Mumford-Shah functional as $\Gamma$-limit of discrete Perona-Malik energies

Massimiliano Morini  
Department of Mathematics  
Carnegie Mellon University  
5000 Forbes Avenue  
Pittsburgh, PA 15213, USA  
morini@asdf.math.cmu.edu

Matteo Negri  
Max Planck Institute for Mathematics in the Sciences  
Inselstrasse 22  
04103 Leipzig, Germany  
negri@mis.mpg.de

Abstract. We prove that a suitable rescaling of biased Perona-Malik energies, defined in the discrete setting, $\Gamma$-converges to an anisotropic version of the Mumford-Shah functional. Numerical results are discussed.

Key words: anisotropic diffusion, image segmentation, $\Gamma$-convergence.

1 Introduction

Many authors tackled in the last years the problem of image segmentation, among them Mumford and Shah (see [13]) suggested a variational approach based on the minimization of the energy

$$
\int_\Omega |\nabla u|^2 \, dx + \mathcal{H}^1(S_u) + \int_\Omega |u - g|^2 \, dx,
$$

while Perona and Malik (see [16]) proposed an evolution equation of the form

$$
\frac{\partial u}{\partial t} = \text{div} \left( \frac{2\nabla u}{1 + |\nabla u|^2} \right) \quad u(x,0) = g(x),
$$

where $g : \Omega \to [0,1]$ is the gray level function of the original image, $u$ represents the segmentation and $S_u$ is considered as the set of contours. Considering (1.2) as the gradient flow of the functional

$$
\int_\Omega \log(1 + |\nabla u|^2) \, dx,
$$

it turns out that the simultaneous smoothing and edge detection effects of the equation depend on the particular structure of the function $\log(1 + t^2)$: the quadratic behavior near the origin
is responsible for the denoising process while the concave and sublinear behavior at infinity is responsible for the edge detection. Moreover this kind of convex-concave potentials has been used also for the variational approximation (in the sense of Γ-convergence) of the Mumford-Shah functional both in the continuous setting (see [5], [12]) and in the discrete one, which we are going to deal with. Considering Gobbino’s paper [12], Chambolle [used in] [7] a functional of the form
\[
F_\varepsilon(u) = \varepsilon^2 \sum_{x \in \Omega \cap \mathbb{Z}^2} \sum_{\xi \in \mathbb{Z}^2} \frac{1}{\varepsilon} f \left( \frac{|u(x + \varepsilon \xi) - u(x)|^2}{\varepsilon |\xi|} \right) \rho(|\xi|) + \varepsilon^2 \sum_{x \in \Omega \cap \mathbb{Z}^2} |u(x) - g(x)|^2, \tag{1.3}
\]
where the function \( f : [0, +\infty) \to [0, +\infty) \) is non-decreasing, continuous and satisfies
\[
f'(0) = 1 \quad \text{and} \quad \lim_{t \to +\infty} f(t) = 1,
\]
while the convolution term \( \rho : \mathbb{Z}^2 \to [0, +\infty) \) is even and satisfies
\[
\rho(0) = 0, \quad \sum_{\xi \in \mathbb{Z}^2} \rho(\xi) < +\infty, \quad \rho(\xi) > 0 \quad \text{if} \quad |\xi| = 1, \quad \text{and} \quad \rho(\xi) = \rho(\xi^\perp). \tag{1.4}
\]
Chambolle proved that the Γ-limit is the anisotropic Mumford-Shah functional given by
\[
c_\rho \int_\Omega |\nabla u|^2 dx + \int_{S_u} \phi(\nu) dH^1, \tag{1.5}
\]
where
\[
c_\rho := \frac{1}{2} \sum_{\xi \in \mathbb{Z}^2} \rho(\xi) \quad \text{and} \quad \phi(\nu) := \sum_{\xi \in \mathbb{Z}^2} \rho(\xi) |\nu \cdot \hat{\xi}|
\]
(\( \hat{\xi} \) stands for \( \xi / |\xi| \)). In our main theorem we prove that (1.5) is the Γ-limit also of the following discrete Perona-Malik energies
\[
F_\varepsilon(u) = \varepsilon^2 \sum_{x \in \Omega \cap \mathbb{Z}^2} \sum_{\xi \in \mathbb{Z}^2} \frac{1}{a_\varepsilon} \log \left( 1 + a_\varepsilon \frac{|u(x + \varepsilon \xi) - u(x)|^2}{\varepsilon^2 |\xi|} \right) \rho(|\xi|) |\xi| + \varepsilon^2 \sum_{x \in \Omega \cap \mathbb{Z}^2} |u(x) - g(x)|^2, \tag{1.6}
\]
where \( a_\varepsilon := \varepsilon \log \frac{1}{\varepsilon} \) and \( \rho \) satisfies (1.4).

In the framework of anisotropic diffusion the functional (1.6) can be seen as a discretization of
\[
G_\varepsilon(u) = \frac{1}{a_\varepsilon} \int_\Omega \log \left( 1 + a_\varepsilon |\nabla u|^2 \right) dx + \int_\Omega |u - g|^2 dx,
\]
whose gradient flow is
\[
\frac{\partial u}{\partial t} = dG_\varepsilon(u) = \text{div} \left( \frac{2\nabla u}{1 + a_\varepsilon |\nabla u|^2} \right) + 2(u - g). \tag{1.7}
\]
The last equation resembles a variant of the Perona-Malik equation, known as biased anisotropic diffusion which was proposed by Nordström in [15].

Concerning the numerical results, which are discussed in detail in the final section, the main advantage of (1.6) is that the solutions obtained by a gradient descent algorithm, which can be interpreted as a discretization of (1.7), are comparable with the ones obtained by a Graduated Non Convexity technique.

2 Notations and statement of the main result

Given a vector \( \tau \in \mathbb{R}^2 \) let

\[
Z_{\tau}^2 = \{ x \in \mathbb{R}^2 : x = m\tau + n\tau^\perp \text{ for } (m, n) \in \mathbb{Z}^2 \},
\]

\[
C_{\tau} = \{ x \in \mathbb{R}^2 : x = s\tau + r\tau^\perp \text{ for } (s, r) \in [0, 1) \times [0, 1) \}.
\]

For every open subset \( A \subset \mathbb{R}^2 \) and for \( y, \tau \in \mathbb{R}^2 \) we denote

\[
l^1((y + Z_{\tau}^2) \cap A) := \left\{ v : (y + Z_{\tau}^2) \cap A \to \mathbb{R} \text{ such that } \sum_{x \in (y + Z_{\tau}^2) \cap A} |v(x)| < +\infty \right\};
\]

in the following every function \( v \in l^1((y + Z_{\tau}^2) \cap A) \) will be identified with the function \( \bar{v} \in L^1(A) \) which takes the constant value \( v(x) \) in \( (x + C_{\tau}) \) if \( x \in (y + Z_{\tau}^2) \cap A \), and zero otherwise. So, having in mind this identification, given a sequence \( v_\varepsilon \in l^1((y_\varepsilon + Z_{\tau, \varepsilon}^2) \cap A) \) and a function \( v \in L^1(A) \), we will often write, with a slight abuse of notation, \( v_\varepsilon \to v \) instead of \( \bar{v}_\varepsilon \to v \) in \( L^1(A) \). Given a vector \( \tau \) we will denote \( \hat{\tau} = \tau/|\tau| \). For all the notations about spaces of special functions of bounded variation we refer to the book [3].

Let \( \Omega \subset \mathbb{R}^2 \) be a bounded open domain with Lipschitz boundary and for every \( \varepsilon > 0 \) consider the following functional

\[
F_\varepsilon(u) := \varepsilon^2 \sum_{x \in \partial^\varepsilon \mathbb{Z}^2} \sum_{\xi \in \mathbb{Z}^2} \frac{1}{a_\varepsilon |\xi|} \log \left( 1 + a_\varepsilon |\xi| \frac{|u(x + \varepsilon \xi) - u(x)|^2}{\varepsilon^2 |\xi|^2} \right) \rho(\xi)
\]

if \( u \in l^1(\varepsilon \mathbb{Z}^2 \cap \Omega) \), and \( F_\varepsilon(u) := +\infty \) otherwise in \( L^1(\Omega) \), where \( a_\varepsilon = \varepsilon \log \frac{1}{\varepsilon} \) and \( \rho : \mathbb{Z}^2 \to [0, +\infty) \) satisfies (1.4).

In this paper we will prove the following theorem.

**Theorem 2.1** The functionals \( F_\varepsilon \) \( \Gamma \)-converge (as \( \varepsilon \to 0 \)) with respect to the \( L^1 \)-norm to the anisotropic Mumford-Shah functional \( F \) given by

\[
F(u) := \left\{ c_p \int_\Omega |\nabla u|^2 \, dx + \int_{\partial^1 u} \phi(\nu_u) \, d\mathcal{H}^1 \text{ if } u \in GSBV(\Omega) \cap L^1(\Omega),
\]

\[
+\infty \text{ otherwise in } L^1(\Omega),
\]

where

\[
c_p := \frac{1}{2} \sum_{\xi \in \mathbb{Z}^2} \rho(\xi) \quad \text{and} \quad \phi(\nu) := \sum_{\xi \in \mathbb{Z}^2} \rho(\xi) |\nu \cdot \hat{\xi}|.
\]

Moreover, every sequence \( (u_\varepsilon) \) satisfying \( \sup_{\varepsilon} (F_\varepsilon(u_\varepsilon) + \|u_\varepsilon\|_\infty) < +\infty \) is strongly precompact in \( L^p(\Omega) \), for \( 1 \leq p < \infty \).
Remark 2.2 It will be clear from the proof that \( F_\varepsilon \) \( \Gamma \)-converges with respect to the \( L^p \)-norm to the functional

\[
F(u) := \begin{cases} 
  c_p \int_{\Omega} |\nabla u|^2 \, dx + \int_{S_u} \phi(\nu_u) \, d\mathcal{H}^1 & \text{if } u \in GSBV(\Omega) \cap L^p(\Omega), \\
  +\infty & \text{otherwise in } L^p(\Omega),
\end{cases}
\]

for every \( 1 \leq p < +\infty \).

The proof of the theorem will be split in the next sections.

3 Estimate from below of the \( \Gamma \)-limit in dimension one

In this section we study the one-dimensional version of the functionals defined above. Given a bounded open subset \( I \subset \mathbb{R} \) we define

\[
I_\varepsilon := \{ x \in I \cap \varepsilon \mathbb{Z} : x + \varepsilon \in I \},
\]

and, for every \( u : I \cap \varepsilon \mathbb{Z} \to \mathbb{R} \), we define

\[
\mathcal{F}_\varepsilon(u, I) := \frac{\varepsilon}{a_\varepsilon} \sum_{x \in I_\varepsilon} \log \left( 1 + a_\varepsilon \frac{|u(x + \varepsilon) - u(x)|^2}{\varepsilon^2} \right),
\]

where, as above, \( a_\varepsilon = \varepsilon \log \frac{1}{\varepsilon} \). As usual we will identify every function \( u : I \cap \varepsilon \mathbb{Z} \to \mathbb{R} \) (briefly \( u \in L^1(I \cap \varepsilon \mathbb{Z}) \)) with the piecewise constant function \( u \) of \( L^1(I) \) given by

\[
u(x) := \begin{cases} 
u \left( \varepsilon \left[ \frac{x}{\varepsilon} \right] \right) & \text{if } \varepsilon \left[ \frac{x}{\varepsilon} \right] \in I, \\
0 & \text{otherwise.}
\end{cases}
\]

Our aim is to prove the following proposition.

Proposition 3.1 Let \( u_\varepsilon \in L^1(I \cap \varepsilon \mathbb{Z}) \) such that \( u_\varepsilon \to u \) in \( L^1(I) \) as \( \varepsilon \to 0^+ \) and \( \sup_\varepsilon \mathcal{F}_\varepsilon(u_\varepsilon) < +\infty \). Then \( u \in SBV(I) \) and

\[
\lim_{\varepsilon \to 0^+} \mathcal{F}_\varepsilon(u_\varepsilon, I) \geq \int_I |u'|^2 \, dx + \mathcal{H}^0(S_u).
\]

Moreover, any sequence \( u_\varepsilon \) satisfying \( \sup_\varepsilon \mathcal{F}_\varepsilon(u_\varepsilon, I) < +\infty \) is strongly precompact in \( L^1(I) \).

We postpone the proof of the proposition after some technical lemmas.

Lemma 3.2 Let \( p(\varepsilon) > 0 \) be such that \( \lim_{\varepsilon \to 0^+} p(\varepsilon) = 0 \) and

\[
\lim_{\varepsilon \to 0^+} \left( p(\varepsilon) \log \frac{1}{\varepsilon} - \log \log \frac{1}{\varepsilon} \right) = +\infty,
\]

and set \( c_\varepsilon := \varepsilon^{p(\varepsilon)} \). Then the following properties hold true:

a) \( \lim_{\varepsilon \to 0^+} c_\varepsilon \log \frac{1}{\varepsilon} = 0; \)
\[ b) \lim_{\varepsilon \to 0^+} \frac{\log \left(1 + a_{\varepsilon} \varepsilon^2 \right)}{\log \frac{1}{\varepsilon}} = 1. \]

**Proof.** It is not difficult to see that

\[
\lim_{\varepsilon \to 0^+} c_{\varepsilon} = 0, \quad \lim_{\varepsilon \to 0^+} \frac{\log c_{\varepsilon}}{\log \frac{1}{\varepsilon}} = 0, \quad \lim_{\varepsilon \to 0^+} \frac{c_{\varepsilon}^2}{\varepsilon} \log \frac{1}{\varepsilon} = +\infty.
\]

From these properties it follows that

\[
\lim_{\varepsilon \to 0^+} \log \left( c_{\varepsilon} \log \frac{1}{\varepsilon} \right) = \log \left( e^{-p(\varepsilon) \log \frac{1}{\varepsilon}} \log \frac{1}{\varepsilon} \right) = -p(\varepsilon) \log \frac{1}{\varepsilon} + \log \log \frac{1}{\varepsilon} = -\infty,
\]

which implies a), and

\[
\lim_{\varepsilon \to 0^+} \frac{\log \left(1 + a_{\varepsilon} \varepsilon^2 \right)}{\log \frac{1}{\varepsilon}} = \lim_{\varepsilon \to 0^+} \frac{\log \left(1 + \log \frac{1}{\varepsilon} + 2 \log c_{\varepsilon} \log \frac{1}{\varepsilon} \right)}{\log \frac{1}{\varepsilon}} = 1,
\]

which gives b). \[\blacksquare\]

**Lemma 3.3** Let \( u_{\varepsilon} \in l^1(I \cap \varepsilon \mathbb{Z}) \) be such that \( \sup_{x} \mathcal{F}_{\varepsilon}(u_{\varepsilon}, I) \leq K < +\infty \) and let \( c_{\varepsilon} \) be as in the previous lemma. Let \( b_{\varepsilon} = (c_{\varepsilon} \log \frac{1}{\varepsilon})^{1/4} \) and consider the following set

\[
D_{\varepsilon} := \left\{ x \in I_{\varepsilon} : \frac{|u_{\varepsilon}(x + \varepsilon) - u_{\varepsilon}(x)|}{\varepsilon} > b_{\varepsilon} \sqrt{a_{\varepsilon}} \right\}.
\]

Then

\[
\lim_{\varepsilon \to 0^+} \mathcal{H}^0(D_{\varepsilon})c_{\varepsilon} = 0.
\]

**Proof.** By our assumptions and recalling the definition of \( a_{\varepsilon} \), we have

\[
K > \frac{\varepsilon}{a_{\varepsilon}} \sum_{x \in D_{\varepsilon}} \log \left(1 + a_{\varepsilon} \frac{|u(x + \varepsilon) - u(x)|^2}{\varepsilon^2} \right) \geq \frac{\log(1 + b_{\varepsilon}^2)}{\log \frac{1}{\varepsilon}} \mathcal{H}^0(D_{\varepsilon})
\]

so that, substituting the expression of \( b_{\varepsilon}^2 \), if \( \varepsilon \) is small enough, from a) of Lemma 3.2 we get

\[
\mathcal{H}^0(D_{\varepsilon})c_{\varepsilon} \leq K \frac{c_{\varepsilon} \log \frac{1}{\varepsilon}}{\log \left(1 + \sqrt{c_{\varepsilon} \log \frac{1}{\varepsilon}} \right)} \leq K' \sqrt{c_{\varepsilon} \log \frac{1}{\varepsilon}} = K' \sqrt{c_{\varepsilon} \log \frac{1}{\varepsilon}}.
\]

Again a) implies now the thesis. \[\blacksquare\]
**Lemma 3.4** Let \( v_\varepsilon \in SBV(I) \) such that \( \lim_{\varepsilon \to 0^+} \|v'_\varepsilon\|_\infty \sqrt{\varepsilon} = 0 \). Then, for every \( \delta > 0 \) there exists \( \varepsilon > 0 \) such that

\[
\frac{1}{a_\varepsilon} \int_I \log(1 + a_\varepsilon |v'_\varepsilon|^2) \, dx \geq (1 - \delta) \int_I |v'_\varepsilon|^2 \, dx,
\]

for every \( \varepsilon \leq \varepsilon_0 \).

**Proof.** Fix \( \delta > 0 \) and note that there exists \( T_\delta > 0 \) such that

\[
\log(1 + a_\varepsilon t^2) \geq (1 - \delta)a_\varepsilon t^2 \quad \forall t \in \left[ 0, \frac{T_\delta}{\sqrt{a_\varepsilon}} \right];
\]

by assumption if \( \varepsilon \) is small enough we have \( \|v'_\varepsilon\|_\infty \leq T_\delta/\sqrt{a_\varepsilon} \) and therefore

\[
\frac{1}{a_\varepsilon} \int_I \log(1 + a_\varepsilon |v'_\varepsilon|^2) \, dx \geq (1 - \delta) \int_I |v'_\varepsilon|^2 \, dx.
\]

\[\blacksquare\]

We are now in a position to prove Proposition 3.1.

**Proof of Proposition 3.1.** Let \( b_\varepsilon \) and \( c_\varepsilon \) be as in Lemma 3.3 and set

\[
B_\varepsilon(u_\varepsilon) := \left\{ x \in I_\varepsilon : \frac{b_\varepsilon}{\sqrt{a_\varepsilon}} < \frac{|u_\varepsilon(x + \varepsilon) - u_\varepsilon(x)|}{\varepsilon} < \frac{c_\varepsilon}{\varepsilon} \right\} = \{ x_\varepsilon^1, x_\varepsilon^2, \ldots, x_\varepsilon^{m_\varepsilon} \},
\]

where \( x_\varepsilon^1 < x_\varepsilon^2 < \cdots < x_\varepsilon^{m_\varepsilon} \) and \( m_\varepsilon := \mathcal{H}^0(B_\varepsilon(u_\varepsilon)) \). Now we want to replace the sequence \( u_\varepsilon \) with a new one \( \tilde{u}_\varepsilon \), still converging to \( u \), such that \( B_\varepsilon(\tilde{u}_\varepsilon) \) is empty and \( \mathcal{F}_\varepsilon(\tilde{u}_\varepsilon, I) \leq \mathcal{F}_\varepsilon(u_\varepsilon, I) \).

Setting \( v_\varepsilon^0 = u_\varepsilon \) and for \( k = 0, \ldots, m_\varepsilon - 1 \) we define by induction the functions

\[
v_\varepsilon^{k+1}(t) := \begin{cases} v_\varepsilon(t) & \text{for } t < x_\varepsilon^{k+1} + \varepsilon, \\ v_\varepsilon^k(t) - \left[ v_\varepsilon^k(x_\varepsilon^{k+1} + \varepsilon) - v_\varepsilon^k(x_\varepsilon^{k+1}) \right] & \text{for } t \geq x_\varepsilon^{k+1} + \varepsilon, \end{cases}
\]

and finally we set \( \tilde{u}_\varepsilon := v_\varepsilon^{m_\varepsilon} \) (see Figure 1). Note that \( |v_\varepsilon^k(x_\varepsilon^{k+1} + \varepsilon) - v_\varepsilon^k(x_\varepsilon^{k+1})| = |u_\varepsilon(x_\varepsilon^{k+1} + \varepsilon) - u_\varepsilon(x_\varepsilon^{k+1})| \). Moreover in this way \( \tilde{u}_\varepsilon(x + \varepsilon) - \tilde{u}_\varepsilon(x) = 0 \) if \( x \in B_\varepsilon(u_\varepsilon) \), while \( \tilde{u}_\varepsilon(x + \varepsilon) - \tilde{u}_\varepsilon(x) = u_\varepsilon(x + \varepsilon) - u_\varepsilon(x) \) for \( x \in I_\varepsilon \setminus B_\varepsilon(u_\varepsilon) \).

First of all, using the fact that for every \( \varepsilon > 0 \) and for every \( k = 0, \ldots, m_\varepsilon - 1 \) we have

\[
\int_I |v_\varepsilon^{k+1} - v_\varepsilon^k| \, dx \leq \int_I |v_\varepsilon^k(x_\varepsilon^{k+1} + \varepsilon) - v_\varepsilon^k(x_\varepsilon^{k+1})| \, |I| = \int_I |u_\varepsilon(x_\varepsilon^{k+1} + \varepsilon) - u_\varepsilon(x_\varepsilon^{k+1})| \, |I| \leq c_\varepsilon |I|,
\]

then we can estimate

\[
\int_I |\tilde{u}_\varepsilon - u_\varepsilon| \, dx \leq \sum_{k=0}^{m_\varepsilon-1} \int_I |v_\varepsilon^{k+1} - v_\varepsilon^k| \, dx \leq \sum_{k=0}^{m_\varepsilon-1} c_\varepsilon |I| \leq \mathcal{H}^0(B_\varepsilon(u_\varepsilon)) c_\varepsilon |I| \leq \mathcal{H}^0(D_\varepsilon) c_\varepsilon |I|.
\]

Therefore, by Lemma 3.3, we get \( \tilde{u}_\varepsilon \to u \) in \( L^1(I) \). Moreover, by construction, we clearly have that \( \mathcal{F}_\varepsilon(\tilde{u}_\varepsilon, I) \leq \mathcal{F}_\varepsilon(u_\varepsilon, I) \).

We set

\[
P_\varepsilon := \left\{ x \in I_\varepsilon : \left| \frac{\tilde{u}_\varepsilon(x + \varepsilon) - \tilde{u}_\varepsilon(x)}{\varepsilon} \right| \leq \frac{b_\varepsilon}{\sqrt{a_\varepsilon}} \right\}
\]
\[ I_{\varepsilon}^2 := \left\{ x \in I_{\varepsilon} : \frac{|\tilde{u}_{\varepsilon}(x + \varepsilon) - \tilde{u}_{\varepsilon}(x)|}{\varepsilon} \geq \frac{c_{\varepsilon}}{\varepsilon} \right\}, \]

and note that \( I_{\varepsilon} = I_{\varepsilon}^1 \cup I_{\varepsilon}^2 \). We call \( w_{\varepsilon} \) the function belonging to \( SBV(I) \) defined by

\[
w_{\varepsilon}(x) := \begin{cases} \tilde{u}_{\varepsilon} \left( \frac{x}{\varepsilon} \right) & \text{if } \varepsilon \left\lfloor \frac{x}{\varepsilon} \right\rfloor \in I_{\varepsilon}^1, \\ \tilde{u}_{\varepsilon} \left( \frac{x}{\varepsilon} \right) + \frac{\tilde{u}_{\varepsilon} \left( \frac{x}{\varepsilon} + \varepsilon \right) - \tilde{u}_{\varepsilon} \left( \frac{x}{\varepsilon} \right)}{\varepsilon} \left( x - \varepsilon \left\lfloor \frac{x}{\varepsilon} \right\rfloor \right) & \text{if } \varepsilon \left\lfloor \frac{x}{\varepsilon} \right\rfloor \in I_{\varepsilon}^2 \text{ or } \varepsilon \left\lfloor \frac{x}{\varepsilon} \right\rfloor + \varepsilon \notin (\varepsilon \mathbb{Z} \cap I), \\ 0 & \text{otherwise.} \end{cases}
\]

Now we can estimate

\[
\mathcal{F}_{\varepsilon}(\tilde{u}_{\varepsilon}, I) \geq \frac{\varepsilon}{a_{\varepsilon}} \sum_{x \in I_{\varepsilon}^1} \log \left( 1 + a_{\varepsilon} \frac{|\tilde{u}_{\varepsilon}(x + \varepsilon) - \tilde{u}_{\varepsilon}(x)|^2}{\varepsilon^2} \right) \\
+ \frac{\varepsilon}{a_{\varepsilon}} \sum_{x \in I_{\varepsilon}^2} \log \left( 1 + a_{\varepsilon} \frac{|\tilde{u}_{\varepsilon}(x + \varepsilon) - \tilde{u}_{\varepsilon}(x)|^2}{\varepsilon^2} \right) \\
\geq \frac{1}{a_{\varepsilon}} \int_{I} \log(1 + a_{\varepsilon} |w_{\varepsilon}'|^2) \, dx + \mathcal{H}^0(I_{\varepsilon}) \mathcal{E}_{a_{\varepsilon}} \log \left( 1 + a_{\varepsilon} \frac{c_{\varepsilon}^2}{\varepsilon^2} \right). \tag{3.3}
\]

Fix \( \delta \in (0, 1) \); recalling (3.1) and the definition of \( a_{\varepsilon} \), by Lemma 3.4 and by \( b) \) of Lemma 3.2, we deduce the existence of \( \tilde{e} \) such that for \( \varepsilon \leq \tilde{e} \)

\[
\mathcal{F}_{\varepsilon}(\tilde{u}_{\varepsilon}, I) \geq (1 - \delta) \left( \int_{I} |w_{\varepsilon}'|^2 \, dx + \mathcal{H}^0(S_{w_{\varepsilon}}) \right); \tag{3.4}
\]

Figure 1: The construction of \( v_{\varepsilon}^{k+1} \) with \( x_{\varepsilon}^{k+1} = 0.4 \).
by Ambrosio’s theorem (see [3] Theorems 4.7 and 4.8) we therefore obtain that \( u \in SBV(I) \)
and
\[
\liminf_{\varepsilon \to 0^+} \mathcal{F}_\varepsilon(u_\varepsilon, I) \geq \liminf_{\varepsilon \to 0^+} \mathcal{F}_\varepsilon(\bar{u}_\varepsilon, I) \geq (1 - \delta) \left( \int_I |u'|^2 \, dx + \mathcal{H}^0(S_u) \right),
\]
which gives the desired inequality since \( \delta \) is arbitrary.

Concerning the last part of the statement, notice that for any energy-bounded sequence \( u_\varepsilon \)
it is possible to construct as above a sequence \( w_\varepsilon \in SBV(I) \)
satisfying
\[
\|u_\varepsilon - w_\varepsilon\|_{L^1} \to 0 \quad \text{and} \quad \mathcal{F}_\varepsilon(u_\varepsilon, I) \geq (1 - \delta) \left( \int_I |w'|^2 \, dx + \mathcal{H}^0(S_{w_\varepsilon}) \right);
\]
the precompactness of \( w_\varepsilon \) and therefore the precompactness of \( u_\varepsilon \) follows again from Ambrosio’s theorem.

We conclude this section with a remark that will be useful in the sequel.

**Remark 3.5** Let \( 0 < t_\varepsilon < \varepsilon \) and for \( u \in l^1((t_\varepsilon + \varepsilon \mathbb{Z}) \cap I) \) define
\[
\tilde{\mathcal{F}}_\varepsilon(u, I) := \frac{\varepsilon}{a_\varepsilon} \sum_{x \in I_{t_\varepsilon}^\varepsilon} \log \left( 1 + a_\varepsilon \frac{|u(x + \varepsilon) - u(x)|^2}{\varepsilon^2} \right),
\]
where
\[
I_{t_\varepsilon}^\varepsilon := \{ x \in I \cap (t_\varepsilon + \varepsilon \mathbb{Z}) : x + \varepsilon \in I \};
\]
then we have that Proposition 3.1 is still valid with \( \tilde{\mathcal{F}}_\varepsilon \) instead of \( \mathcal{F}_\varepsilon \) (without meaningful changes in the proof).

## 4 Estimate from below of the \( \Gamma \)-limit in dimension two

**Lemma 4.1** Let \( u_\varepsilon \in l^1(\varepsilon \mathbb{Z}^2) \) be such that \( u_\varepsilon \to u \) in \( L^1(\mathbb{R}^2) \). For \( y, \xi \in \mathbb{Z}^2 \), let \( v_\varepsilon^{y, \xi} \in l^1(\varepsilon(y + \xi \mathbb{Z})^2) \) be defined as \( v_\varepsilon^{y, \xi}(x) := u_\varepsilon(x) \) for every \( x \in \varepsilon(y + \xi \mathbb{Z})^2 \). Then \( v_\varepsilon^{y, \xi} \to u \) in \( L^1(\mathbb{R}^2) \).

**Proof.** We call \( Q_\xi \) the set of points of \( \mathbb{Z}^2 \) contained in \( C_\xi \), i.e.
\[
Q_\xi := C_\xi \cap \mathbb{Z}^2 = \{ \tau^1, \ldots, \tau^k \},
\]
where \( C_\xi \) is the set defined in (2.2), see Figure 2. For \( j = 1, \ldots, k \) we set \( u_\varepsilon^j(x) = u_\varepsilon(x - \varepsilon \tau^j) \).

Since
\[
\int_{\mathbb{R}^2} |u_\varepsilon^j(x) - u(x)| \, dx \leq \int_{\mathbb{R}^2} |u_\varepsilon(x - \varepsilon \tau^j) - u(x - \varepsilon \tau^j)| \, dx + \int_{\mathbb{R}^2} |u(x - \varepsilon \tau^j) - u(x)| \, dx
\]
\[
= \int_{\mathbb{R}^2} |u_\varepsilon(x) - u(x)| \, dx + \int_{\mathbb{R}^2} |u(x) - u(x)| \, dx,
\]
we have that \( u_\varepsilon^j \to u \) in \( L^1(I) \) as \( \varepsilon \to 0^+ \), for every \( j \in \{1, \ldots, k\} \); therefore, up to passing to a subsequence, we can suppose that
- there exists \( N \subset \mathbb{R}^2 \) with \( L^2(N) = 0 \) such that \( u_\varepsilon^j \to u \) pointwise in \( \mathbb{R}^2 \setminus N \) for \( j = 1, \ldots, k \).
• \(|u^j_\varepsilon| \leq g^j\) almost everywhere where \(g^j\) is an \(L^1\)-function, for \(j = 1, \ldots, k\).

Since for every \(x \in \mathbb{R}^2 \setminus N\) there exists \(j \in \{1, \ldots, k\}\) such that \(v^y_{\varepsilon, \kappa}(x) = u^j_\varepsilon(x)\), we get \(v^y_{\varepsilon, \kappa} \to u\) pointwise in \(\mathbb{R}^2 \setminus N\); moreover \(|v^y_{\varepsilon, \kappa}| \leq g_1 + \ldots + g_k\) and therefore, by the Dominated Convergence Theorem, \(v^y_{\varepsilon, \kappa} \to u\) in \(L^1\). As the same argument can be repeated for every subsequence, the lemma is proved.

We will need also the following lemma.

\[\text{Figure 2: The set } Q_\xi \text{ and the shaded region.}\]

**Lemma 4.2** Let \(Q_\xi\) be the set defined in (4.1). Then \(\mathcal{H}^0(Q_\xi) = |\xi|^2\).

**Proof.** We refer to Figure 2. We associate every point \(x \in Q_\xi\) with the square \(x + [0, 1] \times [0, 1]\). The area of the shaded region, which is the union of such squares, coincides with the cardinality of \(Q_\xi\) and it is clear from the Figure that it is equal to the area of the set \(C_\xi\) (see (2.2)).

Before starting the proof of the \(\Gamma\)-liminf inequality it is convenient to rewrite the functional \(F_\varepsilon\) in such a way that we can apply the one dimensional estimate proved in the previous section. After observing that \(\varepsilon \mathbb{Z}^2\) is the disjoint union of the sets \(\varepsilon (y + \mathbb{Z}_\xi^2)\) for \(y \in Q_\xi\), we can write,
for every \( u \in l^1(\mathbb{Z}^2 \cap \Omega) \),

\[
F_\varepsilon(u) = \varepsilon^2 \sum_{x \in \Omega \cap \mathbb{Z}^2} \sum_{\xi \in \mathbb{Z}^2} \frac{1}{a_\varepsilon|\xi|} \log \left( 1 + a_\varepsilon|\xi| \frac{|u(x + \varepsilon \xi) - u(x)|^2}{\varepsilon^2|\xi|^2} \right) \rho(\xi)
\]

\[
= \sum_{\xi \in \mathbb{Z}^2} \rho(\xi) \sum_{x \in \Omega \cap \mathbb{Z}^2} \varepsilon^2 \frac{1}{a_\varepsilon|\xi|} \log \left( 1 + a_\varepsilon|\xi| \frac{|u(x + \varepsilon \xi) - u(x)|^2}{\varepsilon^2|\xi|^2} \right)
\]

\[
= \sum_{\xi \in \mathbb{Z}^2} \rho(\xi) \sum_{y \in Q_\xi} G^y_{\varepsilon, \xi}(u), \quad (4.2)
\]

where

\[
G^y_{\varepsilon, \xi}(u) := \varepsilon^2 \sum_{x \in (y+\mathbb{Z}^2) \cap \Omega} \frac{1}{a_\varepsilon|\xi|} \log \left( 1 + a_\varepsilon|\xi| \frac{|u(x + \varepsilon \xi) - u(x)|^2}{\varepsilon^2|\xi|^2} \right).
\]

Taking \( u_\varepsilon \) equal to zero in \((\varepsilon \mathbb{Z}^2 \setminus \Omega)\) and \( u \) equal to zero outside \( \Omega \), we can suppose that \( u_\varepsilon \in l^1(\varepsilon \mathbb{Z}^2) \), \( u \in L^1(\mathbb{R}^2) \), and \( u_\varepsilon \to u \) in \( L^1(\mathbb{R}^2) \). If we are able to prove that \( u \in GSBV(\Omega) \) and

\[
\liminf_{\varepsilon \to 0^+} G^y_{\varepsilon, \xi}(u_\varepsilon) \geq \frac{1}{|\xi|^2} \left( \int_\Omega |\nabla u \cdot \hat{\xi}|^2 \, dx + \int_{S_u} |\nu_u \cdot \hat{\xi}| \, dH^1 \right), \quad (4.3)
\]

for every \( \xi \in \mathbb{Z}^2 \) and every \( y \in Q_\xi \), then, by (4.2), Lemma 4.2, (1.4) and (2.4), we have

\[
\liminf_{\varepsilon \to 0^+} F_\varepsilon(u_\varepsilon) \geq \sum_{\xi \in \mathbb{Z}^2} \rho(\xi) \sum_{y \in Q_\xi} \liminf_{\varepsilon \to 0^+} G^y_{\varepsilon, \xi}(u_\varepsilon)
\]

\[
\geq \sum_{\xi \in \mathbb{Z}^2} \rho(\xi) \sum_{y \in Q_\xi} \frac{1}{|\xi|^2} \left( \int_\Omega |\nabla u \cdot \hat{\xi}|^2 \, dx + \int_{S_u} |\nu_u \cdot \hat{\xi}| \, dH^1 \right)
\]

\[
= \sum_{\xi \in \mathbb{Z}^2} \rho(\xi) \left( \int_\Omega |\nabla u \cdot \hat{\xi}|^2 \, dx + \int_{S_u} |\nu_u \cdot \hat{\xi}| \, dH^1 \right)
\]

\[
= \int_\Omega \frac{1}{2} \sum_{\xi \in \mathbb{Z}^2} \rho(\xi) \left( |\nabla u \cdot \hat{\xi}|^2 + |\nabla u \cdot \hat{\xi}^\perp|^2 \right) \, dx + \int_{S_u} \sum_{\xi \in \mathbb{Z}^2} \rho(\xi) |\nu_u \cdot \hat{\xi}| \, dH^1
\]

\[
= c_\rho \int_\Omega |\nabla u|^2 \, dx + \int_{S_u} \phi(\nu_u) \, dH^1. \quad (4.4)
\]

Let us prove (4.3). We denote the hyperplane orthogonal to \( \xi \) by \( \Pi_\xi \) and by \( \Omega_\xi \) the projection of \( \Omega \) on \( \Pi_\xi \). For every \( w \in \Omega_\xi \) we set \( \Omega^w_\xi := \{ x \in \Omega : x = w + t\xi \text{ for } t \in \mathbb{R} \} \) and, given a function \( f \in L^1(\Omega) \), we denote by \( f^w(t) \) its restriction to \( \Omega^w_\xi \). We also need to define the discrete counterparts of the sets \( \Omega_\xi \) and \( \Omega^w_\xi \); for \( y \in Q_\xi \) let \( O^y_{\varepsilon, \xi} = \varepsilon(y + \mathbb{Z^2}) \cap \Omega \) and let \( (O^y_{\varepsilon, \xi})_\xi \) be its projection on \( \Pi_\xi \). For every \( w \in (O^y_{\varepsilon, \xi})_\xi \) let

\[
(O^{y, \xi})^w_\xi := \{ x \in O^{y, \xi} : x = w + t\xi \text{ for } t \in \mathbb{R}, x + \varepsilon \xi \in \Omega \}.
\]
Moreover, let $w \in \Omega_\xi$ such that $w = \tilde{w} + s\varepsilon\xi^\perp$, where $\tilde{w} \in (O_\xi^{\varepsilon,\xi})_\xi$ and $s \in [0, 1)$, and define the section $(O_\xi^{\varepsilon,\xi},_w := (O_\xi^{\varepsilon,\xi})_\xi + s\varepsilon\xi^\perp$.

Given $\mu > 0$ let $\Omega_\mu := \{x \in \Omega : d(x, \partial \Omega) > \mu\}$ and note that for $\varepsilon$ sufficiently small we have

$$\Omega_\mu \subset \bigcup_{x \in O_\xi^{\varepsilon,\xi}} (x + \varepsilon C_\xi),$$

then we can write

$$G_\varepsilon^{\nu,\xi}(u_\varepsilon) = \varepsilon^2 \sum_{w \in (O_\xi^{\varepsilon,\xi})_\xi} \sum_{x \in (O_\xi^{\varepsilon,\xi})^w_\xi} \frac{1}{a_\varepsilon|\xi|} \log \left(1 + a_\varepsilon|\xi| \frac{|u_\varepsilon(x + \varepsilon\xi) - u_\varepsilon(x)|^2}{\varepsilon^2|\xi|^2}\right)$$

$$= \frac{1}{|\xi|^2} \sum_{w \in (O_\xi^{\varepsilon,\xi})_\xi} \sum_{x \in (O_\xi^{\varepsilon,\xi})^w_\xi} \frac{\varepsilon|\xi|}{a_\varepsilon|\xi|} \log \left(1 + a_\varepsilon|\xi| \frac{|u_\varepsilon(x + \varepsilon\xi) - u_\varepsilon(x)|^2}{\varepsilon^2|\xi|^2}\right)$$

$$\geq \frac{1}{|\xi|^2} \int_{(\Omega_\mu)_\xi} \left[ \frac{\varepsilon|\xi|}{a_\varepsilon|\xi|} \sum_{x \in (O_\xi^{\varepsilon,\xi})^w_\xi} \log \left(1 + a_\varepsilon|\xi| \frac{|v_\varepsilon^{\nu,\xi}(x + \varepsilon\xi) - v_\varepsilon^{\nu,\xi}(x)|^2}{\varepsilon^2|\xi|^2}\right) \right] d\mathcal{H}^1(w), \quad (4.5)$$

where $(\Omega_\mu)_\xi$ is the projection of $\Omega_\mu$ on $\Pi_\xi$ and $v_\varepsilon^{\nu,\xi}$ is the sequence defined in Lemma 4.1, which takes the constant value $u_\varepsilon(x)$ in the set $x + \varepsilon C_\xi$ for $x \in O_\xi^{\varepsilon,\xi}$. Set $\eta = \eta(\varepsilon) := \varepsilon|\xi|$, $a_\eta := \eta \log \frac{1}{\eta}$, $w_\eta^{\nu,\xi} := v_\varepsilon^{\nu,\xi}$, and observe that

$$\lim_{\varepsilon \to 0^+} \frac{a_\eta}{a_\varepsilon|\xi|} = 1. \quad (4.6)$$

Fix $\delta \in (0, 1)$, by (4.6) for $\varepsilon$ sufficiently small we have $(2 - \delta)a_\eta > a_\varepsilon|\xi| > \delta a_\eta$, then by (4.5) we obtain

$$G_\varepsilon^{\nu,\xi}(u_\varepsilon) \geq \frac{1}{|\xi|^2} \int_{(\Omega_\mu)_\xi} \left[ \frac{\eta}{(2 - \delta)a_\eta} \sum_{x \in (O_\xi^{\varepsilon,\xi})^w_\xi} \log \left(1 + \delta a_\eta \frac{|w_\eta^{\nu,\xi}(x + \eta \hat{\xi}) - w_\eta^{\nu,\xi}(x)|^2}{\eta^2}\right) \right] d\mathcal{H}^1(w)$$

$$\geq \frac{1}{(2 - \delta)|\xi|^2} \int_{(\Omega_\mu)_\xi} \tilde{\mathcal{F}}_\eta\left((\sqrt{\delta} w_\eta^{\nu,\xi})^w_\xi, (\Omega_\mu)^w_\xi\right) d\mathcal{H}^1(w), \quad (4.7)$$

where $\tilde{\mathcal{F}}_\eta$ is the one dimensional functional defined in Remark 3.5 for a suitable choice of the sequence $t_\eta$. Since $(w_\eta^{\nu,\xi})^w_\xi \to u_\xi^w$ for $\mathcal{H}^1$-a.e. $w \in \Pi_\xi$, as $\eta \to 0$ (thanks to Lemma 4.1), by Proposition 3.1 and Remark 3.5 we deduce

$$\liminf_{\eta \to 0^+} \tilde{\mathcal{F}}_\eta\left((\sqrt{\delta} w_\eta^{\nu,\xi})^w_\xi, (\Omega_\mu)^w_\xi\right) \geq \int_{(\Omega_\mu)^w_\xi} \delta|u_\xi^w|^2 dt + \mathcal{H}^0(S_{u_\xi^w})$$

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and then by Fatou's Lemma we have
\[
\liminf_{\varepsilon \to 0^+} G_\varepsilon^\xi(u_\varepsilon) \geq \frac{1}{(2 - \delta)|\xi|^2} \int_{(\Omega_\varepsilon)_\varepsilon} \left( \int_{(\Omega_\varepsilon)_\varepsilon} \delta|u_\varepsilon|^2 dt + \mathcal{H}^0(S_{u_\varepsilon}) \right) d\mathcal{H}^1(w),
\]
from which (4.3) follows letting \( \delta \uparrow 1, \mu \downarrow 0 \), and applying the well known slicing result due to Ambrosio (see [2]).

## 5 Estimate from above of the \( \Gamma \)-limit

The proof of the \( \Gamma \)-limsup inequality will be based on the density result due to Cortesani and Toader. For the convenience of the reader we recall in the following the statement of their theorem. Let \( \Omega \) be an open bounded subset in \( \mathbb{R}^N \) with Lipschitz boundary and denote by \( \mathcal{W}(\Omega) \) the space of all function \( \varphi \in SBV(\Omega) \) enjoying the following properties:

i) \( \mathcal{H}^{N-1}(\overline{S}_\varphi \setminus S_\varphi) = 0 \);

ii) \( \overline{S}_\varphi \) is the intersection of \( \Omega \) with the union of a finite number of pairwise disjoint \((N - 1)\)-simplices;

iii) \( \varphi \in W^{k,\infty}(\Omega \setminus \overline{S}_\varphi) \) for every \( k \in \mathbb{N} \).

Cortesani and Toader have proved in [8] the following density result.

**Theorem 5.1** Let \( u \in GSBV^p(\Omega) \) \((p > 1)\) and let \( \phi : \mathbb{R} \times S_\varphi^{N-1} \to [0, +\infty) \) be a continuous function increasing in the first variable and even in the second one. Then there exists a sequence \( w_j \in \mathcal{W}(\Omega) \) such that \( w_j \to u \) strongly in \( L^1(\Omega) \), \( \nabla w_j \to \nabla u \) strongly in \( L^p(\Omega, \mathbb{R}^N) \), \( \lim_{j \to +\infty} \| w_j \|_\infty = \| u \|_\infty \) and

\[
\limsup_{j \to +\infty} \int_{S_{w_j}} \phi(|w_j^+-w_j^-|, \nu_{w_j}) \, d\mathcal{H}^{N-1} \leq \int_{S_\varphi} \phi(|u^+-u^-|, \nu_u) \, d\mathcal{H}^{N-1}.
\]

**Remark 5.2** Under the additional assumption that \( 1 < p \leq 2 \) the structure of the jump set of the functions \( w_j \) given by Theorem 5.1 can be further improved by using a capacity argument, see again [8]. In particular for \( N = 2 \) and \( p = 2 \), we can suppose that \( \overline{S}_w \) is made up of a finite family of pairwise disjoint segments compactly contained in \( \Omega \).

Therefore it will be enough to prove the \( \Gamma \)-limsup inequality for a function \( u \in \mathcal{W}(\Omega) \) whose discontinuity set consists of the union of a finite family \( \{S_1, ..., S_k\} \) of disjoint segments compactly contained in \( \Omega \). Let \( \varepsilon_n \to 0 \) and set, for every \( u \in L^1(\Omega) \), \( F''(u) := \Gamma \text{-limsup}_{n \to \infty} F_{\varepsilon_n}(u) \); we want to prove that

\[
F''(u) \leq F(u). \tag{5.1}
\]

Let us consider first the case

\[
S_i \cap \varepsilon_n Z = \emptyset \quad \forall n \in \mathbb{N}, \forall i \in \{1, ..., k\}. \tag{5.2}
\]
Let $u_n \in L^1(\varepsilon_n \mathbb{Z} \cap \Omega)$ be such that $u_n(x) = u(x)$ for every $x \in \varepsilon_n \mathbb{Z}^2 \cap \Omega$. Clearly $u_n \to u$ in $L^1(\Omega)$. As for the proof of the $\Gamma$-liminf inequality, by 4.2 the thesis is achieved once we have shown that

$$\limsup_{n \to \infty} G^{\nu, \xi}_{\varepsilon_n}(u_n) \leq \frac{1}{|\xi|^2} \left( \int_{\Omega} |\nabla u \cdot \dot{\xi}|^2 \, dx + \int_{S_u} |\nu_u \cdot \dot{\xi}| \, d\mathcal{H}^1 \right) \quad \forall \xi \in \mathbb{Z}^2, \forall y \in Q_\varepsilon. \quad (5.3)$$

In the sequel, given $x_1$ and $x_2$ in $\mathbb{R}^2$, we denote by $[x_1, x_2]$ the segment joining the two points. Let us define the following sets:

$$A_n := \{ x \in \varepsilon_n (y + \mathbb{Z}^2_\varepsilon) \cap \Omega : x + \varepsilon_n \xi \in \Omega, [x, x + \varepsilon_n \xi] \cap S_j = \emptyset \text{ for } j = 1, \ldots, k \},$$

and

$$B_n^j := \{ x \in \varepsilon_n (y + \mathbb{Z}^2_\varepsilon) \cap \Omega : x + \varepsilon_n \xi \in \Omega, [x, x + \varepsilon_n \xi] \cap S_j \neq \emptyset \} \quad j = 1, \ldots, k.$$  

Clearly for $n$ large enough, $B_n^i \cap B_n^j = \emptyset$ if $i \neq j$, thus we can write

$$G^{\nu, \xi}_{\varepsilon_n}(u_n) = \varepsilon_n^2 \sum_{x \in A_n} \frac{1}{a_{\varepsilon_n} |\xi|} \log \left( 1 + a_{\varepsilon_n} |\xi| \left| \frac{|u_n(x + \varepsilon_n \xi) - u_n(x)|^2}{\varepsilon_n^2 |\xi|^2} \right| \right)$$

$$+ \sum_{j=1}^k \varepsilon_n^2 \sum_{x \in B_n^j} \frac{1}{a_{\varepsilon_n} |\xi|} \log \left( 1 + a_{\varepsilon_n} |\xi| \left| \frac{|u_n(x + \varepsilon_n \xi) - u_n(x)|^2}{\varepsilon_n^2 |\xi|^2} \right| \right)$$

$$= \frac{1}{|\xi|^2} \int_{\Omega} \frac{1}{a_{\varepsilon_n} |\xi|} \log \left( 1 + a_{\varepsilon_n} |\xi| \left| \frac{|v_n^{\nu, \xi}(x + \varepsilon_n \xi) - v_n^{\nu, \xi}(x)|^2}{\varepsilon_n^2 |\xi|^2} \right| \right) \chi_{(A_n + \varepsilon_n \mathbb{C}_\xi)}$$

$$+ \frac{1}{|\xi|^2} \sum_{j=1}^k \varepsilon_n |\xi| \sum_{x \in B_n^j} \frac{\varepsilon_n |\xi|}{a_{\varepsilon_n} |\xi|} \log \left( 1 + a_{\varepsilon_n} |\xi| \left| \frac{|u_n(x + \varepsilon_n \xi) - u_n(x)|^2}{\varepsilon_n^2 |\xi|^2} \right| \right),$$

where $v_n^{\nu, \xi}$ is the sequence defined in Lemma 4.1, while $C_\xi$ is the set defined in (2.2). It is immediate to see that

$$\chi_{(A_n + \varepsilon_n \mathbb{C}_\xi)} \to \chi_{\Omega \setminus S_u} \quad \text{pointwise.} \quad (5.4)$$

Take $x \in \Omega \setminus S_u$, for $\varepsilon_n$ sufficiently small there exists $y_n \in \varepsilon_n (y + \mathbb{Z}^2_\varepsilon)$ be such that $x \in y_n + \varepsilon_n C_\xi$; by Lagrange’s Theorem it turns out that

$$\log \left( 1 + a_{\varepsilon_n} |\xi| \left| \frac{|v_n^{\nu, \xi}(x + \varepsilon_n \xi) - v_n^{\nu, \xi}(x)|^2}{\varepsilon_n^2 |\xi|^2} \right| \right) = \log \left( 1 + a_{\varepsilon_n} |\xi| \left| \frac{|u(y_n + \varepsilon_n \xi) - u(y_n)|^2}{\varepsilon_n^2 |\xi|^2} \right| \right)$$

$$\leq a_{\varepsilon_n} |\xi||\nabla u(y_n) \cdot \dot{\xi}|^2,$$

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where \( z_n \in [y_n, y_n + \varepsilon_n \xi] \) and therefore \( z_n \to x \). Taking into account the continuity of \( \nabla u \) and recalling (5.4), we deduce that

\[
\limsup_{n \to \infty} I_{n,1} \leq \frac{1}{|\xi|^2} \int_{\Omega} |\nabla u(x) \cdot \xi|^2 \, dx. \tag{5.5}
\]

Moreover, for every \( x \in B_n^j \), we have

\[
\frac{\varepsilon_n |\xi|}{a_{\varepsilon_n} |\xi|} \log \left( 1 + a_{\varepsilon_n} |\xi| \frac{|u_n(x + \varepsilon_n \xi) - u_n(x)|^2}{\varepsilon_n^2 |\xi|^2} \right) \leq \frac{\varepsilon_n}{a_{\varepsilon_n}} \log \left( 1 + a_{\varepsilon_n} \frac{4\|u\|_{L^\infty}}{\varepsilon_n^2 |\xi|} \right) \to 1, \tag{5.6}
\]

where the last limit follows from the definition of \( a_{\varepsilon_n} \). Denote by \( l_\xi(S_j) \) the length of the projection of \( S_j \) on \( \Pi_\xi \); using the fact that \( l_\xi(S_j) = \int_{S_j} \nu_u \cdot \xi \, d\mathcal{H}^1 \), we easily obtain (see Figure 3 below)

\[
\mathcal{H}^0(B_n^j) \leq \frac{l_\xi(S_j)}{\varepsilon_n |\xi|} + 1 \leq \frac{1}{\varepsilon_n |\xi|} \int_{S_j} \nu_u \cdot \xi \, d\mathcal{H}^1 + 1; \tag{5.7}
\]

therefore from (5.6) and (5.7) we get

\[
\limsup_{n \to \infty} I_{2,n} \leq \frac{1}{|\xi|^2} \limsup_{n \to \infty} \sum_{j=1}^k \varepsilon_n |\xi| \mathcal{H}^0(B_n^j) \leq \frac{1}{|\xi|^2} \sum_{j=1}^k \int_{S_j} \nu_u \cdot \xi \, d\mathcal{H}^1 = \frac{1}{|\xi|^2} \int_{S_u} \nu_u \cdot \xi \, d\mathcal{H}^1,
\]

which, combined with (5.5), gives (5.3) and therefore (5.1).

---

**Figure 3:** The projection of \( S_j \) on \( \Pi_\xi \).

If (5.2) is not true we can argue in the following way. We first observe that it is possible to find a sequence \( (\tau_k) \subset \mathbb{R}^2 \) such that \( \tau_k \to 0 \) and \( S_u + \tau_k \) satisfy (5.2) for every \( k \). Let \( u_k(x) := u(x - \tau_k) \), then \( u_k \to u \), \( S_{u_k} = S_u + \tau_k \) satisfies (5.2), and \( F(u_k) \to F(u) \); using the previous step and the semicontinuity of \( F'' \), we have

\[
F''(u) \leq \liminf_{k \to \infty} F''(u_k) \leq \lim_{k \to \infty} F(u_k) = F(u),
\]

which concludes the proof.
6 Compactness

In this section we prove the equicoerciveness of the approximating functionals $F_\varepsilon$. We will use the following $L^1$-precompactness criterion by slicing introduced by Alberti, Bouchitté, Seppecher (see [1]). Using the notations introduced in section 4, given a family $G$ of functions, for every $\xi \in S^{N-1}$ and $w \in \Pi_\xi$ we set $G^w_\xi := \{u^w_\xi : u \in G\}$; moreover we say that a family $G'$ is $\delta$-close to $G$ if $G'$ is contained in a $\delta$-neighborhood of $G$.

**Lemma 6.1** Let $G$ be a family of equiintegrable functions belonging to $L^1(A)$ and assume that there exists a basis of unit vectors $\{\xi_1, \ldots, \xi_N\}$ with the property that for every $i = 1, \ldots, N$, for every $\delta > 0$, there exists a family $G_\delta$ $\delta$-close to $G$ such that $(G_\delta)^w_\xi$ is precompact in $L^1(A^w_\xi)$ for $\mathcal{H}^{N-1}$-a.e. $w \in A_\xi$. Then $G$ is precompact in $L^1(A)$.

**Proposition 6.2** Let $(u_\varepsilon)$ be a sequence of equibounded functions such that $\sup_\varepsilon F_\varepsilon(u_\varepsilon) < M < +\infty$; then $(u_\varepsilon)$ is strongly precompact in $L^p(\Omega)$, for every $p \geq 1$.

**Proof.** Clearly it is enough to prove the precompactness in $L^1$. Let $\{e_1, e_2\}$ be the canonical basis in $\mathbb{R}^2$ and assume for simplicity that $\rho(e_1) \neq 0$, thus from (4.2) for every $y \in Q_{e_1}$ we have

$$M > \sup_\varepsilon F_\varepsilon(u_\varepsilon) \geq \rho(e_1) \sup_\varepsilon G_\varepsilon^{y,e_1}(u_\varepsilon).$$

Considering the proof of the $\Gamma$-liminf inequality, for $\mu > 0$ let $\Omega_\mu := \{x \in \Omega : d(x, \partial \Omega) > \mu\}$. Note that for $\xi = e_i$ (for $i = 1, 2$) the function $w_\eta^{b,\xi}$ coincides with $u_\varepsilon$. Since $\eta := \varepsilon|\xi| = \varepsilon$ we can apply (4.7) with $\delta = 1$, thus we have

$$G_\varepsilon^{0,e_i}(u_\varepsilon) \geq \int_{(\Omega_\mu)_e_i} \left[ \frac{\varepsilon}{a_\varepsilon} \sum \log \left( 1 + a_\varepsilon \frac{|u_\varepsilon(x + \varepsilon e_i) - u_\varepsilon(x)|^2}{\varepsilon^2} \right) \right] d\mathcal{H}^1(w)$$

$$\geq \int_{(\Omega_\mu)_e_i} \tilde{F}_\varepsilon \left( (u_\varepsilon)_e_i, (\Omega_\mu)_e_i \right) d\mathcal{H}^1(w). \quad (6.1)$$

Hence there exist $C > 0$ such that

$$C > \sup_\varepsilon \int_{(\Omega_\mu)_e_i} f_\varepsilon(w) d\mathcal{H}^1(w), \quad (6.2)$$

where we have denoted

$$f_\varepsilon(w) := \tilde{F}_\varepsilon \left( (u_\varepsilon)_e_i, (\Omega_\mu)_e_i \right).$$

Fix $\delta \in (0, 1)$ and choose $k > 0$ so large that

$$C \frac{\sup_\varepsilon \|u_\varepsilon\|_{L^\infty}}{k} \operatorname{diam}(\Omega) < \delta; \quad (6.3)$$

setting $A_{e_i}^k := \{w \in (\Omega_\mu)_e_i : f_\varepsilon(w) > k\}$, by Chebychev Inequality and (6.2), we can estimate

$$|A_{e_i}^k| \leq \frac{1}{k} \int_{(\Omega_\mu)_e_i} f_\varepsilon(w) d\mathcal{H}^1(w) \leq \frac{C}{k}. \quad (6.4)$$

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Let \( z_{\epsilon, \delta} \in \mathcal{L}_{\epsilon}(\Omega_{\mu}) \) be defined by slicing in the following way,

\[
(z_{\epsilon, \delta})_{e_i}^w = \begin{cases} 
(u_{\epsilon})_{e_i}^w & \text{if } w \notin A_{\epsilon, \delta}^k \\
0 & \text{if } w \in A_{\epsilon, \delta}^k.
\end{cases}
\]

Clearly we have

\[
\|z_{\epsilon, \delta} - u_{\epsilon}\|_{L^1} \leq \sup_{\epsilon} \|u_{\epsilon}\|_{\infty} |A_{\epsilon, \delta}^k| \text{diam}(\Omega) \leq \delta,
\]

where the last inequality follows from (6.4) and (6.3). Moreover for \( w \notin A_{\epsilon, \delta}^k \) we have

\[
F_{\epsilon}(z_{\epsilon, \delta})_{e_i}^w \leq f_{\epsilon}(w) \text{ and thus for every } w \in (\Omega_{\mu})_{e_i} \text{ we have } F_{\epsilon}(z_{\epsilon, \delta})_{e_i}^w \leq k.
\]

Therefore \((z_{\epsilon, \delta})_{e_i}^w\), by the one dimensional result, is precompact in \( L^1((\Omega_{\mu})_{e_i}^w) \) for every \( w \in (\Omega_{\mu})_{e_i} \). Thus we have constructed a sequence which is \( \delta \)-close to \((u_{\epsilon})\) and such that the one-dimensional sections in the \( e_i \)-direction are precompact, for \( i = 1, 2 \). By Lemma 6.1 we have the compactness in \( L^1(\Omega_{\mu}) \) and the thesis follows applying a diagonal argument to the sequence of open sets \( \Omega_{\mu_n} \) for \( \mu_n \to 0 \).

7 Some considerations on the numerical results

The choice of the best algorithm for the solution of the Mumford-Shah functional, and the related convergence properties, are difficult problems which are not going to be tackled here. Nevertheless some heuristic considerations and the numerical results themselves seem to suggest the use of functionals like (1.6) instead of (1.3), independently of the choice of the algorithm.

In detail, let \( \Omega = (0,1)^2 \) and \( g : \Omega \to [0,1] \) be the gray level function of the given image; the results presented here are obtained using the functional

\[
F_{\epsilon}(u) = \epsilon^2 \sum_{x \in \Omega \cap \mathbb{Z}^2} \sum_{\xi \in \mathbb{Z}^2} \alpha \frac{a_x}{a_{\epsilon}|\xi|} \log \left( 1 + \frac{\beta a_x |\xi| |u(x + \epsilon \xi) - u(x)|^2}{\epsilon^2 |\xi|^2} \right) \rho(\xi) + \epsilon^2 \sum_{x \in \Omega \cap \mathbb{Z}^2} |u(x) - g(x)|^2,
\]

(7.1)

for some suitable choice of the coefficients \( \alpha > 0 \) and \( \beta > 0 \). The convolution term \( \rho : \mathbb{Z}^2 \to [0, +\infty) \) is not zero only for the nearest neighbors (i.e. \( \xi \in \mathbb{Z}^2 \) such that \( |\xi|_{\infty} = 1 \)) and takes constantly the value \((\sqrt{2} - 1)/2\).

The functional (7.1) \( \Gamma \) converges (as \( \epsilon \to 0 \) and in the strong topology of \( L^2(\Omega) \)) to the anisotropic Mumford-Shah functional

\[
F(u) = c_p \beta \int_{\Omega} |\nabla u|^2 \, dx + \alpha \int_{\Omega} \phi(\nu) \, d\mathcal{H}^1 + \int_{\Omega} |u - g|^2 \, dx,
\]

(7.2)

where the constant \( c_p \) and the anisotropy function \( \phi(\nu) \) are given by

\[
c_p := \frac{1}{2} \sum_{\xi \in \mathbb{Z}^2} \rho(\xi) = 2(\sqrt{2} - 1),
\]

16
\[
\phi(v) := \sum_{\xi \in \mathbb{Z}^2} \frac{\rho(\xi)}{|\langle v, \xi \rangle|} (|v_1| + |v_2| + \sqrt{2/\pi} |v_1 + v_2| + \sqrt{2/\pi} |v_1 - v_2|),
\]
(note that the level curve \(\{ \phi(v) = 1 \} \) is a regular octagon).

It’s well known, see [4], that the minimization of the Mumford-Shah functional produces a filtering effect on the edges of the original image. Indeed the minimizer should preserve only the ones having a contrast greater than a threshold value, which is given (approximately) by \(\sqrt{2\alpha/\beta} \). Nevertheless, at the discrete level, functionals like (1.3), with
\[
f(t) = \frac{2\alpha}{\pi} \arctan \left( \frac{\beta\pi}{2\alpha} t \right),
\]
which is a quite common choice, introduce a local threshold on the contrast which is also related to the presence of some local minimizers and stationary points, see [14]. Indeed, considering in (1.3) the function \(f(t) \approx \min\{\beta t, \alpha\}\), the local behavior of the functional, in the case \(|\xi| = 1\) and \(\rho(\xi) = 1\), becomes
\[
\varepsilon f \left( \frac{|u(x + \varepsilon \xi) - u(x)|^2}{\varepsilon} \right) dx \approx \begin{cases} 
\beta |u(x + \varepsilon \xi) - u(x)|^2 & \text{if } |u(x + \varepsilon \xi) - u(x)|^2 \leq \alpha \varepsilon / \beta \\
\alpha \varepsilon & \text{if } |u(x + \varepsilon \xi) - u(x)|^2 > \alpha \varepsilon / \beta.
\end{cases}
\]

Hence when the contrast \(|u(x + \varepsilon \xi) - u(x)|\) is greater than \(\sqrt{\alpha \varepsilon / \beta}\) the functional behaves as in the case of a discontinuity. This is clearly dangerous when \(\varepsilon\) is small because \(\alpha \varepsilon / \beta \to 0\) and then the functional numerically looks like a constant. This effect justifies the use of special minimizing techniques, like the Graduated Non Convexity algorithm (in the sequel GNC), which basically aims at reducing the non convexity of the function \(f(t)\) by a recursive minimization of the functional for increasing values of the coefficient \(\beta\) (see [4] for details). From this point of view the function
\[
-\frac{\alpha}{\log \varepsilon} \log \left( 1 - \frac{\beta \log \varepsilon}{\alpha} t \right)
\]
is a better choice than \(f(t)\) because its behavior for \(t\) large avoids the introduction of any discrete threshold, while its slope, for small values of \(t\), gives a diffusion smaller than \(\beta\), which can be interpreted as a sort of GNC effect (see Figure 5).
Figure 5: A comparison between $-\frac{\alpha}{\log \varepsilon} \log \left( 1 - \frac{\beta \log \varepsilon}{\alpha} t \right)$ and $\frac{2\alpha}{\pi} \arctan \left( \frac{\beta \pi}{2\alpha} t \right)$.

The numerical results reported here were obtained with coefficient $\beta = 5.54 \cdot 10^{-4}$, $\alpha = 1.18 \cdot 10^{-4}$ (corresponding to a threshold 0.1), $\varepsilon = 1/256$ (since the image has dimension 256 x 256). The minimizers were computed using both GNC and Gradient Descent (shortly GD). In both the algorithms the descent direction is given by $(1/C)\nabla F_t$, where the constant $C = 2(\varepsilon^2 + \beta)$ is a naive value for the second derivative of the functional, and the linesearch strategy is a quadratic backtracking in the segment $[0, 2]$ (see [11] or [10] for details). The images reported in Figure 7 and the values in Table 1 confirm the previous heuristic considerations: using the log function the numerical solutions computed by GNC and GD are almost the same, both in terms of energy values and graphical quality of the segmentations. On the contrary there is much difference when the arctan is employed, in particular the solution obtained by GD presents many false edges, generated by the wrong local threshold, while the one computed by GNC is smoother because for small values of $t$ the diffusion parameter is bigger, since

$$-\frac{\alpha}{\log \varepsilon} \log \left( 1 - \frac{\beta \log \varepsilon}{\alpha} t \right) < \frac{2\alpha}{\pi} \arctan \left( \frac{\beta \pi}{2\alpha} t \right).$$

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Table 1: Comparison of the performances in terms of energy values (computed in the minimizer), and number of iterations, for Graduated Non Convexity (GNC) and Gradient Descent (GD).

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Figure 6: The original image.

Figure 7: The numerical results, in the same order as in Table 1.
References


