^k.$$

If Du is the distributional gradient of a function $u \in L^1_{\text{loc}}$, we write $D_{x'}u = (D_1u, \dots, D_{n-k}u)$ and $D_yu = (D_{n-k+1}u, \dots, D_nu)$. We shall indicate by $B_r(x)$ the open ball in \mathbb{R}^n of radius r centered in x . Occasionally, when balls of different dimensions come into play, we shall specify

the dimension d of a ball by writing $B_r^d(x)$. Finally, we simply write B_r^d if the center of the ball is the origin. The measure of the unit ball in \mathbb{R}^d will be denoted by ω_d .

Let Ω be an open set in \mathbb{R}^n . We recall that a function $u \in L^1(\Omega)$ is said to be of *bounded variation* if its distributional derivative Du is a vector-valued Radon measure in Ω with finite total variation. The set of all functions of bounded variation in Ω will be denoted by $BV(\Omega)$. The space $BV_{\text{loc}}(\Omega)$ of functions of locally bounded variation is defined accordingly. If $u \in BV(\Omega)$, its distributional gradient can be split in the sum of an absolutely continuous part $\nabla u \mathcal{L}^n$ and in a singular part $D^s u$, with respect to the Lebesgue measure. That is,

$$Du = \nabla u \mathcal{L}^n + D^s u.$$

If $u \in BV(\Omega)$ and $D^s u \equiv 0$, then u belongs to the Sobolev space $W^{1,1}(\Omega)$. We recall that in this case the *Lebesgue representative* u^* exists for \mathcal{H}^{n-1} -a.e. point in Ω .

If E is a Borel set, we say that it has *locally finite perimeter in Ω* if its characteristic function χ_E belongs to $BV_{\text{loc}}(\Omega)$. If the total variation $|D\chi_E|(\Omega)$ is finite, E is said to be a set of *finite perimeter in Ω* , or simply a set of finite perimeter, if $\Omega = \mathbb{R}^n$. For a set of finite perimeter, we define the *reduced boundary* $\partial^* E$ of E as the set of all points $x \in \mathbb{R}^n$ such that

$$\nu^E(x) := \lim_{r \rightarrow 0^+} \frac{D\chi_E(B_r(x))}{|D\chi_E|(B_r(x))} \quad \text{exists and belongs to } \mathbb{S}^{n-1}.$$

We shall refer to the vector $\nu^E(x)$ as to the (*generalized*) *inner normal* to $\partial^* E$ at x . From Besicovitch derivation theorem and [1, Theorem 3.59], it follows that $\partial^* E$ is $(n-1)$ -rectifiable and

$$D\chi_E = \nu^E \mathcal{H}^{n-1} \llcorner \partial^* E. \quad (2.1)$$

Given any Borel set $B \subset \mathbb{R}^n$, the *perimeter of E in B* is defined as

$$P(E; B) := |D\chi_E|(B) = \mathcal{H}^{n-1}(\partial^* E \cap B), \quad (2.2)$$

where the second equality follows from (2.1). When $B = \mathbb{R}^n$, we shall simply write $P(E)$, the perimeter of E . In the sequel, given a measurable set E , and $0 \leq a \leq 1$,

$$E^{(a)} := \left\{ x \in \mathbb{R}^n : \lim_{r \rightarrow 0^+} \frac{\mathcal{L}^n(E \cap B_r(x))}{\omega_n r^n} = a \right\}$$

will denote the *set of points with density a with respect to E* . The next result establishes the connection between reduced boundary and densities (see [1, Theorem 3.61]).

Theorem 2.1. *Let $E \subset \mathbb{R}^n$ be a set of finite perimeter. Then,*

$$\partial^* E \subset E^{(1/2)}, \quad \mathcal{H}^{n-1}(E^{(1/2)} \setminus \partial^* E) = 0, \quad \mathcal{H}^{n-1}(\mathbb{R}^n \setminus (E^{(0)} \cup E^{(1)} \cup E^{(1/2)})) = 0.$$

The next proposition is a special case of the coarea formula (see [1, Theorem 2.93]).

Proposition 2.2. *Let E be a set of finite perimeter in \mathbb{R}^n and let $g : \mathbb{R}^n \rightarrow [0, \infty]$ be a Borel function. Then*

$$\int_{\partial^* E} g(x) |\nu_y^E(x)| d\mathcal{H}^{n-1}(x) = \int_{\mathbb{R}^{n-k}} dx' \int_{(\partial^* E)_{x'}} g(x', y) d\mathcal{H}^{k-1}(y).$$

The following proposition gives a link between the k -dimensional sections $y \rightarrow u_{x'}(y)$ and the total variation of the vector measure $D_y u$. Its proof can be obtained as in [1, Theorem 3.103].

Proposition 2.3. *Let $\Omega \subset \mathbb{R}^n$ open, and let $u \in L^1_{\text{loc}}(\Omega)$. For every $x' \in \mathbb{R}^{n-k}$ set $u_{x'}(\cdot) := u(x', \cdot)$. Then*

$$|D_y u|(\Omega) = \int_{\mathbb{R}^{n-k}} |Du_{x'}|(\Omega_{x'}) dx'.$$

Finally, we prove an enhanced version of a result by Vol'pert [15]. We consider this result to be of some interest in itself, regardless of its applications.

Theorem 2.4 (Vol'pert). *Let E be a set of finite perimeter in \mathbb{R}^n . Then, for \mathcal{L}^{n-k} -a.e. $x' \in \mathbb{R}^{n-k}$*

- (i) $E_{x'}$ is a set of finite perimeter in \mathbb{R}^k ;
- (ii) $\mathcal{H}^{k-1}(\partial^*(E_{x'}) \Delta (\partial^* E)_{x'}) = 0$;
- (iii) For \mathcal{H}^{k-1} -a.e. s such that $(x', s) \in (\partial^* E)_{x'} \cap \partial^*(E_{x'})$:
 - (a) $\nu_y^E(x', s) \neq 0$;
 - (b) $\nu_y^E(x', s) = \nu^{E_{x'}}(s) |\nu_y^E(x', s)|$.

In particular, a Borel set $G_E \subset \pi(E)^+$ exists such that $\mathcal{L}^{n-k}(\pi(E)^+ \setminus G_E) = 0$ and (i)–(iii) are fulfilled for every $x' \in G_E$.

Proof. By Proposition 2.3, for every open set $\Omega \subset \mathbb{R}^n$ we have

$$\int_{\mathbb{R}^{n-k}} |D\chi_{E_{x'}}|(\Omega_{x'}) dx' = |D_y \chi_E|(\Omega). \quad (2.3)$$

Since E has finite perimeter, $\int_{\mathbb{R}^{n-k}} |D\chi_{E_{x'}}|(\Omega_{x'}) dx' < \infty$ and of course this implies (i).

To prove (ii), let us set

$$\mathcal{M} := \{B \subset \mathbb{R}^n : B \text{ is a Borel set such that (2.3) holds with } \Omega \text{ replaced by } B\}.$$

One can check that

- (a) $\{B_h\}_{h \in \mathbb{N}} \subset \mathcal{M}$, $B_h \nearrow B \implies B \in \mathcal{M}$;
- (b) $B, B', B \cup B' \in \mathcal{M} \implies B \cap B' \in \mathcal{M}$;
- (c) $B \in \mathcal{M} \implies \mathbb{R}^n \setminus B \in \mathcal{M}$.

Since \mathcal{M} contains all the open subsets of \mathbb{R}^n , from [1, Remark 1.9] we have that \mathcal{M} coincides with the σ -algebra of Borel in \mathbb{R}^n , that is (2.3) holds true also for Borel sets. Thus,

$$\begin{aligned} \int_{\partial^* E} \chi_B(x) |\nu_y^E(x)| d\mathcal{H}^{n-1}(x) &= |D_y \chi_E|(B) = \int_{\mathbb{R}^{n-k}} |D\chi_{E_{x'}}|(B_{x'}) dx' \\ &= \int_{\mathbb{R}^{n-k}} \mathcal{H}^{k-1}(\partial^*(E_{x'}) \cap B_{x'}) dx' = \int_{\mathbb{R}^{n-k}} dx' \int_{\partial^*(E_{x'})} \chi_B(x', y) d\mathcal{H}^{k-1}(y). \end{aligned} \quad (2.4)$$

On the other hand, by Proposition 2.2

$$\int_{\partial^* E} \chi_B(x) |\nu_y^E(x)| d\mathcal{H}^{n-1}(x) = \int_{\mathbb{R}^{n-k}} dx' \int_{(\partial^* E)_{x'}} \chi_B(x', y) d\mathcal{H}^{k-1}(y). \quad (2.5)$$

Let $\{C^h\}_{h \in \mathbb{N}}$ be a countable base for the σ -algebra of Borel in \mathbb{R}^k and A any Borel set in \mathbb{R}^{n-k} . By comparing (2.4) and (2.5) with $B = A \times C^h$, for every $h \in \mathbb{N}$ and for \mathcal{L}^{n-k} -a.e. $x' \in \mathbb{R}^{n-k}$ we get

$$\int_{\partial^*(E_{x'})} \chi_{C^h}(y) d\mathcal{H}^{k-1}(y) = \int_{(\partial^* E)_{x'}} \chi_{C^h}(y) d\mathcal{H}^{k-1}(y),$$

and then the arbitrariness of C^h immediately implies (ii).

By applying (2.4) to the Borel set $Z := \{x \in \partial^* E : \nu_y^E(x) = 0\}$, taking into account (ii), we get

$$\int_{\mathbb{R}^{n-k}} dx' \int_{\partial^*(E_{x'})} \chi_Z(x', y) d\mathcal{H}^{k-1}(y) = \int_{\mathbb{R}^{n-k}} dx' \int_{(\partial^* E)_{x'}} \chi_Z(x', y) d\mathcal{H}^{k-1}(y) = 0.$$

Hence, $\mathcal{H}^{k-1}(Z_{x'}) = 0$ for \mathcal{L}^{n-k} -a.e. $x' \in \mathbb{R}^{n-k}$, that is (iiia).

To show (iiib), we first prove that

$$D_y \chi_E = \mathcal{L}^{n-k} \otimes D\chi_{E_{x'}}. \quad (2.6)$$

Let $\phi \in C_c^1(\mathbb{R}^n)$. Using Fubini Theorem and integrating by parts,

$$\begin{aligned} \int_{\mathbb{R}^n} \phi(x) dD_y \chi_E(x) &= - \int_{\mathbb{R}^n} \nabla_y \phi(x', y) \chi_E(x', y) dx \\ &= - \int_{\mathbb{R}^{n-k}} dx' \int_{\mathbb{R}^k} \chi_{E_{x'}}(y) \nabla_y \phi(x', y) dy = \int_{\mathbb{R}^{n-k}} dx' \int_{\mathbb{R}^k} \phi(x', y) dD\chi_{E_{x'}}(y). \end{aligned}$$

This equality leads to (2.6).

Now, by Proposition 2.2, condition (ii), and relation (2.6), for any Borel set $B \subset \mathbb{R}^n$

$$\begin{aligned} \int_{\mathbb{R}^{n-k}} dx' \int_{\partial^*(E_{x'})} \chi_B(x', y) \chi_{\mathbb{R}^k \setminus Z_{x'}}(y) \nu^{E_{x'}}(y) d\mathcal{H}^{k-1}(y) \\ = \int_{\mathbb{R}^{n-k}} D\chi_{E_{x'}}((B \setminus Z)_{x'}) dx' = D_y \chi_E(B \setminus Z) \\ = \int_{\partial^* E} \chi_{B \setminus Z}(x', y) \nu_y^E(x', y) d\mathcal{H}^{n-1}(x', y) \\ = \int_{\mathbb{R}^{n-k}} dx' \int_{\partial^*(E_{x'})} \chi_B(x', y) \chi_{\mathbb{R}^k \setminus Z_{x'}}(y) \frac{\nu_y^E(x', y)}{|\nu_y^E(x', y)|} d\mathcal{H}^{k-1}(y). \end{aligned}$$

Again by the arbitrariness of B , (iiib) follows. \square

Remark 2.5. Since E^S is symmetric about \mathbb{R}^{n-k} , it is easily checked from the definition of reduced boundary that if $(x', y) \in \partial^* E^S$ and $R : \mathbb{R}^k \rightarrow \mathbb{R}^k$ is a rotation, then $(x', Ry) \in \partial^* E^S$ and

$$\nu_{x'}^{E^S}(x', Ry) = \nu_{x'}^{E^S}(x', y), \quad \nu_y^{E^S}(x', Ry) = R(\nu_y^{E^S}(x', y)).$$

Therefore, (i), (ii) and (iii) of Theorem 2.4 hold in a stronger form: for every $x' \in G_{E^S}$

- (i)_S $E_{x'}^S$ is a k -dimensional ball;
- (ii)_S $\partial(E_{x'}^S) = (\partial^* E^S)_{x'}$;
- (iii)_S For every y such that $(x', y) \in \partial(E_{x'}^S)$:
 - (a) $\nu_y^{E^S}(x', y) \neq 0$;
 - (b) $\nu_y^{E^S}(x', y) = \nu_{x'}^{E^S}(y) |\nu_y^{E^S}(x', y)|$.

Moreover,

$$\mathcal{L}^{n-k}(B_0) = 0, \quad (2.7)$$

where $B_0 := \{x' \in \mathbb{R}^{n-k} : \exists y \in \mathbb{R}^k \text{ s. t. } (x', y) \in \partial^* E^S \text{ and } \nu_y^{E^S}(x', y) = 0\}$.

In view of the above theorem, for every set of finite perimeter E we will use the same notation $\partial^* E_{x'}$ to indicate the sets $(\partial^* E)_{x'}$ and $\partial^*(E_{x'})$ when they coincide modulo \mathcal{H}^{k-1} . Occasionally, we will write $p_E(x') := \mathcal{H}^{k-1}(\partial^* E_{x'})$ to denote their common measure.

The following result is a straightforward variant of [1, Lemma 2.35].

Lemma 2.6. *Let $B \subset \mathbb{R}^n$ be a Borel set, and let $\varphi_h, \varphi : B \rightarrow \mathbb{R}$, $h \in \mathbb{N}$ be summable Borel functions such that $|\varphi_h| \leq |\varphi|$ for every h . Then*

$$\int_B \sup_h \varphi_h dx = \sup \left\{ \sum_{h \in H} \int_{A_h} \varphi_h dx \right\},$$

where the supremum ranges over all finite sets $H \subset \mathbb{N}$ and all finite partitions $\{A_h\}_{h \in \mathbb{N}}$ of B in Borel sets.

3. PROPERTIES OF THE FUNCTION L

This section is a collection of several properties of the function L , which will be used to prove Theorem 1.1. We recall that for every set $E \subset \mathbb{R}^n$, $L : \mathbb{R}^{n-k} \rightarrow [0, \infty]$ is defined as

$$L(x') := \mathcal{L}^k(E_{x'}), \quad \text{for every } x' \in \mathbb{R}^{n-k}.$$

The first important property is that when E is a set of finite perimeter in \mathbb{R}^n , then either E^S is equivalent to \mathbb{R}^n , or L is a function of bounded variation in \mathbb{R}^{n-k} . When $L \in BV(\mathbb{R}^n)$, the measure $DL|_{G_{ES}}$ is absolutely continuous with respect to \mathcal{L}^{n-k} , and it is possible to provide the explicit expression of its corresponding density.

Lemma 3.1. *Let E be any set of finite perimeter in \mathbb{R}^n . Then either $L(x') = \infty$ for \mathcal{L}^{n-k} -a.e. $x' \in \mathbb{R}^{n-k}$, or $L(x') < \infty$ for \mathcal{L}^{n-k} -a.e. $x' \in \mathbb{R}^{n-k}$ and $\mathcal{L}^n(E) < \infty$. Moreover, in the latter case, $L \in BV(\mathbb{R}^{n-k})$ and*

$$|DL|(B) \leq P(E; B \times \mathbb{R}^k) \quad \text{for every Borel set } B \subset \mathbb{R}^{n-k}. \quad (3.1)$$

In addition, for any Borel set $B \subset \mathbb{R}^{n-k}$

$$\begin{aligned} DL(B) &= \int_{\partial^* E \cap (B \times \mathbb{R}^k) \cap \{\nu_y^E = 0\}} \nu_{x'}^E(x', y) d\mathcal{H}^{n-1}(x) \\ &\quad + \int_B dx' \int_{(\partial^* E)_{x'} \cap \{\nu_y^E \neq 0\}} \frac{\nu_{x'}^E(x', y)}{|\nu_y^E(x', y)|} d\mathcal{H}^{k-1}(y), \end{aligned} \quad (3.2)$$

$DL|_{G_{ES}} = \nabla L \mathcal{L}^{n-k}$ and for \mathcal{L}^{n-k} -a.e. $x' \in G_{ES}$

$$\nabla L(x') = \mathcal{H}^{k-1}(\partial^* E_{x'}^S) \frac{\nu_{x'}^{ES}(x')}{|\nu_y^{ES}(x')|} = k \omega_k r^{k-1}(x') \frac{\nu_{x'}^{ES}(x')}{|\nu_y^{ES}(x')|}, \quad (3.3)$$

where we dropped the variable y for functions that are constant in $\partial^* E_{x'}^S$.

Proof. By arguing as in [3, Lemma 3.1] we obtain the first assertion and (3.1), while by arguing as in [3, Lemma 3.2] we have that for any Borel set $B \subset \mathbb{R}^{n-k}$

$$DL(B) = \int_{\partial^* E \cap (B \times \mathbb{R}^k)} \nu_{x'}^E(x', y) d\mathcal{H}^{n-1}(x).$$

Then, formula (3.2) is easily obtained by splitting the integral on the right-hand side of the equation above in an integral over the set $\partial^* E \cap (B \times \mathbb{R}^k) \cap \{\nu_y^E = 0\}$ and an integral over the remaining set $\partial^* E \cap (B \times \mathbb{R}^k) \cap \{\nu_y^E \neq 0\}$. The latter integral is then evaluated using coarea formula.

Finally, from Remark 2.5 it follows that for all $x' \in G_{ES}$ we have

$$\nu_{x'}^{ES}(x', y) \equiv \nu_{x'}^{ES}(x'), \quad |\nu_y^{ES}(x', y)| \equiv |\nu_y^{ES}(x')| > 0,$$

for all $y \in \partial E_{x'}^S$. Therefore, $\partial^* E^S \cap (G_{ES} \times \mathbb{R}^k) \cap \{\nu_y^{ES} = 0\} = \emptyset$, and from (3.2)

$$DL(B) = \int_B \mathcal{H}^{k-1}(\partial^* E_{x'}^S) \frac{\nu_{x'}^{ES}(x')}{|\nu_y^{ES}(x')|} dx', \quad \text{for all Borel sets } B \subset G_{ES}.$$

Hence the absolutely continuous of $DL|_{G_{ES}}$ and (3.3). Since by Theorem 2.4 $L(x') = 0$ for \mathcal{L}^{n-k} -a.e. $x' \in \mathbb{R}^{n-k} \setminus G_{ES}$, $\nabla L \equiv 0$ in $\mathbb{R}^{n-k} \setminus G_{ES}$, thus proving $DL|_{G_{ES}} = \nabla L \mathcal{L}^{n-k}$. \square

Remark 3.2. As hinted in the Introduction, if $k > 1$ the measure

$$\mu(B) = \int_{\partial^* E \cap (B \times \mathbb{R}^k) \cap \{\nu_y^E = 0\}} \nu_{x'}^E(x) d\mathcal{H}^{n-1}(x) \quad (3.4)$$

may contain a non-vanishing absolutely continuous part. As an example, consider the special case where $n \geq 3$ and $E = \{(x', y) \in \mathbb{R} \times \mathbb{R}^{n-1} : 0 < x' < u(y)\}$, with $u : \mathbb{R}^{n-1} \rightarrow [0, +\infty)$ a C^1 function with compact support. Then, the Steiner symmetral E^S of codimension $n-1$ of E is $E^S = \{(x', y) \in \mathbb{R} \times \mathbb{R}^{n-1} : 0 < x' < u^\#(y)\}$, where $u^\#$ is the decreasing rearrangement of u (see [4, Section 2.1]). In this case, the measure μ in (3.4) reduces to

$$\mu(B) = \mathcal{L}^{n-1}(u^{-1}(B) \cap \{\nabla u = 0\}). \quad (3.5)$$

A surprising example given in [2, Section 5.1] shows that, for every $0 < \alpha < 1$, one can always find a $C^{n-2, \alpha}$ function u with compact support in the unit cube Q such that the measure in (3.5) is absolutely continuous with $\mu(Q)$ arbitrarily close to 1. Interestingly, if $u \in C^{n-2, 1}$, then μ is purely singular [2, Theorem 5.2].

Observe that, if $k = 1$, the measure μ in (3.4) is purely singular. In fact, by Vol'pert Theorem, for \mathcal{L}^{n-1} -a.e. $x' \in \mathbb{R}^{n-1}$, we have that $\nu_y^E(x', y) \neq 0$ for all y such that $(x', y) \in \partial^* E$. Therefore, the projection of $\{x \in \partial^* E : \nu_y^E(x) = 0\}$ on \mathbb{R}^{n-1} has zero Lebesgue measure.

The following lemma shows that if E has finite perimeter, the same is true for E^S . Moreover, it provides a first estimate of $P(E^S)$.

Lemma 3.3. *Let E be any set of finite perimeter in \mathbb{R}^n having finite measure. Then, also E^S has finite perimeter and*

$$P(E^S; B \times \mathbb{R}^k) \leq |DL|(B) + |D_y \chi_{E^S}|(B \times \mathbb{R}^k) \quad (3.6)$$

for every Borel set $B \subset \mathbb{R}^{n-k}$.

Proof. The proof is based on the same argument of [3, Lemma 3.5]. Consider a sequence $\{L_j\}_{j \in \mathbb{N}}$ of non-negative functions belonging to $C_c^1(\mathbb{R}^{n-k})$ such that $L_j \rightarrow L$ \mathcal{L}^{n-k} -a.e. in \mathbb{R}^{n-k} and $|DL_j|(\mathbb{R}^{n-k}) \rightarrow |DL|(\mathbb{R}^{n-k})$. Subsequently, denote with E_j^S and r_j the set and the function defined as E^S and r , respectively, with L_j in place of L . Let $\Omega \subset \mathbb{R}^{n-k}$ be an open set and let $\phi = (\phi_1, \dots, \phi_n) \in C_c^1(\Omega \times \mathbb{R}^k, \mathbb{R}^n)$. First of all, let us show that for every $j \in \mathbb{N}$ we have

$$\sum_{i=1}^{n-k} \int_{\Omega \times \mathbb{R}^k} \chi_{(E_j^S)_{x'}}(y) \frac{\partial \phi_i}{\partial x_i}(x', y) dx \leq |DL_j|(\pi(\text{supp } \phi)), \quad (3.7)$$

whenever $\|\phi\|_\infty \leq 1$. Define the function $V \in C_c^1(\Omega; \mathbb{R}^{n-k})$ as

$$V_i(x') := \int_{B_{r_j(x')}^k} \phi_i(x', y) dy, \quad i = 1, \dots, n-k.$$

By differentiating the latter with respect to x_i and using spherical coordinates (ρ, σ) ,

$$\begin{aligned}
\frac{\partial V_i}{\partial x_i}(x') &= \frac{\partial}{\partial x_i} \int_0^{r_j(x')} \left[\int_{\mathbb{S}^{k-1}} \phi_i(x', \rho \sigma) d\sigma \right] \rho^{k-1} d\rho \\
&= [r_j(x')]^{k-1} \frac{\partial r_j}{\partial x_i}(x') \int_{\mathbb{S}^{k-1}} \phi_i(x', r_j(x') \sigma) d\sigma \\
&\quad + \int_0^{r_j(x')} \rho^{k-1} \frac{\partial}{\partial x_i} \left[\int_{\mathbb{S}^{k-1}} \phi_i(x', \rho \sigma) d\sigma \right] d\rho \\
&= \frac{\partial L_j}{\partial x_i}(x') \frac{1}{k \omega_k} \int_{\mathbb{S}^{k-1}} \phi_i(x', r_j(x') \sigma) d\sigma + \int_{B_{r_j(x')}^k} \frac{\partial \phi_i}{\partial x_i}(x', y) dy,
\end{aligned} \tag{3.8}$$

since $r_j^k = L_j(x')/\omega_k$. By applying the classical divergence theorem, $\int_{\Omega} \operatorname{div} V dx' = 0$. Thus, adding relation (3.8) for $i = 1, \dots, n-k$, and integrating over Ω

$$\begin{aligned}
&\sum_{i=1}^{n-k} \int_{\Omega \times \mathbb{R}^k} \chi_{(E_j^S)_{x'}}(y) \frac{\partial \phi_i}{\partial x_i}(x', y) dx \\
&= - \int_{\pi(\operatorname{supp} \phi)} \sum_{i=1}^{n-k} \frac{\partial L_j}{\partial x_i}(x') \frac{1}{k \omega_k} \left[\int_{\mathbb{S}^{k-1}} \phi_i(x', r_j(x') \sigma) d\sigma \right] dx' \\
&= - \int_{\pi(\operatorname{supp} \phi)} \frac{1}{k \omega_k} \left[\int_{\mathbb{S}^{k-1}} \sum_{i=1}^{n-k} \frac{\partial L_j}{\partial x_i}(x') \phi_i(x', r_j(x') \sigma) d\sigma \right] dx' \\
&\leq \int_{\pi(\operatorname{supp} \phi)} \frac{|\nabla L_j(x')|}{k \omega_k} \left[\int_{\mathbb{S}^{k-1}} \|\phi(x', r_j(x') \sigma)\|_{L^\infty} d\sigma \right] dx'.
\end{aligned}$$

If $\|\phi\|_\infty \leq 1$, (3.7) is obtained. Hence,

$$\begin{aligned}
\int_{\Omega \times \mathbb{R}^k} \chi_{E_j^S} \operatorname{div} \phi dx &= \int_{\Omega} \left[\int_{\mathbb{R}^k} \chi_{(E_j^S)_{x'}} \operatorname{div}_{x'} \phi_{x'} dy \right] dx' + \int_{\Omega \times \mathbb{R}^k} \chi_{E_j^S} \operatorname{div}_y \phi_y dx \\
&\leq |DL_j|(\pi(\operatorname{supp} \phi)) + \int_{\Omega \times \mathbb{R}^k} \chi_{E_j^S} \operatorname{div}_y \phi_y dx.
\end{aligned} \tag{3.9}$$

Integrating by parts the following applies:

$$\int_{\Omega \times \mathbb{R}^k} \chi_{E_j^S} \operatorname{div}_y \phi_y dx = \int_{\Omega} dx' \int_{\partial B_{r_j(x')}^k} \phi \cdot \frac{y}{r_j(x')} d\mathcal{H}^{k-1}(y) \leq \int_{\Omega} \mathcal{H}^{k-1}(\partial B_{r_j(x')}^k) dx'.$$

Then, using isoperimetric inequality and coarea formula,

$$\limsup_j \left| \int_{\Omega \times \mathbb{R}^k} \chi_{E_j^S} \operatorname{div}_y \phi_y dx \right| \leq \int_{\Omega} \mathcal{H}^{k-1}(\partial^* B_{r(x')}^k) dx' \leq \int_{\Omega} \mathcal{H}^{k-1}(\partial^* E_{x'}) dx' \leq P(E).$$

Moreover, since $\chi_{E_j^S} \rightarrow \chi_{E^S}$ \mathcal{L}^n -a.e. by the definition of E_j^S , and $\pi(\operatorname{supp} \phi)$ is a compact subset of Ω , passing to the limsup in (3.9) as $j \rightarrow \infty$ we have

$$\int_{\Omega \times \mathbb{R}^k} \chi_{E^S} \operatorname{div} \phi dx \leq |DL|(\operatorname{supp} \phi) + P(E), \tag{3.10}$$

thus proving that E^S has finite perimeter. Having obtained (3.10), we can go back to (3.9) and, passing to the limit again, we conclude that

$$\begin{aligned} \int_{\Omega \times \mathbb{R}^k} \chi_{E^S} \operatorname{div} \phi \, dx &\leq |DL|(\operatorname{supp} \phi) + \int_{\Omega \times \mathbb{R}^k} \chi_{E^S} \operatorname{div}_y \phi_y \, dx \\ &\leq |DL|(\Omega) + |D_y \chi_{E^S}|(\Omega \times \mathbb{R}^k). \end{aligned}$$

From the last inequality, we infer that (3.6) holds whenever B is an open set, and therefore, when B is any Borel set. \square

Proof of Theorem 1.1. If $L = \infty$ \mathcal{L}^{n-k} -a.e. in \mathbb{R}^{n-k} , then E^S is equivalent to \mathbb{R}^n ; it follows that $P(E^S; B \times \mathbb{R}^k) = 0$ for every Borel set $B \subset \mathbb{R}^{n-k}$ and (1.1) is trivially satisfied. If not, by Lemma 3.1, we may assume that $L < \infty$ \mathcal{L}^{n-k} -a.e. in \mathbb{R}^{n-k} . Let G_{E^S} be the set associated with E^S , as in Theorem 2.4, and let B be a Borel subset of \mathbb{R}^{n-k} . We shall prove inequality (1.1) when either $B \subset \mathbb{R}^{n-k} \setminus G_{E^S}$, or $B \subset G_{E^S}$. The general case is obtained by splitting B into $B \setminus G_{E^S}$ and $B \cap G_{E^S}$.

Firstly assume that $B \subset \mathbb{R}^{n-k} \setminus G_{E^S}$. Observe that, by (2.1), Proposition 2.2 and Theorem 2.4(ii),

$$|D_y \chi_{E^S}|(B \times \mathbb{R}^k) = \int_{\partial^* E^S \cap (B \times \mathbb{R}^k)} |\nu_y^{E^S}| \, d\mathcal{H}^{n-1}(x) = \int_B \mathcal{H}^{k-1}(\partial^* E_{x'}^S) \, dx'.$$

Since $\mathcal{L}^{n-k}(\pi(E)^+ \cap B) = \mathcal{L}^{n-k}(G_{E^S} \cap B) = 0$, the last integral equals

$$\int_{(\mathbb{R}^{n-k} \setminus \pi(E)^+) \cap B} \mathcal{H}^{k-1}(\partial^* E_{x'}^S) \, dx',$$

and vanishes. Therefore, by (3.1) and (3.6), we obtain (1.1):

$$P(E^S; B \times \mathbb{R}^k) \leq |DL|(B) \leq P(E; B \times \mathbb{R}^k). \quad (3.11)$$

Suppose now that $B \subset G_{E^S}$. Since $\mathcal{L}^{n-k}(G_E \Delta G_{E^S}) = 0$ and $\nu_y^E(x', y) \neq 0$ \mathcal{H}^{k-1} -a.e. in $\partial^* E_{x'}$ for all $x' \in G_E$, we have

$$\begin{aligned} P(E; B \times \mathbb{R}^k) &= P(E; (B \times \mathbb{R}^k) \cap \{\nu_y^E = 0\}) + P(E; (B \times \mathbb{R}^k) \cap \{\nu_y^E \neq 0\}) \\ &= P(E; (B \times \mathbb{R}^k) \cap \{\nu_y^E = 0\}) + \int_{\partial^* E \cap (B \times \mathbb{R}^k) \cap \{\nu_y^E \neq 0\}} d\mathcal{H}^{n-1}(x) \\ &= P(E; (B \times \mathbb{R}^k) \cap \{\nu_y^E = 0\}) + \int_B dx' \int_{\partial^* E_{x'}} \frac{1}{|\nu_y^E(x', y)|} d\mathcal{H}^{k-1}(y) \\ &= P(E; (B \times \mathbb{R}^k) \cap \{\nu_y^E = 0\}) + \int_B dx' \int_{\partial^* E_{x'}} \sqrt{1 + \sum_{i=1}^{n-k} \left(\frac{\nu_i^E(x', y)}{|\nu_y^E(x', y)|} \right)^2} d\mathcal{H}^{k-1}(y), \end{aligned}$$

where the second equality is due to (2.2), the third one to Proposition 2.2 applied to the function $\chi_{B \times \mathbb{R}^k}(x)/|\nu_y^E(x)|$, and the fourth one to the fact that ν^E is a unit vector. Now, applying Jensen's inequality to the strictly convex function

$$f(z) := \sqrt{1 + |z|^2}, \quad z \in \mathbb{R}^{n-k}, \quad (3.12)$$

we get

$$\begin{aligned}
P(E; B \times \mathbb{R}^k) &\geq P(E; (B \times \mathbb{R}^k) \cap \{\nu_y^E = 0\}) \\
&+ \int_B \sqrt{\left(\int_{\partial^* E_{x'}} d\mathcal{H}^{k-1} \right)^2 + \sum_{i=1}^{n-k} \left(\int_{\partial^* E_{x'}} \frac{\nu_i^E(x', y)}{|\nu_y^E(x', y)|} d\mathcal{H}^{k-1}(y) \right)^2} dx' \quad (3.13) \\
&= P(E; (B \times \mathbb{R}^k) \cap \{\nu_y^E = 0\}) + \int_B \sqrt{p_E^2(x') + |g(x')|^2} dx',
\end{aligned}$$

where we have set, for \mathcal{L}^{n-k} -a.e. $x' \in B$,

$$g(x') := \int_{\partial^* E_{x'}} \frac{\nu_{x'}^E(x', y)}{|\nu_y^E(x', y)|} d\mathcal{H}^{k-1}(y).$$

Note that since $B \subset G_{ES}$, from Lemma 3.1 it follows that $DL \llcorner B$ is absolutely continuous. We claim that

$$\int_B \sqrt{p_E^2(x') + |\nabla L(x')|^2} dx' \leq P(E; (B \times \mathbb{R}^k) \cap \{\nu_y^E = 0\}) + \int_B \sqrt{p_E^2(x') + |g(x')|^2} dx'. \quad (3.14)$$

To prove this claim note that, by duality, we can write the function f in (3.12) as

$$f(z) = \sqrt{1 + |z|^2} = \sup_h \left\{ z \cdot w_h + \sqrt{1 - |w_h|^2} \right\}, \quad z \in \mathbb{R}^{n-k},$$

where $\{w_h\}_h$ is a countable dense set in B_1^{n-k} . Let now $\{A_h\}_{h \in H}$ be a finite partition of Borel sets of B . Recalling (3.2), we have

$$\begin{aligned}
&\sum_{h \in H} \int_{A_h} \left(\nabla L(x') \cdot w_h + p_E(x') \sqrt{1 - |w_h|^2} \right) dx' \\
&= \sum_{h \in H} \left(\int_{\partial^* E \cap (A_h \times \mathbb{R}^k) \cap \{\nu_y^E = 0\}} w_h \cdot \nu_{x'}^E(x', y) d\mathcal{H}^{n-1}(x) + \int_{A_h} \left(g(x') \cdot w_h + p_E(x') \sqrt{1 - |w_h|^2} \right) dx' \right) \\
&\leq \sum_{h \in H} \left(\int_{\partial^* E \cap (A_h \times \mathbb{R}^k) \cap \{\nu_y^E = 0\}} |\nu_{x'}^E(x', y)| d\mathcal{H}^{n-1}(x) + \int_{A_h} p_E(x') \left(\frac{g(x')}{p_E(x')} \cdot w_h + \sqrt{1 - |w_h|^2} \right) dx' \right) \\
&\leq \sum_{h \in H} \left(P(E; (A_h \times \mathbb{R}^k) \cap \{\nu_y^E = 0\}) + \int_{A_h} p_E(x') \left(\sqrt{1 + \frac{|g(x')|^2}{p_E^2(x')}} \right) dx' \right) \\
&= P(E; (B \times \mathbb{R}^k) \cap \{\nu_y^E = 0\}) + \int_B \sqrt{p_E^2(x') + |g(x')|^2} dx'.
\end{aligned}$$

From the previous inequality, applying Lemma 2.6 to the functions

$$\varphi_h(x') = p_E(x') \left(\frac{\nabla L(x')}{p_E(x')} \cdot w_h + \sqrt{1 - |w_h|^2} \right)$$

we have (3.14). Note that when $E = E^S$, Remark 2.5 and Lemma 3.1 lead to

$$\begin{aligned}
P(E^S; B \times \mathbb{R}^k) &= \int_B dx' \int_{\partial^* E_{x'}^S} \sqrt{1 + \sum_{i=1}^{n-k} \left(\frac{\nu_i^{E^S}(x', y)}{|\nu_y^{E^S}(x', y)|} \right)^2} d\mathcal{H}^{k-1}(y) \\
&= \int_B dx' \int_{\partial^* E_{x'}^S} \sqrt{1 + \frac{|\nabla L(x')|^2}{|\mathcal{H}^{k-1}(\partial B_{r(x')})|^2}} d\mathcal{H}^{k-1}(y) \\
&= \int_B \sqrt{p_{E^S}^2(x') + |\nabla L(x')|^2} dx'.
\end{aligned} \tag{3.15}$$

Owing to the isoperimetric inequality in \mathbb{R}^k , we have $p_{E^S}(x') \leq p_E(x')$ for \mathcal{L}^{n-k} -a.e. $x' \in \mathbb{R}^{n-k}$. Hence, combining (3.13), (3.14) and (3.15)

$$\begin{aligned}
P(E^S; B \times \mathbb{R}^k) &\leq \int_B \sqrt{p_E^2(x') + |\nabla L(x')|^2} dx' \\
&\leq P(E; (B \times \mathbb{R}^k) \cap \{\nu_y^E = 0\}) + \int_B \sqrt{p_E^2(x') + |g(x')|^2} dx' \\
&\leq P(E; B \times \mathbb{R}^k),
\end{aligned}$$

thus proving (1.1).

Now, moving on to the case of equality. If $P(E) = P(E^S)$, then inequality (1.1) implies

$$P(E^S; B \times \mathbb{R}^k) = P(E; B \times \mathbb{R}^k),$$

for every Borel set $B \subset \mathbb{R}^{n-k}$. On the other hand, as shown above, by taking $B = G_{E^S}$, we have

$$\begin{aligned}
P(E^S; G_{E^S} \times \mathbb{R}^k) &= \int_{G_{E^S}} \sqrt{p_{E^S}(x')^2 + |\nabla L(x')|^2} dx' \\
&\leq \int_{G_{E^S}} \sqrt{p_E^2(x') + |\nabla L(x')|^2} dx' \\
&\leq P(E; (G_{E^S} \times \mathbb{R}^k) \cap \{\nu_y^E = 0\}) + \int_{G_{E^S}} \sqrt{p_E^2(x') + |g(x')|^2} dx' \\
&\leq P(E; G_{E^S} \times \mathbb{R}^k).
\end{aligned}$$

All inequalities in the previous chain must therefore hold as equalities. The former of these inequalities entails that $p_E(x') = p_{E^S}(x')$ for \mathcal{L}^{n-k} -a.e. $x' \in \mathbb{R}^{n-k}$, thereby implying that $E_{x'}$ is equivalent to a k -dimensional ball of radius $r(x')$.

The fact that the second inequality as well must hold as an equality, implies that the Jensen's inequality in (3.13) holds as an equality too. By the strict convexity of the function in (3.12), this also implies that for \mathcal{L}^{n-k} -a.e. $x' \in G_{E^S}$

$$y \mapsto \frac{\nu_{x'}^E(x', y)}{|\nu_y^E(x', y)|} \text{ is } \mathcal{H}^{k-1}\text{-a.e. constant in } \partial^* E_{x'}.$$

Since ν^E is a unit vector, we also have that for \mathcal{L}^{n-k} -a.e. $x' \in G_{E^S}$

$$y \mapsto |\nu_y^E(x', y)| \text{ is } \mathcal{H}^{k-1}\text{-a.e. constant in } \partial^* E_{x'},$$

so that (b) follows. \square

An inspection of the above proof leads to the following result which, regardless of its relation to Theorem 1.1, proves to be of some interest in itself.

Proposition 3.4. *Let $E \subset \mathbb{R}^n$ be a set of finite measure and perimeter in \mathbb{R}^n , and let $f : \mathbb{R}^{n-k} \rightarrow [0, \infty]$ be a Borel function. Then,*

$$\int_{\partial^* E} f(x') d\mathcal{H}^{n-1}(x) \geq \int_{\mathbb{R}^{n-k}} f(x') \sqrt{p_E(x')^2 + |\nabla L(x')|^2} dx' + \int_{\mathbb{R}^{n-k}} f(x') d|D^s L|(x'), \quad (3.16)$$

with equality if $E = E^S$.

Proof. It is enough to show (3.16) in the case $f = \chi_B$, with $B \subset \mathbb{R}^{n-k}$ arbitrary Borel set.

In the case $B \subset \mathbb{R}^{n-k} \setminus G_{E^S}$, thanks to Lemma 3.1

$$|DL|(B) = |D^s L|(B). \quad (3.17)$$

Therefore, (3.16) follows from (3.1), observing that p_E vanishes \mathcal{L}^{n-k} -a.e. in B . Conversely, when $B \subset G_{E^S}$, by Lemma 3.1 $|D^s L|(B) = 0$, and (3.16) follows from (3.13) and (3.14).

In the case $E = E^S$, the opposite of inequality (3.16) follows from (3.15), when $B \subset G_{E^S}$, and from (3.11) and (3.17), when $B \subset \mathbb{R}^{n-k} \setminus G_{E^S}$. \square

We conclude this section by providing two additional results related to conditions (1.2) and (1.3), which extend [3, Propositions 1.2 and 4.2], respectively. The proof of Proposition 3.5 is obtained with arguments similar to the ones used in [3]. However, the proof of Proposition 3.6 requires a different idea, due to the phenomena occurring when $k > 1$ and described in the Introduction.

Proposition 3.5. *Let Ω be an open subset of \mathbb{R}^{n-k} and let E be any set of finite perimeter in \mathbb{R}^n , with $\mathcal{L}^n(E \cap (\Omega \times \mathbb{R}^k)) < \infty$. Then, the following conditions are equivalent:*

- (i) $\mathcal{H}^{n-1}(\{x \in \partial^* E^S : \nu_y^{E^S}(x) = 0\} \cap (\Omega \times \mathbb{R}^k)) = 0$;
- (ii) $L \in W^{1,1}(\Omega)$;
- (iii) $P(E^S; B \times \mathbb{R}^k) = 0$ for every Borel set $B \subset \Omega$ such that $\mathcal{L}^{n-k}(B) = 0$.

Proof. (i) \implies (ii). It follows observing that, by (3.2), $DL(B) = 0$ if $B \subset \mathbb{R}^{n-k}$ is a Borel set with zero \mathcal{L}^{n-k} -measure.

(ii) \implies (iii). This is a consequence of the fact that for E^S the equality holds in (3.16).

(iii) \implies (i). From (2.7) it follows that

$$\mathcal{H}^{n-1}(\{x \in \partial^* E^S : \nu_y^{E^S}(x) = 0\} \cap (\Omega \times \mathbb{R}^k)) \leq P(E^S; B_0 \times \mathbb{R}^k) = 0.$$

\square

Proposition 3.6. *Let E and Ω be as in Proposition 3.5. If*

$$\mathcal{H}^{n-1}(\{x \in \partial^* E : \nu_y^E(x) = 0\} \cap (\Omega \times \mathbb{R}^k)) = 0, \quad (3.18)$$

Then

$$\mathcal{H}^{n-1}(\{x \in \partial^* E^S : \nu_y^{E^S}(x) = 0\} \cap (\Omega \times \mathbb{R}^k)) = 0. \quad (3.19)$$

Conversely, if E satisfies $P(E; \Omega \times \mathbb{R}^k) = P(E^S; \Omega \times \mathbb{R}^k)$ and (3.19) holds, then (3.18) does as well.

Proof. If (3.18) holds, then arguing as in the proof of the implication (i) \implies (ii) in Proposition 3.5, we have that $L \in W^{1,1}(\Omega)$. Therefore, (3.19) follows by Proposition 3.5 again.

Let us now show that (3.19) implies (3.18) when $P(E; \Omega \times \mathbb{R}^k) = P(E^S; \Omega \times \mathbb{R}^k)$. First, we recall that this implication is proven, in [3, Lemma 4.2], when $k = 1$. So, we have to deal only with the case $k > 1$.

To this aim, we start by proving that if $F \subset \mathbb{R}^n$ is a set of finite perimeter satisfying (1.3) and such that almost every section $F_{x'}$ is a k -dimensional ball, then

$$\mathcal{H}^{n-1}(\{x \in \partial^* F : \nu_{y_i}^F(x) = 0\} \cap (\Omega \times \mathbb{R}^k)) = 0, \quad i = 1, \dots, k. \quad (3.20)$$

Set $B^i := \{x \in \partial^* F : \nu_{y_i}^F(x) = 0\} \cap (\Omega \times \mathbb{R}^k)$. From Theorem 2.4, $B_{x'}^i = \{y \in \partial^* F_{x'} : \nu_{y_i}^{F_{x'}}(y) = 0\}$ up to a \mathcal{H}^{k-1} -negligible set. Since $F_{x'}$ is (equivalent to) a k -dimensional ball, $\mathcal{H}^{k-1}(B_{x'}^i) = 0$. From (1.3) and using the coarea formula, we have

$$\mathcal{H}^{n-1}(B^i) = \mathcal{H}^{n-1}(B^i \cap \{x \in \partial^* F : \nu_y^F(x) \neq 0\}) = \int_{\mathbb{R}} dx' \int_{\partial^* F_{x'} \cap B_{x'}^i} \frac{d\mathcal{H}^{k-1}(y)}{|\nu_y^F(x', y)|} = 0.$$

Let us set $E^{S_0} = E$ and for every $i = 1, \dots, k$, let us indicate by E^{S_i} the (1-codimensional) Steiner symmetral of $E^{S_{i-1}}$ with respect to the hyperplane $y_i = 0$. Clearly, E^{S_k} is equivalent to E^S in $\Omega \times \mathbb{R}^k$. By applying repeatedly Theorem 1.1 (for the 1-codimensional Steiner symmetrization) we get that

$$P(E^S; \Omega \times \mathbb{R}^k) = P(E^{S_{k-1}}; \Omega \times \mathbb{R}^k) = \dots = P(E^{S_1}; \Omega \times \mathbb{R}^k) = P(E; \Omega \times \mathbb{R}^k).$$

From assumption (3.19) and (3.20) $\mathcal{H}^{n-1}(\{x \in \partial^* E^S : \nu_{y_k}^{E^S}(x) = 0\} \cap (\Omega \times \mathbb{R}^k)) = 0$. Therefore, since the assertion is true for $k = 1$, we deduce that $\mathcal{H}^{n-1}(\{x \in \partial^* E^{S_{k-1}} : \nu_{y_k}^{E^{S_{k-1}}}(x) = 0\} \cap (\Omega \times \mathbb{R}^k)) = 0$ and, a fortiori, $\mathcal{H}^{n-1}(\{x \in \partial^* E^{S_{k-1}} : \nu_y^{E^{S_{k-1}}}(x) = 0\} \cap (\Omega \times \mathbb{R}^k)) = 0$. Iterating this argument (3.18) follows. \square

4. REGULARITY OF THE BARYCENTER OF THE SECTIONS

The next definition has an important role in describing the properties of a set of finite perimeter.

Definition 4.1. The *barycenter of the sections* of a set $E \subset \mathbb{R}^n$ is the function $b : \mathbb{R}^{n-k} \rightarrow \mathbb{R}^k$ defined as

$$b(x') := \begin{cases} \frac{1}{L(x')} \int_{E_{x'}} y \, dy & \text{if } 0 < L(x') < \infty \text{ and } |y| \in L^1(E_{x'}), \\ 0 & \text{otherwise.} \end{cases}$$

The following example shows that in general b is not summable, even if E is a set of finite perimeter.

Example 4.2. Let $n = 3, k = 2$. Consider the set $E \subset \mathbb{R}^3$ defined by

$$E = \{(x', y) \in \mathbb{R} \times \mathbb{R}^2 : |y - b(x')| < r(x'), \quad x' \in (-1, 1)\},$$

where $b(x') = (0, 1/|x'|)$ and $r(x') = |x'|^2$. Then,

$$\begin{aligned} P(E; (-1, 1) \times \mathbb{R}^2) &= 2 \int_0^1 r(x') \, dx' \int_0^{2\pi} \sqrt{1 + |b'(x') \cdot (\cos \theta, \sin \theta) + r'(x')|^2} \, d\theta \\ &= 2 \int_0^1 r(x') \, dx' \int_0^{2\pi} \sqrt{1 + \left| -\frac{1}{|x'|^2} \sin \theta + 2x' \right|^2} \, d\theta < \infty. \end{aligned}$$

Therefore, E is a set of finite perimeter in $(-1, 1) \times \mathbb{R}^2$. Nevertheless, $b \notin L_{loc}^1((-1, 1); \mathbb{R}^2)$.

The next result is the key point of the paper, and concerns the regularity of the barycenter. First, we introduce some notation. Given $i = 1, \dots, n - k$, for all x in \mathbb{R}^{n-k} we write $\hat{x}_i := (x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_{n-k})$. If $\Omega \subset \mathbb{R}^{n-k}$ is an open set, Ω_i denotes its projection on $x_i = 0$. Moreover, if f is a function defined in Ω , we set $f_{\hat{x}_i} := f|_{\Omega \cap R_{\hat{x}_i}}$, where $R_{\hat{x}_i}$ is the straight line

passing through \hat{x}_i and orthogonal to the hyperplane $x_i = 0$. In order to simplify the notation, we shall drop the subscript i when it is clear from the context.

Theorem 4.3. *Let $E \subset \mathbb{R}^n$ and let $\Omega \subset \mathbb{R}^{n-k}$ be an open set such that E has finite perimeter in $\Omega \times \mathbb{R}^k$, and $E_{x'}$ is equivalent to a k -dimensional ball for \mathcal{L}^{n-k} -a.e. $x' \in \Omega$. Assume that conditions (1.3) and (1.4) hold, and fix $i \in \{1, \dots, n-k\}$. Then, for \mathcal{H}^{n-k-1} -a.e. $\hat{x}_i \in \Omega_i$, $b_{\hat{x}_i} \in W_{\text{loc}}^{1,1}(\Omega \cap R_{\hat{x}_i}, \mathbb{R}^k)$ and for \mathcal{L}^1 -a.e. $x_i \in \Omega \cap R_{\hat{x}_i}$*

$$b'_{\hat{x}_i}(x_i) = \frac{1}{L_{\hat{x}_i}^*(x_i)} \int_{\partial^* E_{x'}} (y - b_{\hat{x}_i}(x_i)) \frac{\nu_i^E(x', y)}{|\nu_y^E(x', y)|} d\mathcal{H}^{k-1}(y). \quad (4.1)$$

In addition, when $k = 1$, $b \in W_{\text{loc}}^{1,1}(\Omega)$ and $\|\nabla b\|_{L^1(\Omega; \mathbb{R}^{n-1})} \leq P(E; \Omega \times \mathbb{R})/2$.

Proof. Since our argument is local, we may assume without loss of generality that Ω is bounded and that E has finite perimeter. Note that $L \in W^{1,1}(\Omega)$, as consequence of Propositions 3.5 and 3.6. We divide the proof in to several steps and we consider only direction $i = 1$, the other directions being analogous.

Step 1. Assume $k = n - 1$ and b locally bounded.

With no loss of generality, we may also assume that L coincides with its Lebesgue representative. By (1.4) L is a strictly positive, absolutely continuous function in Ω . Therefore, since b is locally bounded, for any interval I compactly contained in Ω the set $E \cap (I \times \mathbb{R}^{n-1})$ is essentially bounded.

Let us introduce the auxiliary function $f(x') := \int_{E_{x'}} y dy \in \mathbb{R}^{n-1}$. Let $\phi \in C_c^1(I)$, and let $\{\psi_j\}_{j \in \mathbb{N}}$ be a sequence in $C_c^1(\mathbb{R}^{n-1}, [0, 1])$ pointwise converging to 1. Using (2.1) and (1.3), by the dominated convergence theorem and coarea formula we obtain

$$\begin{aligned} - \int_I \phi'(x') f(x') dx' &= - \lim_{j \rightarrow \infty} \int_{I \times \mathbb{R}^{n-1}} y \phi'(x') \psi_j(y) \chi_E(x', y) dx' dy \\ &= \lim_{j \rightarrow \infty} \int_{I \times \mathbb{R}^{n-1}} y \phi(x') \psi_j(y) dD_1 \chi_E(x) = \int_{I \times \mathbb{R}^{n-1}} y \phi(x') dD_1 \chi_E(x) \\ &= \int_{\partial^* E \cap (I \times \mathbb{R}^{n-1})} y \phi(x') \nu_1^E(x', y) d\mathcal{H}^{n-1}(x) = \int_I \phi(x') dx' \int_{\partial^* E_{x'}} y \frac{\nu_1^E(x', y)}{|\nu_y^E(x', y)|} d\mathcal{H}^{n-2}(y). \end{aligned} \quad (4.2)$$

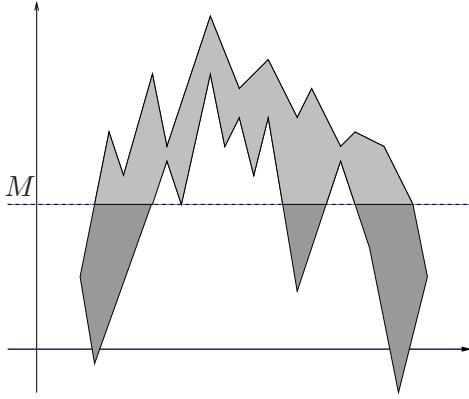
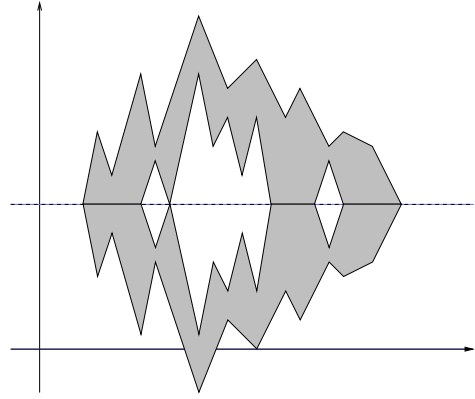
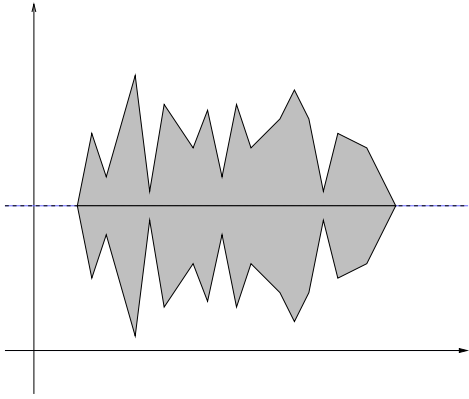
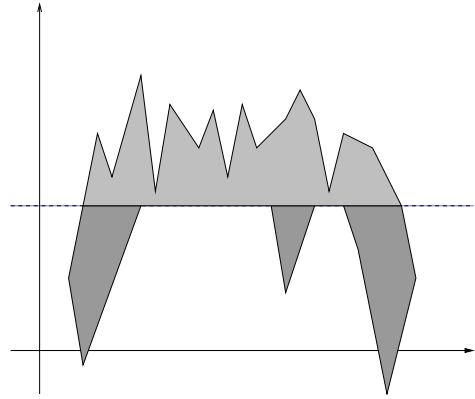
Being I and ϕ arbitrary, f belongs to $W_{\text{loc}}^{1,1}(\Omega; \mathbb{R}^{n-1})$.

Consequently, $b = f/L$ belongs to $W_{\text{loc}}^{1,1}(\Omega; \mathbb{R}^{n-1})$, and by (3.2) and (4.2) we get

$$\begin{aligned} b'(x') &= \left(\frac{1}{L} f \right)'(x') = - \frac{L'(x')}{|L(x')|^2} f(x') + \frac{1}{L(x')} \int_{\partial^* E_{x'}} y \frac{\nu_1^E(x', y)}{|\nu_y^E(x', y)|} d\mathcal{H}^{n-2}(y) \\ &= - \frac{b(x')}{L(x')} \int_{\partial^* E_{x'}} \frac{\nu_1^E(x', y)}{|\nu_y^E(x', y)|} d\mathcal{H}^{n-2}(y) + \frac{1}{L(x')} \int_{\partial^* E_{x'}} y \frac{\nu_1^E(x', y)}{|\nu_y^E(x', y)|} d\mathcal{H}^{n-2}(y) \\ &= \frac{1}{L(x')} \int_{\partial^* E_{x'}} (y - b(x')) \frac{\nu_1^E(x', y)}{|\nu_y^E(x', y)|} d\mathcal{H}^{n-2}(y). \end{aligned}$$

Step 2. In order to remove the boundedness condition on b , we shall first examine the case $n = 2$, $k = 1$.

Our strategy consists of shrinking the set E in a suitable way: given $M > 0$, we translate every segment $E_{x'}$ contained in the half plane $y > M$ (respect. in $y < -M$) until it touches the line $y = M$ (respect. $y = -M$). To be more precise, defining the truncation $b^M :=$

FIGURE 4. The sets E and F_+^M FIGURE 5. The set $F_+^M \cup \tilde{F}_+^M$ FIGURE 6. The set G FIGURE 7. The sets G_+^M and E^M

$(b \wedge (L/2 + M)) \vee (-L/2 - M)$, we set

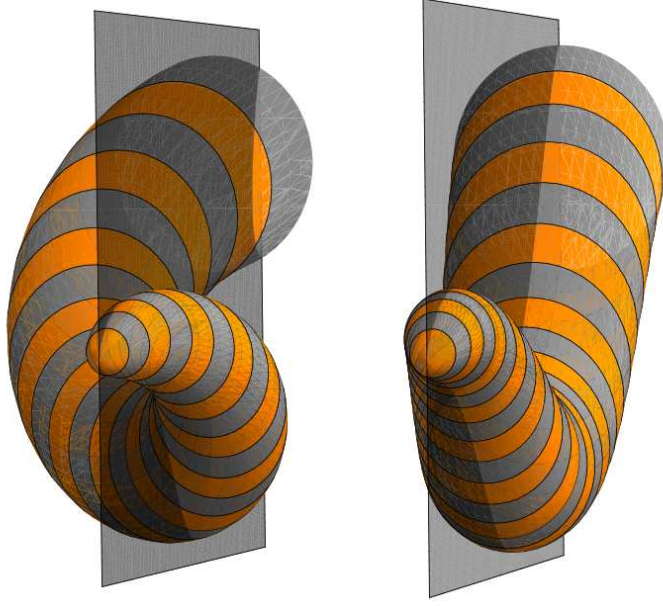
$$E^M := \bigcup_{x' \in \Omega} (b^M(x') - L(x')/2, b^M(x') + L(x')/2).$$

We want to show that E^M has finite perimeter and satisfies condition (1.3). This, thanks to Step 1, shall imply that $b^M \in W_{\text{loc}}^{1,1}(\Omega)$.

We set $F_+^M := \{(x', y) \in E : y > M\}$ (Figure 4) and consider its reflection with respect to the line $y = M$, i.e., $\tilde{F}_+^M := \{(x', y) \in \mathbb{R}^2 : (x', 2M - y) \in F_+^M\}$ (Figure 5). Let G be the set obtained by $F_+^M \cup \tilde{F}_+^M$ through a Steiner symmetrization with respect $y = M$ (Figure 6), and let $G_+^M := \{(x', y) \in G : y > M\}$ (Figure 7).

Observing that $(F_+^M \cup \tilde{F}_+^M)^{(1/2)} \cap \{y = M\} \subset E^{(1/4)}$ and that by Theorem 2.1 $\mathcal{H}^1(E^{(1/4)}) = 0$, we have $P(F_+^M \cup \tilde{F}_+^M) \leq 2P(E)$. Bearing in mind that the Steiner symmetrization decreases the perimeter, we get $P(G) \leq 2P(E)$. Since all sections $G_{x'}$ of G are segments, it is easy to verify that $\partial^* G_+^M \cap \{y = M\}$ is contained (up to an \mathcal{H}^1 -negligible set) in the projection of $\partial^* G \cap \{y > M\}$ on $\{y = M\}$. Therefore, $P(G_+^M) \leq P(G) \leq 2P(E)$.

Let us now repeat the same procedure on the set $F_-^M := \{(x', y) \in E : y < -M\}$ in a symmetric way with respect to the line $y = -M$, thus obtaining another set G_-^M such that

FIGURE 8. The sets E (left) and E^{S_1} (right)

$P(G_-^M) \leq 2P(E)$. Since by construction

$$E^M = (E \setminus (F_+^M \cup F_-^M)) \cup G_+^M \cup G_-^M,$$

we have that $P(E^M) \leq 5P(E)$ and condition (1.3) is still satisfied thanks to Proposition 3.6.

Define now the functions

$$h_{\pm}(x') := b(x') \pm L(x')/2 \quad \text{and} \quad h_{\pm}^M(x') := b^M(x') \pm L(x')/2.$$

From the Poincaré-Wirtinger inequality in $W^{1,1}(I)$

$$\int_I |h_{\pm}^M - \int_I h_{\pm}^M| dx' \leq c(I) \int_I |(h_{\pm}^M)'| dx'$$

and the bound $\int_I |(h_+^M)'| + |(h_-^M)'| dx' \leq P(E^M) \leq 5P(E)$, we have that, up to a (not relabeled) subsequence, $h_{\pm}^M - \int_I h_{\pm}^M$ converges in $BV(I)$ as $M \rightarrow \infty$. Considering that $h_{\pm}^M \rightarrow h_{\pm}$ pointwise, h_{\pm} belong to $BV(I)$, and so does b . Since b is locally bounded, by Step 1 we conclude that $b \in W_{\text{loc}}^{1,1}(\Omega)$ and (4.1) holds.

Step 3. Now we move to the case $n > 2$, $k = n - 1$. Here our strategy is to rearrange the set E , keeping fixed the $(n - 1)$ -th component of the barycenter, so that the $(n - 1)$ -dimensional balls $E_{x'}$ are all centered in the plane $y_1 = \dots = y_{n-2} = 0$.

We keep the same definition of E^{S_i} used in the proof of Proposition 3.6. A pictorial example of E^{S_1} is given in Figure 8. From (3.20) and Proposition 3.6 we get that $\mathcal{H}^{n-1}(\{x \in \partial^* E^{S_1} : \nu_{y_1}^{E^{S_1}} = 0\} \cap (\Omega \times \mathbb{R}^{n-1})) = 0$ and a fortiori (1.3) holds. Moreover, for \mathcal{L}^1 -a.e. $x' \in \Omega$ the section $(E^{S_1})_{x'}$ is a ball obtained by translating $E_{x'}$. This fact implies in particular that also E^{S_1} satisfies (1.4) and that for a.e. $x' \in \Omega$ the barycenter of the section $(E^{S_1})_{x'}$ is given by $(0, b_2(x'), \dots, b_{n-1}(x'))$.

Iterating this procedure with respect to the $n - 3$ variables y_2, \dots, y_{n-2} we obtain the set $E^{S_{n-2}}$, which is equivalent in $\Omega \times \mathbb{R}^{n-1}$ to

$$F := \left\{ (x', y) \in \Omega \times \mathbb{R}^{n-1} : x' \in \Omega, y \in B_{r(x')}^{n-1}((0, \dots, 0, b_{n-1}(x'))) \right\},$$

with $\mathcal{H}^{n-1}(\{x \in \partial^* F : \nu_{y_{n-1}}^F(x) = 0\} \cap (\Omega \times \mathbb{R}^{n-1})) = 0$. From Theorem 2.4, for \mathcal{L}^{n-2} -a.e. $y' = (y_1, \dots, y_{n-2}) \in \mathbb{R}^{n-2}$ the section $F_{y'} := \{(x', y_{n-1}) \in \mathbb{R}^2 : (x', y', y_{n-1}) \in F\}$ has finite perimeter. Moreover,

$$\mathcal{H}^1(\{(x', y_{n-1}) \in \partial^* F_{y'} : \nu_{y_{n-1}}^{F_{y'}}(x', y_{n-1}) = 0\} \cap (\Omega \times \mathbb{R})) = 0. \quad (4.3)$$

Indeed, setting $A := \{x \in \partial^* F : \nu_{y_{n-1}}^F(x) = 0\} \cap (\Omega \times \mathbb{R}^{n-1})$, by coarea formula we have $\mathcal{H}^1(A_{y'}) = 0$ for \mathcal{L}^{n-2} -a.e. y' , while again by Theorem 2.4 the section $A_{y'}$ is equivalent to $\{(x', y_{n-1}) \in \partial^* F_{y'} : \nu_{y_{n-1}}^{F_{y'}}(x', y_{n-1}) = 0\} \cap (\Omega \times \mathbb{R})$.

Let I be an interval compactly included in Ω . Since $r = r(x')$ is continuous, there exists a $\rho > 0$ such that $I \times B_\rho^{n-2}$ is included in the projection of F on the hyperplane $y_{n-1} = 0$. For all $y' \in B_\rho^{n-2}$ and $x' \in I$ the 1-dimensional section $(F_{y'})_{x'}$ of $F_{y'}$ is a segment of \mathcal{L}^1 -measure equal to $2\sqrt{r(x')^2 - |y'|^2} > 0$ and its barycenter coincides with $b_{n-1}(x')$. Therefore, $b_{n-1} \in W_{\text{loc}}^{1,1}(\Omega)$ by Step 2 and (4.1) holds by Step 1.

Step 4. The remaining case, i.e., $k < n - 1$, shall be addressed through a slicing argument.

Let $Q = \prod_{i=1}^{n-k} I_i$ be a cube compactly included in Ω . For any $\hat{x} \in \prod_{i=2}^{n-k} I_i$ set $E_{\hat{x}} = \{(x_1, y) \in \mathbb{R}^{k+1} : (x_1, \hat{x}, y) \in E\}$. Let us now prove that for \mathcal{L}^{n-k-1} -a.e. $\hat{x} \in \prod_{i=2}^{n-k} I_i$ the slice $E_{\hat{x}} \subset \mathbb{R}^{k+1}$ satisfies the same assumptions of the sets considered in steps 2 and 3. Indeed, the section $E_{\hat{x}}$ has finite perimeter in \mathbb{R}^{k+1} thanks to Theorem 2.4, and $(E_{\hat{x}})_{x_1} = E_{(x_1, \hat{x})}$ is a k -dimensional ball for \mathcal{L}^1 -a.e. $x_1 \in I_1$. Moreover, the same argument used to prove (4.3) shows that $\mathcal{H}^k(\{(x_1, y) \in \partial^* E_{\hat{x}} : \nu_y^{E_{\hat{x}}}(x_1, y) = 0\} \cap (I_1 \times \mathbb{R}^k)) = 0$. Finally, by [1, Theorem 3.108], $(L^*)_{\hat{x}}$ is the Lebesgue representative of $L_{\hat{x}}$. Therefore, $(L^*)_{\hat{x}} > 0$ in I_1 , given that the projection on $\prod_{i=2}^{n-k} I_i$ of the set $\{x' \in \Omega : L^*(x') = 0\}$ is \mathcal{L}^{n-k-1} -negligible.

By Steps 2 and 3 we conclude that $b_{\hat{x}} \in W^{1,1}(I_1; \mathbb{R}^k)$. Using again Theorem 2.4 we get (4.1):

$$\begin{aligned} b'_{\hat{x}}(x_1) &= \frac{1}{L_{\hat{x}}^*(x_1)} \int_{\partial^*(E_{\hat{x}})_{x_1}} (y - b_{\hat{x}}(x_1)) \frac{\nu_1^{E_{\hat{x}}}(x_1, y)}{|\nu_y^{E_{\hat{x}}}(x_1, y)|} d\mathcal{H}^{k-1}(y) \\ &= \frac{1}{L^*(x_1, \hat{x})} \int_{(\partial^* E)_{(x_1, \hat{x})}} (y - b(x_1, \hat{x})) \frac{\nu_1^E(x_1, \hat{x}, y)}{|\nu_y^E(x_1, \hat{x}, y)|} d\mathcal{H}^{k-1}(y). \end{aligned}$$

Step 5. We shall now move on to the special case $k = 1$, to prove that $b \in W_{\text{loc}}^{1,1}(\Omega)$ and $\|\nabla b\|_{L^1(\Omega; \mathbb{R}^{n-1})} \leq P(E; \Omega \times \mathbb{R})/2$. From (4.1), we get

$$\int_{\prod_{i=2}^{n-1} I_i} d\hat{x} \int_{I_1} |b'_{\hat{x}}(x_1)| dx_1 \leq \frac{1}{2} \int_Q dx' \int_{(\partial^* E)_{x'}} \frac{|\nu_1^E(x', y)|}{|\nu_y^E(x', y)|} d\mathcal{H}^0(y) \leq \frac{1}{2} P(E; Q \times \mathbb{R}). \quad (4.4)$$

Clearly, the same bound holds also if we consider the slices of b with respect to all the remaining directions. Fix now $T > 0$ and consider the truncated function $b^T := (b \wedge T) \vee (-T)$. From (4.4) we have that

$$\int_Q |(b_{\hat{x}}^T)'| dx_1 d\hat{x} \leq \frac{1}{2} P(E; Q \times \mathbb{R}).$$

The same holds for the other directions. As consequence of [1, Theorem 3.105], b^T belongs to $W^{1,1}(Q)$. Therefore, by using Poincaré-Wirtinger inequality as in step 2 and letting $T \rightarrow \infty$, we

get that $b \in W^{1,1}(Q)$. Finally, from [1, Theorem 3.107] follows that for \mathcal{L}^{n-1} -a.e. $x' \in \Omega$

$$\nabla b(x') = \frac{1}{L(x')} \int_{\partial^* E_{x'}} (y - b(x')) \frac{\nu_{x'}^E(x', y)}{|\nu_y^E(x', y)|} d\mathcal{H}^0(y).$$

In particular, this implies $\|\nabla b\|_{L^1(\Omega; \mathbb{R}^{n-1})} \leq P(E; \Omega \times \mathbb{R})/2$. \square

The next example shows that the regularity of the barycenter in general is only local.

Example 4.4. Let $n = 3$, $k = 2$. Let $\Omega = \pi^+(E) = (0, 1)$, and let E be the set of \mathbb{R}^3 given by

$$E = \{(x', y) \in \mathbb{R} \times \mathbb{R}^2 : |y - b(x')| < r(x'), \quad x' \in (0, 1)\},$$

where $r(x') = |x'|^2$ and $b : (0, 1) \rightarrow \mathbb{R}^2$ is given by

$$b(x') := \left(0, \sin \frac{1}{x'}\right), \quad x' \in (0, 1).$$

We observe that $L(x') = \pi|x'|^4$ is strictly positive in $(0, 1)$. Moreover,

$$\begin{aligned} P(E; (0, 1) \times \mathbb{R}^2) &= \int_0^1 r(x') dx' \int_0^{2\pi} \sqrt{1 + |b'(x') \cdot (\cos \theta, \sin \theta) + r'(x')|^2} d\theta \\ &\leq 2\pi \int_0^1 r(x') \sqrt{1 + 2|b'(x')|^2 + 2|r'(x')|^2} dx_1 < \infty, \end{aligned}$$

so that E is a set of finite perimeter in $(0, 1) \times \mathbb{R}^2$. In addition, $L \in W^{1,1}(0, 1)$ and conditions (1.3) and (1.4) are satisfied. Nevertheless, $b \notin W^{1,1}((0, 1); \mathbb{R}^2)$.

The next example shows that in codimension higher than 1, despite being absolutely continuous on almost every 1-dimensional section of Ω (see Theorem 4.3 above), the barycenter may not be in $W_{\text{loc}}^{1,1}(\Omega; \mathbb{R}^2)$.

Example 4.5. Let $n = 4$, $k = 2$. Let $L : B_1^2 \rightarrow [0, \infty)$ be a continuous function vanishing only at the origin, and such that $L \in W^{1,1}(B_1^2)$, while \sqrt{L} does not belong to $BV(U)$ for any neighborhood U of the origin. For example, one can take $L(x') = |x'|^{3/4}[2 + \sin(1/|x'|^2)]$. Set

$$E = \{(x', y) \in \mathbb{R}^2 \times \mathbb{R}^2 : |(y - b(x'))| < r(x'), \quad x' \in B_1^2\},$$

where $r(x') = \sqrt{L(x')/\pi}$ and $b(x') := (0, r(x'))$. Moreover,

$$P(E; B_1^2 \times \mathbb{R}^2) = \int_{B_1^2} r(x') dx' \int_0^{2\pi} \sqrt{1 + |\nabla r(x')|^2 (1 + \sin \theta)^2} d\theta < \infty,$$

so that E is a set of finite perimeter. Note that conditions (1.3) and (1.4) are satisfied. Nevertheless, $b \notin BV_{\text{loc}}(B_1^2; \mathbb{R}^2)$.

Once Theorem 4.3 is established, Theorem 1.2 follows at once.

Proof of Theorem 1.2. By Theorem 1.1 we have that for \mathcal{L}^{n-k} -a.e. $x' \in \Omega$ the section $E_{x'}$ is a k -dimensional ball and that $\nu_{x'}^E(x', \cdot)$ and $|\nu_y^E(x', \cdot)|$ are constant \mathcal{H}^{k-1} -a.e. in $\partial^* E_{x'}$. Moreover, condition (1.3) holds due to Proposition 3.6. Let $i = 1, \dots, n - k$. By formula (4.1), we have that for \mathcal{H}^{n-k-1} -a.e. $\hat{x}_i \in \Omega_i$ and for \mathcal{L}^1 -a.e. $x_i \in \Omega \cap R_{\hat{x}_i}$

$$b'_{\hat{x}_i}(x_i) = \frac{1}{L_{\hat{x}_i}^*(x_i)} \frac{\nu_i^E(x')}{|\nu_y^E(x')|} \int_{\partial^* E_{x'}} (y - b_{\hat{x}_i}(x_i)) d\mathcal{H}^{k-1}(y) = 0,$$

where we dropped the variable y for functions that are constant in $\partial^* E_{x'}$. Arguing as we did in Step 5 of the proof of Theorem 4.3, we conclude that $b \in W_{\text{loc}}^{1,1}(\Omega; \mathbb{R}^k)$, and is, therefore, constant. \square

5. QUANTITATIVE ESTIMATES IN THE CONVEX CASE

The present section focuses on a quantitative version of inequality (1.1) in the case of convex sets, in the spirit of [5, 8, 10]. Throughout the whole section, given $E \subset \mathbb{R}^n$, L and b will always denote the measure and the barycenter of the 1-dimensional sections respectively. In the special case, where all the sections are segments, we will also use the notation

$$h_{\pm} := b \pm \frac{L}{2},$$

so that

$$E = \{(x', y) \in \mathbb{R}^{n-1} \times \mathbb{R} : h_-(x') \leq y \leq h_+(x')\}.$$

Clearly, if E is convex, h_- is convex and h_+ is concave.

Lemma 5.1. *Let $E \subset \mathbb{R}^n$ be an open, bounded and convex set, such that $E \subset B_R(\bar{x})$ for some $\bar{x} \in \mathbb{R}^n$ and $R > 0$. Then, setting $\Omega = \pi^+(E)$*

$$|\nabla L(x')| \leq \frac{2R}{\text{dist}(x', \partial\Omega)},$$

for every $x' \in \Omega$ such that L is differentiable at x' .

Proof. Let $x' \in \Omega$ such that L is differentiable at x' and suppose, by contradiction, that

$$|\nabla L(x')| > \frac{2R}{\text{dist}(x', \partial\Omega)}.$$

Setting $\nu := -\frac{\nabla L(x')}{|\nabla L(x')|}$, we have

$$\frac{\partial L}{\partial \nu}(x') < -\frac{2R}{\text{dist}(x', \partial\Omega)}. \quad (5.1)$$

Defining $z := x' + \text{dist}(x', \partial\Omega) \nu$, we have $z \in \overline{\pi^+(E)}$. Moreover, since $L|_{[x', z]}$ is concave we get

$$L(z) - L(x') = \int_0^{\text{dist}(x', \partial\Omega)} \frac{\partial L}{\partial \nu}(x' + s\nu) ds \leq \int_0^{\text{dist}(x', \partial\Omega)} \frac{\partial L}{\partial \nu}(x') ds < -2R,$$

which is impossible. \square

The next result is a particular case of a more general one proved in [6]. We provide the proof as well, for the sake of completeness.

Proposition 5.2 (Weighted Poincaré inequality). *Let $\Omega \subset \mathbb{R}^n$, $n \geq 1$, be an open set, and let $\bar{x}, \tilde{x} \in \Omega$ and $0 < r \leq R < \infty$ be such that $B_r(\bar{x}) \subset \Omega \subset B_R(\tilde{x})$. Assume that Ω is star-shaped with respect to $B_r(\bar{x})$. Then,*

$$\int_{\Omega} |f(x) - f_{B_{\frac{r}{2}}(\bar{x})}| dx \leq \left(\frac{4R}{r}\right)^{n+1} \int_{\Omega} |\nabla f(x)| \text{dist}(x, \partial\Omega) dx \quad \text{for all } f \in W^{1,1}(\Omega),$$

where $f_{B_{\frac{r}{2}}(\bar{x})}$ denotes the average of f on $B_{\frac{r}{2}}(\bar{x})$.

Proof. Without any loss of generality, we can assume that $\bar{x} = 0$. We can also assume that $f \in C^1$, since the general case can be obtained by a density argument. For every $y \in \Omega$ and $z \in B_{\frac{r}{2}}$ we have

$$f(y) - f(z) = \int_0^1 \nabla f((1-s)y + sz) \cdot (y - z) ds.$$

Multiplying by $\frac{1}{\mathcal{L}^n(B_{\frac{r}{2}})}\chi_{B_{\frac{r}{2}}}(z)$ and integrating over z

$$f(y) - f_{B_{\frac{r}{2}}} = \frac{1}{\mathcal{L}^n(B_{\frac{r}{2}})} \int_{\Omega} \int_0^1 \nabla f((1-s)y + sz) \cdot (y-z) \chi_{B_{\frac{r}{2}}}(z) ds dz.$$

Making the change of variable

$$x = (1-s)y + sz,$$

we get

$$f(y) - f_{B_{\frac{r}{2}}} = \frac{1}{\mathcal{L}^n(B_{\frac{r}{2}})} \int_{\Omega} \int_0^1 \nabla f(x) \cdot \frac{y-x}{s} \chi_{B_{\frac{r}{2}}}\left(\frac{x-(1-s)y}{s}\right) \frac{1}{s^n} ds dx.$$

Passing to the absolute value and integrating with respect to y

$$\int_{\Omega} |f(y) - f_{B_{\frac{r}{2}}}| dy \leq \frac{1}{\mathcal{L}^n(B_{\frac{r}{2}})} \int_{\Omega} \int_{\Omega} \int_0^1 |\nabla f(x)| \chi_{B_{\frac{r}{2}}}\left(\frac{x-(1-s)y}{s}\right) \frac{|x-y|}{s^{n+1}} ds dx dy.$$

Let us show that

$$|x-y| \leq \frac{4R}{r} \text{dist}(x, \partial\Omega).$$

This inequality is trivial when $n=1$. Otherwise, if $y \notin B_{\frac{r}{2}}(z)$, by indicating with C the convex hull of $B_{\frac{r}{2}}(z) \cup \{y\}$, we have

$$|x-y| = \frac{2|z-y|}{r} \text{dist}(x, \partial C) \leq \frac{4R}{r} \text{dist}(x, \partial\Omega)$$

because $C \subset \Omega$. Conversely, if $y \in B_{\frac{r}{2}}(z)$, we have $|x-y| \leq \text{dist}(x, \partial\Omega) \leq \frac{4R}{r} \text{dist}(x, \partial\Omega)$. Therefore, setting $A(x) := \{y \in \Omega : |x-y| \leq \frac{4R}{r} \text{dist}(x, \partial\Omega)\}$ and interchanging the order of integration

$$\int_{\Omega} |f(y) - f_{B_{\frac{r}{2}}}| dy \leq \frac{1}{\mathcal{L}^n(B_{\frac{r}{2}})} \int_{\Omega} |\nabla f(x)| \int_{A(x)} \int_0^1 \chi_{B_{\frac{r}{2}}}\left(\frac{x-(1-s)y}{s}\right) \frac{|x-y|}{s^{n+1}} ds dy dx.$$

Also, for x and y fixed, the only values of s for which we have contribution are such that

$$\frac{|x-y|}{s} - |y| \leq \frac{|x-(1-s)y|}{s} \leq \frac{r}{2}, \quad \text{and so } s \geq \frac{|x-y|}{\frac{r}{2} + |y|} \geq \frac{|x-y|}{2R}.$$

Hence,

$$\begin{aligned} \int_{\Omega} |f(y) - f_{B_{\frac{r}{2}}}| dy &\leq \frac{1}{\mathcal{L}^n(B_{\frac{r}{2}})} \int_{\Omega} |\nabla f(x)| \int_{A(x)} |x-y| \int_{\frac{|x-y|}{2R}}^1 \frac{1}{s^{n+1}} ds dy dx \\ &= \frac{1}{n\mathcal{L}^n(B_{\frac{r}{2}})} \int_{\Omega} |\nabla f(x)| \int_{A(x)} \frac{1}{|x-y|^{n-1}} [(2R)^n - |x-y|^n] dy dx \\ &\leq \frac{(2R)^n}{n\mathcal{L}^n(B_{\frac{r}{2}})} \int_{\Omega} |\nabla f(x)| \int_{A(x)} \frac{1}{|x-y|^{n-1}} dy dx \\ &= \frac{(2R)^n}{\mathcal{L}^n(B_{\frac{r}{2}})} \int_{\Omega} |\nabla f(x)| \int_0^{\frac{4R}{r} \text{dist}(x, \partial\Omega)} \omega_n d\rho dx \\ &= \left(\frac{4R}{r}\right)^{n+1} \int_{\Omega} |\nabla f(x)| \text{dist}(x, \partial\Omega) dx. \end{aligned}$$

□

Corollary 5.3. *Let $\Omega \subset \mathbb{R}^n$ be a bounded and connected Lipschitz domain. Given an open and not empty set $S \subset\subset \Omega$, there is a positive constant c_P , intrinsically depending only on Ω and S , such that*

$$\int_{\Omega} |f(x) - f_S| dx \leq c_P \int_{\Omega} |\nabla f(x)| \text{dist}(x, \partial\Omega) dx \quad \text{for all } f \in W^{1,1}(\Omega), \quad (5.2)$$

where f_S denotes the average of f on S .

Proof. We decompose the domain Ω in a finite number of domains Ω^i , each star-shaped with respect to an open ball $B^i \subset\subset \Omega^i$ (see [14, Section 1.1.9, Lemma 1]). We also consider a connected open set $U \subset\subset \Omega$, having the cone property and such that $S \subset U$ and $B^i \subset U$ for every i .

By the Poincaré-Wirtinger inequality in $W^{1,1}(U)$ (see [14, Lemma 1.1.11]), it follows that

$$\begin{aligned} |f_S - f_{B^i}| &= \left| \frac{1}{\mathcal{L}^n(S)} \int_S (f - f_{B^i}) dx \right| \leq \frac{1}{\mathcal{L}^n(S)} \int_U |f - f_{B^i}| dx \\ &\leq \frac{c_i}{\mathcal{L}^n(S)} \int_U |\nabla f| dx \leq \frac{c_i}{\mathcal{L}^n(S) \text{dist}(U, \partial\Omega)} \int_{\Omega} |\nabla f| \text{dist}(x, \partial\Omega) dx \end{aligned}$$

for suitable $c_i = c_i(B^i, U)$. Then, Proposition 5.2 leads

$$\int_{\Omega} |f - f_S| dx \leq \sum_i \left[\int_{\Omega^i} |f - f_{B^i}| dx + \mathcal{L}^n(\Omega^i) |f_S - f_{B^i}| \right] \leq c_P \int_{\Omega} |\nabla f| \text{dist}(x, \partial\Omega) dx.$$

□

We can now prove Theorem 1.4.

Proof of Theorem 1.4. For the sake of simplification, we set $\Omega = \pi^+(\mathring{E}^S)$, and indicate by r_0 and R_0 the inner radius and the outer radius of E^S , respectively. We shall prove that

$$\inf_{y \in \mathbb{R}} \mathcal{L}^n(E^S \Delta (E + (0, y))) \leq 2R_0 \left(\frac{4R_0}{r_0} \right)^n \sqrt{P(E) - P(E^S)} \sqrt{P(E)}. \quad (5.3)$$

First of all, let us observe that $b \in W_{\text{loc}}^{1,1}(\Omega)$ by Theorem 4.3. Then, the difference $P(E) - P(E^S)$ is greater than or equal to

$$\begin{aligned} &\int_{\Omega} \left[\sqrt{1 + \left| \nabla b + \frac{1}{2} \nabla L \right|^2} + \sqrt{1 + \left| \nabla b - \frac{1}{2} \nabla L \right|^2} - 2\sqrt{1 + \frac{1}{4} |\nabla L|^2} \right] dx' \\ &= 2 \int_{\Omega} \frac{\sqrt{\left(1 + \frac{1}{4} |\nabla L|^2 + |\nabla b|^2\right)^2 - (\nabla L \cdot \nabla b)^2} - \left(1 + \frac{1}{4} |\nabla L|^2 - |\nabla b|^2\right)}{\sqrt{1 + \left| \nabla b + \frac{1}{2} \nabla L \right|^2} + \sqrt{1 + \left| \nabla b - \frac{1}{2} \nabla L \right|^2} + 2\sqrt{1 + \frac{1}{4} |\nabla L|^2}} dx' = 2 \int_{\Omega} \frac{N}{D} dx', \end{aligned} \quad (5.4)$$

where N and D denote the numerator and the denominator of the second integrand, respectively. Using Hölder inequality, we get

$$\begin{aligned} \sqrt{P(E) - P(E^S)} \sqrt{P(E) + P(E^S)} &= \sqrt{P(E) - P(E^S)} \left(\int_{\Omega} D dx' \right)^{\frac{1}{2}} \geq \sqrt{2} \int_{\Omega} \sqrt{N} dx' \\ &= \sqrt{2} \int_{\Omega} \sqrt{\sqrt{\left(1 + \frac{1}{4} |\nabla L|^2 + |\nabla b|^2\right)^2 - (\nabla L \cdot \nabla b)^2} - \left(1 + \frac{1}{4} |\nabla L|^2 - |\nabla b|^2\right)} dx'. \end{aligned}$$

Rationalizing, the latter term becomes

$$\begin{aligned}
& \sqrt{2} \int_{\Omega} \frac{\sqrt{4\left(1 + \frac{1}{4}|\nabla L|^2\right)|\nabla b|^2 - (\nabla b \cdot \nabla L)^2}}{\sqrt{\sqrt{\left(1 + \frac{1}{4}|\nabla L|^2 + |\nabla b|^2\right)^2 - (\nabla L \cdot \nabla b)^2} + \left(1 + \frac{1}{4}|\nabla L|^2 - |\nabla b|^2\right)}} dx' \\
& \geq 2\sqrt{2} \int_{\Omega} \frac{|\nabla b|}{\sqrt{\sqrt{\left(1 + \frac{1}{4}|\nabla L|^2 + |\nabla b|^2\right)^2 - (\nabla L \cdot \nabla b)^2} + \left(1 + \frac{1}{4}|\nabla L|^2 - |\nabla b|^2\right)}} dx'.
\end{aligned} \tag{5.5}$$

In order to estimate the last integral, we introduce the function $f : [-a, a] \rightarrow [0, \infty)$ defined as

$$f(z) := \sqrt{a^2 - z^2}.$$

Using Taylor expansion, we have

$$\begin{aligned}
f(z) &= f(0) + zf'(0) + z^2 \int_0^1 (1-t)f''(tz) dt = a - z^2 \int_0^1 \frac{(1-t)a^2}{[a^2 - t^2 z^2]^{\frac{3}{2}}} dt \\
&\leq a - \frac{z^2}{a} \int_0^1 (1-t) dt = f(0) - \frac{z^2}{2a}.
\end{aligned}$$

Applying the previous inequality at \mathcal{L}^{n-1} -a.e. every $x' \in \Omega$ with

$$a := 1 + \frac{1}{4}|\nabla L(x')|^2 + |\nabla b(x')|^2, \quad z = \nabla L(x') \cdot \nabla b(x'),$$

and using Lemma 5.1 with E replaced by E^S we have

$$\begin{aligned}
& \sqrt{\left(1 + \frac{1}{4}|\nabla L|^2 + |\nabla b|^2\right)^2 - (\nabla L \cdot \nabla b)^2} + \left(1 + \frac{1}{4}|\nabla L|^2 - |\nabla b|^2\right) \\
& \leq 2 + \frac{1}{2}|\nabla L|^2 \frac{1 + \frac{1}{4}|\nabla L|^2 + \sin^2 \theta |\nabla b|^2}{1 + \frac{1}{4}|\nabla L|^2 + |\nabla b|^2} \leq 2 + \frac{1}{2}|\nabla L|^2 \\
& \leq 2 \frac{R_0^2}{\text{dist}^2(\cdot, \partial\Omega)} + \frac{1}{2}|\nabla L|^2 \leq \frac{4R_0^2}{\text{dist}^2(\cdot, \partial\Omega)},
\end{aligned} \tag{5.6}$$

where θ is the angle between $\nabla L(x')$ and $\nabla b(x')$. Plugging (5.6) into inequality (5.5) we obtain

$$\sqrt{P(E) - P(E^S)} \sqrt{P(E) + P(E^S)} \geq \frac{\sqrt{2}}{R_0} \int_{\Omega} |\nabla b(x')| \text{dist}(x', \partial\Omega) dx'.$$

Let b_0 denote the average of b in $B_{r_0/2}^{n-1}(x_0)$. Since the outer radius of Ω is not greater than R_0 , using Proposition 5.2,

$$\begin{aligned}
& \sqrt{P(E) - P(E^S)} \sqrt{P(E) + P(E^S)} \geq \frac{\sqrt{2}}{R_0} \left(\frac{r_0}{4R_0}\right)^n \int_{\Omega} |b - b_0| dx' \\
& \geq \frac{1}{R_0 \sqrt{2}} \left(\frac{r_0}{4R_0}\right)^n \int_{\Omega} \mathcal{L}^1(E_{x'}^S \Delta(E_{x'} - b_0)) dx' \\
& = \frac{1}{R_0 \sqrt{2}} \left(\frac{r_0}{4R_0}\right)^n \mathcal{L}^n(E^S \Delta(E - (0, b_0))),
\end{aligned}$$

thus obtaining (5.3).

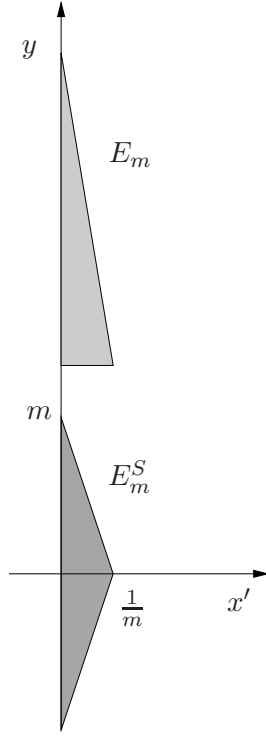


FIGURE 9

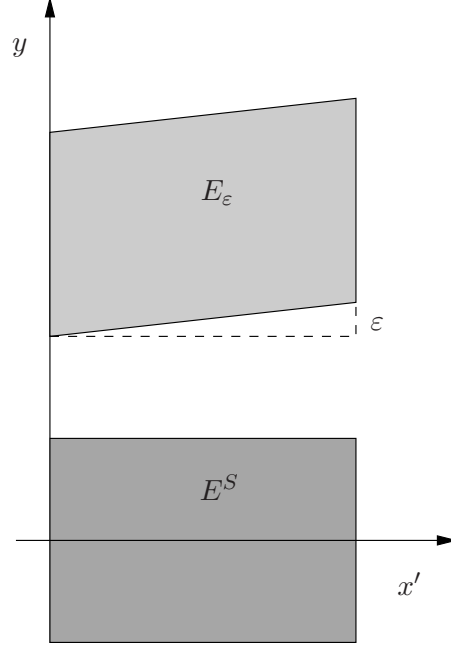


FIGURE 10

To conclude the proof observe that, if $D(E) > 1$ then $\lambda(E) < 2\sqrt{D(E)}$, since we always have $\lambda(E) < 2$. On the other hand, if $D(E) \leq 1$, since $P(E) \leq 2P(E^S)$, from (5.3) we easily obtain

$$\lambda(E) \leq 4^n 2\sqrt{2} R_0 \left(\frac{R_0}{r_0} \right)^n \sqrt{D(E)} \frac{P(E^S)}{\mathcal{L}^n(E^S)}.$$

Hence estimate (1.6), observing that $P(E^S) \leq n\omega_n R_0^{n-1}$ and $\mathcal{L}^n(E^S) \geq \omega_n r_0^n$. \square

Example 5.4. Let E_m be a rectangle triangle with basis $1/m$ and height $2m$ as in Figure 9. It can be computed that $\lambda(E_m) = 2/5$ for every $m \in \mathbb{N}$. On the other hand, $D(E_m) \rightarrow 0$ as $m \rightarrow \infty$. This shows that the dependence on the eccentricity of the constant in the right-hand side of (1.6) cannot be avoided.

Example 5.5. Consider the rectangle $E^S := (0, a) \times (-b/2, b/2)$ and, for $\varepsilon > 0$, the parallelogram E_ε as in Figure 10. One can compute that $\lambda(E_\varepsilon) = \varepsilon/2b$, and $D(E_\varepsilon) \approx \varepsilon^2/(2a^2 + 2ab)$. This shows that the exponent $1/2$ in (1.4) is optimal.

Remark 5.6. We can weaken the convexity assumption on E^S in Theorem 1.4 by merely requiring that $|\nabla L| \leq c' \text{dist}^{-1}(\cdot, \partial\Omega)$ for some $c' \equiv c'(E^S)$, and (using Corollary 5.3) that $\Omega = \pi^+(E^S)$ is a bounded and connected Lipschitz domain. In this case we get a generic constant $c \equiv c(n, E^S)$ instead of $4^{n+1}n\mathcal{E}^{2n}$ in the right-hand side of (1.6). In the example shown in Figure 2 the constant c blows up because c' does, as the slope of the dotted segments increases. On the other hand, in the example in Figure 3, the constant c_P in (5.2) blows up when the neck in $\Omega = \pi^+(E)$ (the area colored in black) shrinks.

Remark 5.7. Note that the first integral in (5.4) is equal to $P(E; \Omega \times \mathbb{R}) - P(E^S; \Omega \times \mathbb{R})$, whit $\Omega = \pi^+(\dot{E}^S)$. Therefore, (1.6) holds in a slightly stronger form, with $D(E)$ replaced by

$$\frac{1}{P(E^S)} [P(E; \Omega \times \mathbb{R}) - P(E^S; \Omega \times \mathbb{R})].$$

Following the terminology introduced in [10], we say that a set $E \subset \mathbb{R}^n$ is n -symmetric if it is symmetric with respect to the n coordinate hyperplanes. The next lemma shows that for a convex n -symmetric set E the asymmetry index $A(E)$ can be obtained by choosing the ball centered at the origin.

Lemma 5.8. *Let $E \subset \mathbb{R}^n$, $n \geq 1$, be an n -symmetric bounded convex set with $\mathcal{L}^n(E) = \mathcal{L}^n(B_r)$ for some $r > 0$. Then*

$$\min_{x \in \mathbb{R}^n} \mathcal{L}^n(E \Delta B_r(x)) = \mathcal{L}^n(E \Delta B_r).$$

Proof. Given $F, G \subset \mathbb{R}^n$, by comparing their sections it is easy to prove that $\mathcal{L}^n(F^S \cap G^S) \geq \mathcal{L}^n(F \cap G)$ for any k -codimensional Steiner symmetrization.

Note now that E is stable for the 1-dimensional Steiner symmetrizations with respect to $x_1 = 0, \dots, x_n = 0$. Hence

$$\mathcal{L}^n(E \cap B_r(x)) \leq \mathcal{L}^n(E \cap B_r(x_1, \dots, x_{n-1}, 0)) \leq \dots \leq \mathcal{L}^n(E \cap B_r).$$

□

We can finally prove the quantitative estimate for the Steiner symmetrization in the case $2 \leq k \leq n-1$.

Proof of Theorem 1.5. As in the previous case, we can assume without loss of generality that $\Delta(E) \leq 1$, hence $P(E) \leq 2P(E^S)$.

Step 1. Define E^{S_i} as in the proof of Proposition 3.6. Denote also by r_0 and R_0 the inner radius and the outer radius of E , respectively. Since the Steiner symmetrization decreases the outer radius, while increases the inner radius, by formula (5.3), we get that for every $i = 1, \dots, k$ we have

$$\mathcal{L}^n(E^{S_{i-1}} \Delta E^{S_i}) \leq 2R_0 \left(\frac{4R_0}{r_0} \right)^n \sqrt{P(E^{S_{i-1}}) - P(E^{S_i})} \sqrt{P(E^{S_{i-1}})}.$$

up to a suitable translation in the direction of the y_i axis. By triangular inequality

$$\begin{aligned} \mathcal{L}^n(E \Delta E^{S_k}) &\leq \sum_{i=1}^k \mathcal{L}^n(E^{S_{i-1}} \Delta E^{S_i}) \leq 2R_0 \left(\frac{4R_0}{r_0} \right)^n \sqrt{P(E)} \sum_{i=1}^k \sqrt{P(E^{S_{i-1}}) - P(E^{S_i})} \\ &\leq 2\sqrt{k}R_0 \left(\frac{4R_0}{r_0} \right)^n \sqrt{P(E)} \sqrt{P(E) - P(E^S)}. \end{aligned} \tag{5.7}$$

Step 2. We now estimate the measure of the symmetric difference of E^S and E^{S_k} . To this aim, for \mathcal{L}^{n-k} -a.e. $x' \in \mathbb{R}^{n-k}$ we set

$$f_k(x') := p_{E^{S_k}}^2(x') + |\nabla L(x')|^2, \quad f(x') := p_{E^S}^2(x') + |\nabla L(x')|^2,$$

Using Corollary 3.4, setting $\Omega = \pi^+(E^S)$ and using Hölder inequality the difference $P(E^{S_k}) - P(E^S)$ can be estimated by

$$\begin{aligned} \int_{\Omega} \left(\sqrt{f_k(x')} - \sqrt{f(x')} \right) dx' &= \int_{\Omega} \frac{p_{E^{S_k}}^2(x') - p_{E^S}^2(x')}{\sqrt{f_k(x')} + \sqrt{f(x')}} dx' \\ &\geq \frac{\left(\int_{\Omega} \sqrt{p_{E^{S_k}}^2(x') - p_{E^S}^2(x')} dx' \right)^2}{\int_{\Omega} \left(\sqrt{f_k(x')} + \sqrt{f(x')} \right) dx'} \geq \frac{\left(\int_{\Omega} \sqrt{p_{E^{S_k}}^2(x') - p_{E^S}^2(x')} dx' \right)^2}{2P(E^{S_k})}. \end{aligned}$$

Thus,

$$\begin{aligned} \sqrt{2P(E^{S_k})} \sqrt{P(E^{S_k}) - P(E^S)} &\geq \int_{\Omega} \sqrt{p_{E^{S_k}}^2(x') - p_{E^S}^2(x')} dx' \\ &= \int_{\Omega} \sqrt{\frac{p_{E^{S_k}}(x') - p_{E^S}(x')}{p_{E^S}(x')}} \sqrt{p_{E^S}(x')(p_{E^{S_k}}(x') + p_{E^S}(x'))} dx' \\ &\geq \sqrt{2} \int_{\Omega} p_{E^S}(x') \sqrt{\frac{p_{E^{S_k}}(x') - p_{E^S}(x')}{p_{E^S}(x')}} dx' = \sqrt{2} k \omega_k \int_{\Omega} r^{k-1}(x') \sqrt{\frac{p_{E^{S_k}}(x') - p_{E^S}(x')}{p_{E^S}(x')}} dx'. \end{aligned}$$

Thanks to Lemma 5.8, for \mathcal{L}^{n-k} -a.e. every $x' \in \Omega$ we can use the quantitative isoperimetric inequality (1.5) in dimension k , to get

$$\begin{aligned} \sqrt{P(E)} \sqrt{P(E) - P(E^S)} &\geq \frac{k \omega_k}{\gamma_k} \int_{\Omega} r^{k-1}(x') \frac{\mathcal{L}^n(E_{x'}^{S_k} \Delta E_{x'}^S)}{\mathcal{L}^n(E_{x'}^S)} dx' \\ &= \frac{k}{\gamma_k} \int_{\Omega} \frac{1}{r(x')} \mathcal{L}^n(E_{x'}^{S_k} \Delta E_{x'}^S) dx' \geq \frac{k}{\gamma_k R_0} \mathcal{L}^n(E^{S_k} \Delta E^S). \end{aligned}$$

Combining this estimate with (5.7) and arguing as in the final part of the proof of Theorem 1.4 we easily obtain

$$\lambda(E) \leq c \mathcal{E}^{2n} \sqrt{D(E)},$$

with

$$c = n \left(2^{2n+1} \sqrt{2k} + \frac{\gamma_k \sqrt{2}}{k} \right). \quad (5.8)$$

□

Finally, we highlight that for convex sets the measure of the symmetric difference $E \Delta E^S$ is equivalent to the Hausdorff distance $d_{\mathcal{H}}(E, E^S)$ between E and E^S . In the spirit of [9], by combining the quantitative isoperimetric inequalities (1.6) and (1.7) with [7, Lemma 4.2], we have the following corollary. Here $\text{diam}(E)$ is the diameter of the set E .

Corollary 5.9. *Let $1 \leq k \leq n-1$, $\mathcal{E} \geq 1$. There exists a constant $c \equiv c(k, n)$, such that, if E^S is a Steiner symmetric convex set, then*

$$d_{\mathcal{H}}(E, E^S) \leq c \text{diam}(E) \mathcal{E}^2 (D(E))^{\frac{1}{2n}}$$

for every convex set $E \subset \mathbb{R}^n$ with eccentricity not greater than \mathcal{E} whose Steiner symmetral is E^S .

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