On lower semicontinuity in BH^p and 2-quasiconvexification

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Abstract

It is proved that if $u \in BH^p(\Omega;\mathbb{R}^d)$, with $p > 1$, if $\{u_n\}$ is bounded in $BH^p(\Omega;\mathbb{R}^d)$, $|D_s^2 u_n|(\Omega) \to 0$, and if $u_n \to u$ in $W^{1,1}(\Omega;\mathbb{R}^d)$, then

$$\int_\Omega f(x,u(x),\nabla u(x),\nabla^2 u(x))\,dx \leq \liminf_{n\to+\infty} \int_\Omega f(x,u_n(x),\nabla u_n(x),\nabla^2 u_n(x))\,dx$$

provided $f(x,u,\xi,\cdot)$ is 2-quasiconvex and satisfies some appropriate growth and continuity condition. Characterizations of the 2-quasiconvex envelope when admissible test functions belong to BH^p are provided.

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1 Introduction

In a recent paper Ambrosio [5] proved the following result:

Theorem 1.1 [5, Thm. 4.3] *Let $\Omega \subset \mathbb{R}^N$ be an open set and let*

$$f : \Omega \times \mathbb{R}^N \times \mathbb{R}^{d\times N} \to [0,+\infty)$$

be a Carathéodory function, quasiconvex in ξ*, and such that*

$$|\xi|^p \leq f(x,u,\xi) \leq a(x) + \Psi(|u|)(1+|\xi|^p)$$

for all $(x,u,\xi) \in \Omega \times \mathbb{R}^N \times \mathbb{R}^{d\times N}$*, where* $p>1$*,* $a \in L^1(\Omega)$ *and* $\Psi : [0,\infty) \to [0,\infty)$ *is continuous. Then for every* $u \in SBV(\Omega;\mathbb{R}^d)$ *and any sequence* $\{u_n\} \subset SBV(\Omega;\mathbb{R}^d)$ *converging to* u *in* $L^1_{\text{loc}}(\Omega;\mathbb{R}^d)$ *and such that*

$$\sup_n \mathcal{H}^{N-1}(S(u_n)) < \infty \tag{1.1}$$

we have

$$\int_\Omega f(x,u,\nabla u)\,dx \leq \liminf_{n\to\infty} \int_\Omega f(x,u_n,\nabla u_n)\,dx.$$

Theorem 1.1 extends to the space SBV a classical result obtained by Acerbi and Fusco [1] in the Sobolev space $W^{1,p}$. The approach is based on a careful articulation of the blow-up method developed by Fonseca and Müller [27] together with a Lusin type theorem which allows to approximate SBV functions with Lipschitz functions. This approximation result was first established by Liu [34] in the Sobolev setting and is the key ingredient also in the argument used in [1].

Theorem 1.1 was later improved by Kristensen [32] who studied normal integrands f, possibly unbounded from below, and weakened condition (1.1) to read

$$\sup_n \int_{S(u_n)} \theta(|u_n^+ - u_n^-|)\,d\mathcal{H}^{N-1} < \infty,$$

where $\theta : [0,\infty) \to [0,\infty)$ is a concave, nondecreasing function such that

$$\lim_{t\to 0^+} \frac{\theta(t)}{t} = \infty.$$

The proof in [32] is based on Young measures and on the Hodge Decomposition Theorem.

The aim of this paper is to extend Theorem 1.1 to functionals depending also on the second gradient, i.e., functionals of the form

$$J(u) := \int_\Omega f(x,u,\nabla u,\nabla^2 u)\,dx \tag{1.2}$$

where f is a 2-quasiconvex integrand. We recall that a function $f : \mathbb{S}^{d\times N\times N} \to \mathbb{R}$ is *2-quasiconvex* if

$$\int_Q f(\Lambda + \nabla^2\psi)\,dx \geq f(\Lambda)$$

for every $\Lambda \in \mathbb{S}^{d\times N\times N}$, and for every $\psi \in W_0^{2,\infty}(\Omega;\mathbb{R}^d)$, where $Q := (-1/2,1/2)^N$ and $\mathbb{S}^{d\times N\times N}$ is the space of third order symmetric tensors.

The notion of 2-quasiconvexity was introduced by Meyers [36] and extends the concept of quasiconvexity, which is due to Morrey [37], to integrands which depend on second orders gradients.

Our main motivation to study this class of functionals when ∇u may have jumps derived from the theory of *second order structured deformations* (SOSD). The notion of *first order structured deformation* is due to Del Piero and Owen [19], and the energetics of these deformations was studied by Choksi and Fonseca [15]. Owen and Paroni [38] extended this framework to encompass second order derivatives and they introduced SOSD: a SOSD is a quadruple (κ, g, G, Σ) satisfying some technical conditions preventing interpenetration of matter, as well as regularity properties of the fields away from κ, where κ is the disarrangement site, g is the transplacement and G and Σ are tensor fields with properties similar to those of ∇g and $\nabla^2 g$, respectively. A pair (κ, u) is a simple deformation from a region Ω if $\kappa \subset \Omega$ has zero volume, u is injective and is a "piece-wise" classical deformation from $\Omega \backslash \kappa$. Owen and Paroni [38] showed that a SOSD (κ, g, G, Σ) may be approximated by simple deformations (κ_n, u_n) i.e., $\kappa = \liminf \kappa_n$, $u = \lim u_n$, $g = \lim \nabla u_n$ and $f = \lim \nabla^2 u_n$, where the latter three limits are taken in the L^∞ sense, while $\liminf \kappa_n := \cup_{n=1}^{\infty} \cap_{m=n}^{\infty} \kappa_m$ (for an analog of this approximation result for first order structured deformations and within SBV see [15]). In light of this approximation, we may view $G(x)$ as the local deformation at x without including the effects of discontinuities of the transplacement u_n at the disarrangement site κ_n for the approximating simple deformation (κ_n, u_n). A similar interpretation holds for Σ. The energy associated to a simple deformation is

$$\begin{aligned} E(u, \kappa) \quad &:= \quad \int_\Omega W(x, u, \nabla u, \nabla^2 u)\, dx + \int_\kappa \psi(x, [u], \nu)\, d\mathcal{H}^{N-1} \\ &\qquad + \int_\kappa \eta(x, [\nabla u], \nu)\, d\mathcal{H}^{N-1}, \end{aligned} \tag{1.3}$$

where the first term is the bulk energy of the material in the placement Ω, the second and third terms take into account the contribution of the surface energy due to slips and separation and to interfaces between two phases of material, respectively. In (1.3) ν represents the normal to κ and $[\cdot]$ denotes the jump, while $\mathcal{H}^{N-1}$ stands for $N-1$ Hausdorff measure. By means of this energy we define the energy of a SOSD as the energetically most economical way to attain it from simple deformations, i.e.,

$$I(\kappa, g, G, \Sigma) := \inf\{\liminf E(u_n, \kappa_n) : (k_n, u_n) \text{ approximates } (\kappa, g, G, \Sigma)\}. \tag{1.4}$$

The characterization of (1.4) by means of an integral representation requires a good handle of the lower semicontinuity properties of the bulk contribution (1.2) (see [15] for the case with only first derivatives). Here, we search for conditions ensuring that if $u_n, u \in BH^p(\Omega; \mathbb{R}^d)$ and if $u_n \to u$ in $W^{1,1}(\Omega; \mathbb{R}^d)$ then $J(u) \leq \liminf_{n \to +\infty} J(u_n)$. We recall that the space of *Bounded Hessian* functions, BH, was introduced by Demengel [20], [21] (see also Temam [41], Carriero, Leaci and Tomarelli [13]), and is defined as the set of functions in $W^{1,1}$ whose Hessian, in the sense of distributions, is a finite Radon measure. Also BH^p, for $p > 1$, is the space of functions $u \in BH$ such that $\nabla^2 u \in L^p$.

One of the principal results of this paper is the following theorem:

Theorem 1.2 *Let $\Omega \subset \mathbb{R}^N$ be an open bounded set and let*

$$f : \Omega \times \mathbb{R}^N \times \mathbb{R}^{d \times N} \times \mathbb{S}^{d \times N \times N} \to [0, +\infty)$$

be a normal integrand, 2−quasiconvex in Λ, and such that

$$|\Lambda|^p \leq f(x, u, \xi, \Lambda) \ \leq a\,(x, u, \xi)\,(1 + |\Lambda|^p) \tag{1.5}$$

for $\mathcal{L}^N$ a.e. $x \in \Omega$ and all $(u, \xi, \Lambda) \in \mathbb{R}^N \times \mathbb{R}^{d \times N} \times \mathbb{S}^{d \times N \times N}$, where $a(x, u, \xi)$ is a non-negative constant, and $p > 1$. Then for every $u \in W^{1,1}(\Omega; \mathbb{R}^d)$ and any sequence $\{u_n\} \subset BH(\Omega; \mathbb{R}^d)$ converging to u in $W^{1,1}(\Omega; \mathbb{R}^d)$ and such that

$$\left|D^2_s u_n\right|(\Omega) \to 0 \tag{1.6}$$

we have

$$\int_\Omega f(x, u, \nabla u, \nabla^2 u)\, dx \leq \liminf_{n \to \infty} \int_\Omega f(x, u_n, \nabla u_n, \nabla^2 u_n)\, dx.$$

We remark that no smoothness nor integrability properties are required from the function $(x, u, \xi) a(x, u, \xi)$ that appears in the upper bound (1.5). All that is needed is that $a(x, u, \xi)$ be defined, finite, and non-negative for $\mathcal{L}^N$ a.e. $x \in \Omega$ and all $(u, \xi, \Lambda) \in \mathbb{R}^N \times \mathbb{R}^{d\times N} \times \mathbb{S}^{d\times N\times N}$. In this generality, Theorem 1.2 is new even in the Sobolev setting. Indeed, we have

Corollary 1.3 *Let $\Omega \subset \mathbb{R}^N$ be an open bounded set and let*

$$f : \Omega \times \mathbb{R}^N \times \mathbb{R}^{d\times N} \times \mathbb{S}^{d\times N\times N} \to [0, +\infty)$$

be a normal integrand, $2-$quasiconvex in Λ, and such that

$$0 \leq f(x, u, \xi, \Lambda) \ \leq a\,(x, u, \xi)\,(1 + |\Lambda|^p)$$

for $\mathcal{L}^N$ a.e. $x \in \Omega$ and all $(u, \xi, \Lambda) \in \mathbb{R}^N \times \mathbb{R}^{d\times N} \times \mathbb{S}^{d\times N\times N}$, where $a(x, u, \xi)$ is a non-negative constant, and $p > 1$. Then for every $u \in W^{2,p}(\Omega; \mathbb{R}^d)$ and any sequence $\{u_n\} \subset W^{2,p}(\Omega; \mathbb{R}^d)$ weakly converging to u in $W^{2,p}(\Omega; \mathbb{R}^d)$ we have

$$\int_\Omega f(x, u, \nabla u, \nabla^2 u)\, dx \leq \liminf_{n\to\infty} \int_\Omega f(x, u_n, \nabla u_n, \nabla^2 u_n)\, dx.$$

Corollary 1.3 was first proved by Meyers [36] for integrands satisfying strong continuity conditions, which essentially exclude genuine dependence on lower order terms, and using results of Agmon, Douglis and Nirenberg [2] concerning Poisson kernels for elliptic equations. Fusco [30], under the same set of hypotheses, later gave a simpler proof using De Giorgi's Slicing Lemma.

More recently Guidorzi and Poggiolini [31] established Corollary 1.3 for Carathéodory integrands under the Lipschitz condition

$$|f(x, u, \xi, \Lambda) - f(x, u, \xi, \Lambda_1)| \leq C(1 + |\Lambda|^{p-1} + |\Lambda_1|^{p-1})|\Lambda - \Lambda_1|.$$

As proved in [31], this Lipschitz type condition is automatically satisfied for $2-$quasiconvex integrands which satisfy the growth condition

$$0 \leq f(x, u, \xi, \Lambda) \ \leq C(1 + |\Lambda|^p)$$

(see also [39] for a significantly simpler proof valid within the realm of k-quasiconvexity for any $k \in \mathbb{N}$).

The lower semicontinuity result in [31] is based on the approximation of the integrand f with a nondecreasing sequence of $2-$quasiconvex integrands each of them independent of (x, u, ξ) and convex for large values of Λ. This technique was developed by Marcellini [35] and relies on regularity theory.

Yet another proof of Corollary 1.3 was given by Braides, Fonseca and Leoni in [12], who obtained a general relaxation result in $W^{2,p}(\Omega; \mathbb{R}^d)$ with respect to weak convergence (see also [24] for some related results). The methods used there are based on the general setting of A-quasiconvexity, as introduced by Fonseca and Müller in [29]. The argument exploits the use of Young measures (see also the work of Balder [6] and of Kristensen [32]) together with the blow-up method introduced by Fonseca and Müller in [27].

Both approaches in [12] and [31] present difficulties when one tries to extend them to the space BH. Indeed, Marcellini's method allows one to replace general integrands $f = f(x, u, \nabla u, \nabla^2 u)$ with integrands of the form $f = f(\nabla^2 u)$, but then one is still left with the problem of approximating the admissible sequences in $BH(\Omega; \mathbb{R}^d)$ with sequences in $W^{2,p}(\Omega; \mathbb{R}^d)$.

On the other hand, the $A-$quasiconvexity method used in [12] strongly relies on the underlying PDEs which characterize the space $W^{2,p}(\Omega; \mathbb{R}^d)$. However, in a recent paper extending a result of Alberti [3] we have shown that in the passage from the Sobolev spaces to the space BH the Hessian matrix $D^2 u$ remains symmetric but it may loose, in general, the PDE constraint curl$= 0$ More precisely, we have proved that

Theorem 1.4 [26, Thm. 1.4] *Let Ω be an open subset of $\mathbb{R}^N$ and let $f \in L^1\left(\Omega; \mathbb{R}^{N\times N}_{\text{sym}}\right)$. Then there exists $u \in BH(\Omega)$ and a constant $C > 0$ depending only on N such that*

$$D^2 u = f\,\mathcal{L}^N + [\nabla u] \otimes \nu_{\nabla u}\,\mathcal{H}^{N-1} \lfloor S(\nabla u),$$

and

$$\int_\Omega |u| + |\nabla u|\, dx + \int_{S(\nabla u)\cap\Omega} |[\nabla u]|\, d\mathcal{H}^{N-1} \leq C \int_\Omega |f|\, dx.$$

In spite of this fact, and in view of condition (1.6), it is still possible to approximate BH^p functions by $W^{2,\infty}$ functions. We establish these approximation results in Section 3 following Ambrosio [5] lead and Acerbi and Fusco [1] ideas, via maximal functions. In Section 4 we prove Theorem 1.2 and, as a consequence, in Section 6 we find a new characterization of the quasiconvexification of an integrand with $p-$growth, namely Theorems 6.2, 6.3.

Ongoing work addresses the relaxation problem described in (1.4) in the space BH. A function $u \in BH$ has no jump discontinuities, hence the second integral on the right hand side of (1.3) is equal to zero, and thence the disarrangement site κ can be taken to be the jump set of the gradient of u. A relaxed version (suitable for BH functions) of the approximation of SOSD by means of simple deformations will rest on Theorem 1.4. An integral representation of (1.4) will be obtained using the global method of relaxation introduced by Bouchitté, Fonseca and Mascarenhas [9]. Generalizations to the space SBV^2, introduced by Carriero, Leaci and Tomarelli [14], will take into account also the jumps in the function, thus completing the study of the full energy (1.3).

2 Preliminaries

Let Ω be an open bounded subset, let B denote the open ball centered at the origin with radius 1, and let $B(x,r)$ be the open ball centered at x and with radius r, i.e. $B(x,r) := x + rB$. Q stands for the open cube $\left(-\frac{1}{2}, \frac{1}{2}\right)^N$, and $Q(x,r)$ is the cube centered at x with side length r. For a given set $U \subset \mathbb{R}^N$ we denote by $\mathcal{H}^{N-1}(U)$ its $(N-1)$-dimensional Hausdorff measure, and by $\mathcal{L}^N(U)$ its Lebesgue outer measure; we denote $\omega_N := \mathcal{L}^N(B)$. $\mathbb{S}^{d\times N\times N}$ is the set of third-order symmetric constant tensors, i.e., $\Lambda \in \mathbb{S}^{d\times N\times N}$ if its components Λ_{ijk} satisfy $\Lambda_{ijk} = \Lambda_{ikj}$ for every $i = 1, \ldots, d$ and every $j, k = 1, \ldots, N$.

We recall the definition of the space of functions of *bounded variation* in Ω with values in $\mathbb{R}^d$,

$$BV(\Omega;\mathbb{R}^d) := \{u \in \mathrm{L}^1(\Omega;\mathbb{R}^d) : Du \text{ is a finite Radon measure}\},$$

where $Du = (D_i u_j)_{i=1,\ldots,N, j=1,\ldots d}$ is the distributional derivative of u. For general BV space theory we refer to Braides [11], Evans and Gariepy [23], Ziemer [42]. We represent by ∇u the density of the absolutely continuous part of Du with respect to the Lebesgue measure (or Radon-Nikodym derivative), and $S(u)$ is the jump set, i.e., the set of points x where the approximate upper limit $u_j^+(x)$ is different from the approximate lower limit $u_j^-(x)$ (see Evans and Gariepy [23], Section 5.9); the approximate upper limits are oriented with respect to a chosen normal ν_u to $S(u)$. We set $[u](x) := u^+(x) - u^-(x)$, the difference between the trace of u at $x \in S(u)$, and we denote by $C(u)$ the Cantor part of the measure Du. The following decomposition holds:

$$Du = \nabla u\,\mathcal{L}^N \lfloor \Omega + D_s u = \nabla u\,\mathcal{L}^N \lfloor \Omega + [u] \otimes \nu_u\,\mathcal{H}^{N-1} \lfloor S(u) + C(u), \tag{2.1}$$

where $D_s u$ denotes the singular part of the measure Du with respect to the Lebesgue measure. Following De Giorgi and Ambrosio [18], we define the space of *special functions of bounded variation*, $SBV(\Omega;\mathbb{R}^d)$, as the space of all functions $u \in BV(\Omega;\mathbb{R}^d)$ with $C(u) = 0$.

We now introduce the space of *functions with bounded Hessian*

$$\begin{aligned} BH(\Omega;\mathbb{R}^d) &:= \{u \in \mathrm{W}^{1,1}(\Omega;\mathbb{R}^d) : D^2u \text{ is a finite Radon measure}\} \\ &= \{u \in \mathrm{L}^1(\Omega;\mathbb{R}^d) : Du \in BV(\Omega;\mathbb{R}^{d\times N})\}, \end{aligned}$$

where D^2u denotes the distributional Hessian of u. For various properties of the space BH, we refer to Demengel [20], [21], Carriero, Leaci and Tomarelli [13] and Temam [41]. We recall that if $u \in BH(\Omega;\mathbb{R}^d)$ then $Du = \nabla u$ and $[u](x) = 0$ for $\mathcal{H}^{N-1}$-a.e. $x \in \Omega$. Moreover,

Theorem 2.1 [Demengel [20], [21]] Let $\Omega \subset \mathbb{R}^N$ be a Lipschitz, bounded open set. Then

$$BH(\Omega) \subset W^{1,p}(\Omega)$$

with continuous embedding if $p \leq \frac{N}{N-1}$; the embedding is compact if $p < \frac{N}{N-1}$.

We shall also have occasion to use the following two theorems.

Theorem 2.2 (Interpolation inequality) *Let $\Omega \subset \mathbb{R}^N$ be a Lipschitz, bounded open set. Then for every $\varepsilon > 0$ there is a constant $C = C(\varepsilon)$ such that*

$$\|\nabla u\|_{L^1(\Omega)} \leq C\|u\|_{L^1(\Omega)} + \varepsilon|D^2u|(\Omega)$$

for all $u \in BH(\Omega)$.

Proof. The proof follows easily from Theorem 2.1 and a standard argument by contradiction. See, for instance, Lemma 4.2.2 of Ziemer [42], where the result is proven for Sobolev functions. ■

In view of (2.1), if $u \in BH(\Omega;\mathbb{R}^d)$ then

$$D^2u = \nabla^2 u\,\mathcal{L}^N\lfloor\Omega + [\nabla u]\otimes\nu_{\nabla u}\,\mathcal{H}^{N-1}\lfloor S(\nabla u) + C(\nabla u),$$

where $\nabla^2 u$ is the density of the absolutely continuous part of D^2u with respect to $\mathcal{L}^N$ and $[\nabla u] = (\nabla u)^+ - (\nabla u)^-$. Since D^2u is a symmetric distribution, it follows that $\nabla^2u \in \mathbb{S}^{d\times N\times N}$ and $[\nabla u] = a \otimes \nu_{\nabla u}$ for some $a \in L^1(\Omega;\mathbb{R}^d)$. We now define for $1 < p < +\infty$

$$BH^p(\Omega;\mathbb{R}^d) := \{u \in BH(\Omega;\mathbb{R}^d) : \nabla^2 u \in L^p(\Omega;\mathbb{S}^{d\times N\times N})\}.$$

Theorem 2.3 *If $u \in BH(\Omega;\mathbb{R}^d)$ then*

$$\lim_{\varepsilon\to 0^+}\frac{1}{\varepsilon}\fint_{B(x,\varepsilon)}\left|\nabla u(y) - \nabla u(x) - \nabla^2u(x)(y-x)\right|\,dy = 0,$$

and

$$\lim_{\varepsilon\to 0^+}\frac{1}{\varepsilon^2}\fint_{B(x,\varepsilon)}\left|u(y) - u(x) - \nabla u(x)(y-x) - \frac{1}{2}\nabla^2u(x)(y-x,y-x)\right|\,dy = 0,$$

for $\mathcal{L}^N$ a.e. $x \in \Omega$.

Proof. The first identity follows immediately by applying Theorem 1 of Section 6.1 of Evans and Gariepy [23] to ∇u, while the second identity can be proved with obvious extensions to second order derivatives of the arguments used in that theorem. ■

Next we recall a well known criteria for equi-integrability.

Theorem 2.4 [Dunford and Pettis] *Let $\Omega \subset \mathbb{R}^N$ be a bounded, measurable set, and let $\{f_j\}$ be a bounded sequence in $L^1(\Omega)$. The following statements are equivalent:*

1. *There exists a subsequence of* $\{f_j\}$ *which is weakly converging in* $L^1(\Omega)$;

2.

$$\lim_{k\to+\infty}\sup_j \int_{\Omega\cap\{|f_j|>k\}} |f_j|\,dx = 0;$$

3. *for all* $\varepsilon > 0$ *there exists* $\delta > 0$ *such that*

$$\sup_j \int_E |f_j|\,dx \le \varepsilon$$

for all measurable sets $E \subset \Omega$ *with* $\mathcal{L}^N(E) \le \delta$.

The lemma below was proved by Fonseca and Müller [29] using Young measures.

Theorem 2.5 *Let* $1 < p < +\infty$, *let* $\{u_n\}$ *be a bounded sequence in* $L^p(\Omega;\mathbb{R}^d)$. *For* $\lambda > 0$ *consider the truncation* $\tau_\lambda : \mathbb{R}^N \to \mathbb{R}^d$ *given by*

$$\tau_\lambda(x) := \begin{cases} x & \text{if } |x| \le \lambda, \\ \lambda\frac{x}{|x|} & \text{if } |x| > \lambda. \end{cases}$$

Then there exists a subsequence of $\{u_n\}$ *(not relabeled) and an increasing sequence* $\lambda_n \to +\infty$ *such that the truncated sequence* $\{\tau_{\lambda_n} \circ u_n\}$ *is p-equi-integrable and*

$$\|\tau_{\lambda_n} \circ u_n - u_n\|_{L^q(\Omega)} \to 0 \qquad \text{for all } 1 \le q < p.$$

Moreover, if $\varepsilon_n > 0$ *is any decreasing sequence approaching zero, we may choose the sequence* $\{\lambda_n\}$ *so that*

$$\lim_{n\to+\infty} \lambda_n\varepsilon_n = 0.$$

Meyers [36] extended the notion of quasiconvexity introduced by Morrey [37] to functions depending on derivatives of order higher than one. In particular for second order derivatives the definition reads as

Definition 2.6 *Let* $U \subset \mathbb{R}^N$ *be an open, bounded set with* $\mathcal{L}^N(\partial U) = 0$. *A function* $f : \mathbb{S}^{d\times N\times N} \to \mathbb{R}$ *is said to be 2-quasiconvex if*

$$\int_U f(\Lambda + \nabla^2\psi)\,dx \ge \mathcal{L}^N(U)f(\Lambda)$$

for every constant $\Lambda \in \mathbb{S}^{d\times N\times N}$ *and for every* $\psi \in W_0^{2,\infty}(U;\mathbb{R}^d)$.

As in the case of first order gradients (see [17]), it can be shown that this definition does not depend on the choice of the open set U (see [36]).

3 Approximation of BH^p, $p > 1$, functions by $W^{2,\infty}$ functions

Throughout this section we will use ideas of Acerbi and Fusco [1] and Liu [34] (see also Ambrosio [5]).

If μ is a (nonnegative) Radon measure in $\mathbb{R}^N$ we define the *maximal function* of μ by

$$M(\mu)(x) := \sup_{r>0} \frac{\mu(B(x,r))}{\omega_N r^N}, \qquad \text{for } x \in \mathbb{R}^N.$$

If the measure $\mu = u\,\mathcal{L}^N$, with $u \geq 0$, then we simply write $M(u)$ in place of $M(u\mathcal{L}^N)$. It is well known that if $u \in L^p(\mathbb{R}^N)$ for $p > 1$ then

$$\|M(u)\|_{L^p(\mathbb{R}^N)} \leq C\|u\|_{L^p(\mathbb{R}^N)}, \tag{3.1}$$

for some constant C which does not depend of u. Hereafter for $u \in BH(\mathbb{R}^N)$ and for $\lambda > 0$, we set

$$H_\lambda := \{x : M(|D^2u|)(x) \leq 2\lambda\}.$$

The following lemmas are extensions of classical results (see [5] and [40]).

Lemma 3.1 *Let $u \in BH^p(\mathbb{R}^N)$, for $p > 1$, and let A be a measurable set in $\mathbb{R}^N$. Then there exists a constant C, which depends only on N, such that*

$$\mathcal{L}^N(A \backslash H_\lambda) \leq \int_{A\cap\{|M(\nabla^2 u)|>\lambda\}} \frac{1}{\lambda^p} M(|\nabla^2 u|)^p\, dx + \frac{C}{\lambda}|D^2_s u|(\mathbb{R}^N).$$

Proof. For every $x \in \{M(|D^2_s u|) > \lambda\}$ there exists a radius $r_x > 0$ such that $|D^2_s u|(B(x, r_x)) \geq \lambda\omega_N r_x^N$. Hence, by Vitali's Covering Theorem we may find a countable family of mutually disjoint closed balls $\{\overline{B}(x_i, r_i)\} \subset \{\overline{B}(x, r_x) : x \in \{M(|D^2_s u|) > \lambda\}\}$ such that $\{x : M(|D^2_s u|)(x) > \lambda\} \subset \cup_i \overline{B}(x_i, 5r_i)$. Thus

$$\begin{aligned} \lambda\mathcal{L}^N(\{x : M(|D^2_s u|)(x) \geq \lambda\}) &\leq \lambda \sum_i \mathcal{L}^N(B(x_i, 5r_i)) = 5^N \sum_i \lambda\omega_N r_i^N \\ &\leq 5^N\omega_N \sum_i |D^2_s u|\,(B(x_i, r_i)) \leq 5^N\omega_N|D^2_s u|(\mathbb{R}^N). \end{aligned} \tag{3.2}$$

Since $|D^2u| = |\nabla^2 u|\mathcal{L}^N + |D^2_s u|$ we have that $M(|D^2u|) \leq M(|\nabla^2 u|) + M(|D^2_s u|)$, and thus

$$A \cap \{M(|D^2u|) > 2\lambda\} \subset (A \cap \{M(|\nabla^2 u|) > \lambda\}) \cup \{M(|D^2_s u|) > \lambda\}.$$

Hence, by (3.2) we find

$$\begin{aligned} \mathcal{L}^N(A\backslash H_\lambda) &\leq \mathcal{L}^N(A \cap \{M(|\nabla^2 u|) > \lambda\}) + \mathcal{L}^N(\{M(|D^2_s u|) > \lambda\}) \\ &\leq \int_{A\cap\{M(|\nabla^2 u|)>\lambda\}} \frac{1}{\lambda^p} M(|\nabla^2 u|)^p\, dx + \frac{5^N\omega_N}{\lambda}|D^2_s u|(\mathbb{R}^N). \end{aligned}$$

■

Lemma 3.2 *Let $u \in BH(\mathbb{R}^N)$ be given. There exist a constant K, depending only on N, and a set $E = E(u)$ with $\mathcal{L}^N(E) = 0$ such that if $\lambda > 0$ then*

$$\frac{|u(x) - u(y) - \nabla u(y)(x - y)|}{|x - y|^2} \leq K\lambda$$

and

$$\frac{|\nabla u(x) - \nabla u(y)|}{|x - y|} \leq K\lambda$$

for every $x, y \in H_\lambda\backslash E$ with $x \neq y$.

Proof. Fix $\lambda > 0$ and let η be the standard symmetric mollifier. Let E be the complement in Ω of the intersection of the set of Lebesgue points of u and ∇u, i.e., $E := \Omega \backslash (L_u \cap L_{\nabla u})$, where L_u and $L_{\nabla u}$ are the set of Lebesgue points of u and ∇u, respectively. Define $\eta_\varepsilon := \eta(x/\varepsilon)/\varepsilon^N$ and $u_\varepsilon := \eta_\varepsilon * u$. Applying the identity for C^∞ functions

$$g(1) - g(0) - g'(0) = \int_0^1 \int_0^s g''(t)\, dt\, ds$$

to $g(t) := u_\varepsilon(y + t(x-y))$, we find

$$u_\varepsilon(x) - u_\varepsilon(y) - \nabla u_\varepsilon(y)(x-y) = \int_0^1 \int_0^s \nabla^2 u_\varepsilon(y + t(x-y))(x-y) \cdot (x-y)\, dt\, ds,$$

from which we conclude that

$$\frac{|u_\varepsilon(x) - u_\varepsilon(y) - \nabla u_\varepsilon(y)(x-y)|}{|x-y|^2} \leq \int_0^1 \int_0^s |\nabla^2 u_\varepsilon|(y + t(x-y))\, dt\, ds.$$

Now integrating and applying Fubini's theorem we find

$$\begin{aligned} \fint_{B(y,r)} \frac{|u_\varepsilon(x) - u_\varepsilon(y) - \nabla u_\varepsilon(y)(x-y)|}{|x-y|^2}\, dx &\leq \int_0^1 \int_0^s \fint_{B(y,r)} |\nabla^2 u_\varepsilon|(y + t(x-y))\, dx\, dt\, ds \\ &= \int_0^1 \int_0^s \fint_{B(y,tr)} |\nabla^2 u_\varepsilon|(x)\, dx\, dt\, ds. \end{aligned}$$

Note that from the inequality $\int_{B(y,s)} |\nabla^2 u_\varepsilon|\, dx \leq |D^2 u|(B(y, s+\varepsilon))$ it follows that

$$\limsup_{\varepsilon \to 0+} \int_{B(y,s)} |\nabla^2 u_\varepsilon|\, dx \leq |D^2 u|(\overline{B}(y,s)).$$

In view of the lower semicontinuity of the variation we deduce that

$$|D^2 u|(B(y,s)) \leq \liminf_{\varepsilon \to 0+} \int_{B(y,s)} |\nabla^2 u_\varepsilon|\, dx$$

and hence $\int_{B(y,s)} |\nabla^2 u_\varepsilon|\, dx \to |D^2 u|(B(y,s))$ for every $s > 0$ such that $|D^2 u|(\partial B(y,s)) = 0$. Therefore, letting ε go to zero and applying Dominated Convergence Theorem we obtain

$$\fint_{B(y,r)} \frac{|u(x) - u(y) - \nabla u(y)(x-y)|}{|x-y|^2}\, dx \leq \int_0^1 \int_0^s \frac{|D^2 u|(B(y,tr))}{\mathcal{L}^N(B(y,tr))}\, dt\, ds$$

for every $y \notin E$, where we have used the fact that $u_\varepsilon \to u$ in $W^{1,1}(\mathbb{R}^N)$. Similarly, it can be shown that

$$\fint_{B(y,r)} \frac{|\nabla u(x) - \nabla u(y)|}{|x-y|}\, dx \leq \int_0^1 \frac{|D^2 u|(B(y,tr))}{\mathcal{L}^N(B(y,tr))}\, dt \quad \text{if } y \notin E.$$

For every positive number k, define

$$S_{k,r}(y) := \left\{ x \in B(y,r) : \frac{|u(x) - u(y) - \nabla u(y)(x-y)|}{|x-y|^2} \geq k\lambda \right\},$$

and

$$T_{k,r}(y) := \left\{ x \in B(y,r) : \frac{|\nabla u(x) - \nabla u(y)|}{|x-y|} \geq k\lambda \right\}.$$

If $y \in H_\lambda \backslash E$ then we have

$$\begin{aligned} k\lambda \frac{\mathcal{L}^N(S_{k,r}(y))}{\mathcal{L}^N(B(y,r))} &\le \frac{1}{\mathcal{L}^N(B(y,r))} \int_{S_{k,r}(y)} \frac{|u(x)-u(y)-\nabla u(y)(x-y)|}{|x-y|^2}\,dx \\ &\le \fint_{B(y,r)} \frac{|u(x)-u(y)-\nabla u(y)(x-y)|}{|x-y|^2}\,dx \\ &\le \int_0^1 \int_0^s \frac{|D^2 u|(B(y,tr))}{\mathcal{L}^N(B(y,tr))}\,dt\,ds \le \int_0^1 \int_0^s 2\lambda\,dt\,ds = \lambda, \end{aligned}$$

and so

$$\mathcal{L}^N(S_{k,r}(y)) \le \frac{1}{k}\mathcal{L}^N(B(y,r)).$$

Similarly, for $y \in H_\lambda \backslash E$ we can show that

$$\mathcal{L}^N(T_{k,r}(y)) \le \frac{2}{k}\mathcal{L}^N(B(y,r)).$$

Denote by γ_N the measure of the intersection of two balls of radius 1 in $\mathbb{R}^N$ whose centers are also at distance 1. Fix $x, y \in H_\lambda \backslash E$ and set $r := |y-x|$. Choose $k := 8\omega_N/\gamma_N$, so that

$$\mathcal{L}^N(S_{k,r}(x) \cup S_{k,r}(y) \cup T_{k,r}(x) \cup T_{k,r}(y)) \le \frac{6}{k}\omega_N r^N = \frac{3}{4}\gamma_N r^N < \mathcal{L}^N(B(x,r) \cap B(y,r)).$$

We may, therefore, choose a $z \in (B(x,r) \cap B(y,r)) \backslash (S_{k,r}(x) \cup S_{k,r}(y) \cup T_{k,r}(x) \cup T_{k,r}(y))$, and conclude that

$$\frac{|u(z)-u(y)-\nabla u(y)(z-y)|}{|z-y|^2} \le k\lambda, \qquad \frac{|u(z)-u(x)-\nabla u(x)(z-x)|}{|z-x|^2} \le k\lambda,$$
$$\frac{|\nabla u(z)-\nabla u(y)|}{|z-y|} \le k\lambda, \qquad \frac{|\nabla u(z)-\nabla u(x)|}{|z-x|} \le k\lambda.$$

Whence, by the triangle inequality we have

$$|\nabla u(x) - \nabla u(y)| \le |\nabla u(z) - \nabla u(y)| + |\nabla u(z) - \nabla u(x)| \le 2k\lambda|x-y|, \tag{3.3}$$

and

$$\begin{aligned} |u(x)-u(y)-\nabla u(y)(x-y)| &\le |u(z)-u(y)-\nabla u(y)(z-y)| \\ &\quad + |u(z)-u(x)-\nabla u(x)(z-x)| \\ &\quad + |\nabla u(y)(z-y) - \nabla u(x)(z-x) - \nabla u(y)(x-y)| \\ &\le 2k\lambda|x-y|^2 + |\nabla u(y) - \nabla u(x)||z-x| \le 4k\lambda|x-y|^2, \end{aligned}$$

where we have used (3.3). ■

Definition 3.3 [Stein [40] p. 176] *Let $F \subset \mathbb{R}^N$. We say that a function $u : F \to \mathbb{R}$ belongs to* $\mathrm{Lip}(2, F)$ *if there exists a function $U : F \to \mathbb{R}^N$ and a constant $K > 0$ such that*

$$|u(x)| \le K, \quad |U(x)| \le K, \tag{3.4}$$

and

$$\frac{|U(x)-U(y)|}{|x-y|} \le K, \quad \frac{|u(x)-u(y)-U(y)(x-y)|}{|x-y|^2} \le K, \tag{3.5}$$

for every $x, y \in F$ with $x \neq y$.

Note that, unless F is open, a function u does not determine uniquely the function U and so when we speak of an element of $\mathrm{Lip}(2,F)$ we refer to the couple (u,U). Also the smallest constant $K>0$ for which (3.4) and (3.5) hold can be taken as norm in the space $\mathrm{Lip}(2,F)$ and will be denoted

$$\|u\|_{\mathrm{Lip}(2,F)}\,.$$

It can be shown that with this norm the space $\mathrm{Lip}(2,F)$ becomes a Banach space. Moreover, if F is open then the space $\mathrm{Lip}(2,F)$ can actually be identified with the Sobolev space $W^{2,\infty}(F)$.

We now state and prove an extension theorem of Whitney type.

Theorem 3.4 *Let $F\subset\mathbb{R}^N$ be a Lebesgue measurable set. Then for every $u\in\mathrm{Lip}(2,F)$ there exists $v\in W^{2,\infty}(\mathbb{R}^N)$ such that $v(x)=u(x)$ for $\mathcal{L}^N$ a.e. $x\in F$ and*

$$\|v\|_{W^{2,\infty}(\mathbb{R}^N)}\le C\|u\|_{\mathrm{Lip}(2,F)},$$

where $C>0$ depends only on N.

Proof. In view of the inner regularity of the Lebesgue measure, we may find a sequence of closed sets $\{F_n\}$ such that $F_n\subset F_{n+1}\subset F$ and

$$\mathcal{L}^N\left(F\backslash\bigcup_{n=1}^{\infty}F_n\right)=0.$$

By Theorem 4 of Section 2.3 of Chapter VI in [40] for each $n\in\mathbb{N}$ there exists a continuous operator $\mathcal{E}_n$ of $\mathrm{Lip}(2,F_n)$ into $W^{2,\infty}(\mathbb{R}^N)$ such that for every $u\in\mathrm{Lip}(2,F_n)$ we have $\mathcal{E}_n(u)=u$ on F_n. The norm of the operator $\mathcal{E}_n$ has a bound independent of F_n.

Fix $u\in\mathrm{Lip}(2,F)$ and for each $n\in\mathbb{N}$ let $v_n:=\mathcal{E}_n\left(u|_{F_n}\right)$. Then

$$\|v_n\|_{W^{2,\infty}(\mathbb{R}^N)}\le\|\mathcal{E}_n\|\;\|u|_{F_n}\|_{\mathrm{Lip}(2,F_n)}\le C\|u\|_{\mathrm{Lip}(2,F)}.$$

Thus we may extract a subsequence (not relabelled) of $\{v_n\}$ such that $v_n\overset{*}{\rightharpoonup}v$ in $W^{2,\infty}(\mathbb{R}^N)$ and v_n converges to v pointwise almost everywhere in $\mathbb{R}^N$. Without loss of generality, we may assume that v_n converges to v pointwise everywhere in $\mathbb{R}^N$. For any fixed

$$x\in\bigcup_{n=1}^{\infty}F_n$$

we may find $n_x\in\mathbb{N}$ such that $x\in F_n$ for all $n\ge n_x$. Since $v_n(x)=u(x)$ for all $n\ge n_x$ by letting $n\to\infty$ we obtain that

$$v(x)=u(x)\text{ for all }x\in\bigcup_{n=1}^{\infty}F_n$$

and hence almost everywhere in F. Moreover, by the lower semicontinuity of the norm

$$\|v\|_{W^{2,\infty}(\mathbb{R}^N)}\le\liminf_{n\to\infty}\|v_n\|_{W^{2,\infty}(\mathbb{R}^N)}\le C\|u\|_{\mathrm{Lip}(2,F)}.$$

This completes the proof. ■

We now state and prove an approximation theorem.

Theorem 3.5 *Let $p > 1$ and consider a function $u \in BH^p(\mathbb{R}^N;\mathbb{R}^d)$, with $\operatorname{supp} u \subset B(0,R)$ for some $R > 0$. Then there exists a constant $\alpha > 0$ depending only on N, such that for every*

$$\lambda > \max\left\{1, \frac{\alpha}{R^N}\left(\int_{\mathbb{R}^N} |\nabla^2 u|^p\, dx + |D_s^2 u|(\mathbb{R}^N)\right)\right\}$$

there exists a function $v_\lambda \in W^{2,\infty}(\mathbb{R}^N;\mathbb{R}^d)$ such that

$$\|v_\lambda\|_{W^{2,\infty}(\mathbb{R}^N;\mathbb{R}^d)} \leq C\lambda, \quad u(x) = v_\lambda(x) \text{ for } \mathcal{L}^N \text{ a.e. } x \in H_\lambda$$

where the constant $C > 0$ depends only on R and N.

Proof. By arguing component by component, we may reduce to the scalar-valued case where $u \in BH(\mathbb{R}^N;\mathbb{R})$. Let $A_R := B(0,2R)\backslash B(0,R)$. By Lemma 3.1 and (3.1), for every $\lambda > 1$ we have that

$$\begin{aligned}\mathcal{L}^N(A_R\backslash H_\lambda) &\leq \int_{\mathbb{R}^N} \frac{1}{\lambda^p} M(|\nabla^2 u|)^p\, dx + \frac{C}{\lambda}|D_s^2 u|(\mathbb{R}^N)\\ &\leq \frac{C}{\lambda}\left(\int_{\mathbb{R}^N} |\nabla^2 u|^p\, dx + |D_s^2 u|(\mathbb{R}^N)\right)\\ &< \mathcal{L}^N(A_R)\end{aligned}$$

where $C = C(N)$, provided

$$\lambda > \frac{C}{\mathcal{L}^N(A_R)}\left(\int_{\mathbb{R}^N} |\nabla^2 u|^p\, dx + |D_s^2 u|(\mathbb{R}^N)\right). \tag{3.6}$$

Fix any $\lambda > 1$ which satisfies (3.6). Then $\mathcal{L}^N(A_R \cap H_\lambda) > 0$, and by Lemma 3.2 we may find $y \in (A_R \cap H_\lambda)\backslash E(u)$ and $K = K(N)$ such that

$$\frac{|u(x) - u(y) - \nabla u(y)(x-y)|}{|x-y|^2} \leq K\lambda, \quad \frac{|\nabla u(x) - \nabla u(y)|}{|x-y|} \leq K\lambda$$

for every $x \in H_\lambda\backslash E(u)$ with $x \neq y$. Since $\operatorname{supp} u \subset B(0,R)$ we have that $u(y) = 0$ and $\nabla u(y) = 0$, and thus

$$|u(x)| \leq K\lambda|x-y|^2 \leq 9KR^2\lambda, \quad |\nabla u(x)| \leq K\lambda\,|x-y| \leq 3KR\lambda$$

for every $x \in (H_\lambda\backslash E)\cap B(0,R)$. As $u \equiv 0$ outside $B(0;R)$ the previous inequalities actually hold in $H_\lambda\backslash E$. Again by Lemma 3.2, and in view of Definition 3.3, this implies that $u \in \operatorname{Lip}(2, H_\lambda\backslash E(u))$ and thus the desired result follows from Theorem 3.4 and Lemma 3.1. ■

Theorem 3.6 *Let Ω be an open, bounded subset in $\mathbb{R}^N$ with smooth boundary, and let $\{u_n\} \subset BH^p(\Omega;\mathbb{R}^d)$, $p > 1$, be a sequence converging in $W^{1,1}(\Omega;\mathbb{R}^d)$ to some function u, and such that*

$$\sup_n \int_\Omega |\nabla^2 u_n|^p\, dx < +\infty$$

and

$$|D_s^2 u_n|(\Omega) \to 0 \quad \text{as} \quad n \to +\infty. \tag{3.7}$$

Then there exists a subsequence $\{u_{n_k}\}$ and a sequence $\{v_k\} \subset W^{2,\infty}(\Omega;\mathbb{R}^d)$ such that $\{v_k\}$ converges to u weakly in $W^{2,p}(\Omega;\mathbb{R}^d)$, $\{|\nabla^2 v_k|^p\}$ is equi-integrable, and

$$\mathcal{L}^N(\{x \in \Omega : v_k(x) \neq u_{n_k}(x)\}) \to 0 \quad \text{as } k \to +\infty.$$

If, in addition, $u \in W^{2,\infty}(\Omega;\mathbb{R}^d)$ then $v_k = u$ in a neighborhood of $\partial\Omega$. Moreover, if $\varepsilon_k \searrow 0$, as $k \to +\infty$, it is possible to construct $\{v_k\}$ so that $\|v_k\|_{W^{1,\infty}(\mathbb{R}^N;\mathbb{R}^d)} \leq C\Lambda_k$ for some sequence $\Lambda_k \nearrow +\infty$ such that $\Lambda_k\varepsilon_k \to 0$ as $k \to +\infty$.

Proof. Let $\Omega \subset\subset B(0,R)$, for some $R > 0$. Since Ω has smooth boundary we can extend the functions u_n in such a way that the extensions $U_n \in BH^p(\mathbb{R}^N;\mathbb{R}^d)$ have all compact support $\operatorname{supp} U_n \subset B(0,R)$, and (see [13])

$$\int_{\mathbb{R}^N} (|U_n|+|\nabla U_n|+|\nabla^2 U_n|^p)\,dx+|D_s^2 U_n|(\mathbb{R}^N) \le C(N,d)\int_\Omega (|u_n|+|\nabla u_n|+|\nabla^2 u_n|^p)\,dx+|D_s^2 u_n|(\Omega),$$

$$|D_s^2 U_n|(\mathbb{R}^N) \to 0 \quad \text{as } n \to +\infty.$$

Since $p > 1$, by (3.1) we have that

$$\sup_n \int_{\mathbb{R}^N} M(|\nabla^2 U_n|)^p\,dx \le C \sup_n \int_{\mathbb{R}^N} |\nabla^2 U_n|^p\,dx < +\infty, \tag{3.8}$$

and thus if $\{\varepsilon_n\}$ is a sequence of real numbers such that $\varepsilon_n \searrow 0$, by Theorem 2.5 we may find a sequence $\lambda_n \nearrow +\infty$ and a subsequence (not relabeled) of $\{U_n\}$ such that

$$\lambda_n^{p-1}|D_s^2 U_n|(\mathbb{R}^N) \to 0, \qquad \lambda_n^2 \varepsilon_n \to 0,$$

as $n \to +\infty$, and $\{\tau_{\lambda_n} \circ M(|\nabla^2 U_n|)\}$ is p-equi-integrable. Moreover, without loss of generality, we may assume that for every $n \in \mathbb{N}$

$$\lambda_n > \max\left\{1, \frac{\alpha}{R^N} \sup_j \left(\int_{\mathbb{R}^N} |\nabla^2 U_j|^p\,dx + |D_s^2 U_j|(\mathbb{R}^N)\right)\right\},$$

and so by Theorem 3.5 and Lemma 3.1 for every n there exists a $w_n \in W^{2,\infty}(\mathbb{R}^N;\mathbb{R}^d)$ such that

$$\|w_n\|_{W^{2,\infty}(\mathbb{R}^N;\mathbb{R}^d)} \le C\lambda_n, \quad U_n(x) = w_n(x) \text{ for } \mathcal{L}^N \text{ a.e. } x \in H_n,$$

and

$$\mathcal{L}^N(\mathbb{R}^N \setminus H_n) \le \int_{\{M(|\nabla^2 U_n|)>\lambda_n\}} \frac{1}{\lambda_n^p} M(|\nabla^2 U_n|)^p\,dx + \frac{C}{\lambda_n}|D_s^2 U_n|(\mathbb{R}^N), \tag{3.9}$$

where

$$H_n := \{x \in \mathbb{R}^N : M(|D^2 U_n|) \le 2\lambda_n\}$$

and $C = C(N,R)$. Since $\|w_n\|_{W^{2,\infty}(\mathbb{R}^N;\mathbb{R}^d)} \le C\lambda_n$, on $\{M(|\nabla^2 U_n|) \ge \lambda_n\}$ we have

$$|\nabla^2 w_n| \le C\lambda_n \le C\,\tau_{\lambda_n} \circ M(|\nabla^2 U_n|),$$

while on $H_n \cap \{M(|\nabla^2 U_n|) < \lambda_n\}$

$$|\nabla^2 w_n| = |\nabla^2 U_n| \le M(|\nabla^2 U_n|) = \tau_{\lambda_n} \circ M(|\nabla^2 U_n|)|.$$

Hence to prove the equi-integrability of $\{|\nabla^2 w_n|^p\}$ it remains to study the sequence on the set $\{M(|\nabla^2 U_n|) < \lambda_n\}\setminus H_n$. By (3.2) we find

$$\begin{aligned}\int_{\{M(|\nabla^2 U_n|)<\lambda_n\}\setminus H_n} |\nabla^2 w_n|^p\,dx &\le C\lambda_n^p \mathcal{L}^N(\{M(|\nabla^2 U_n|) < \lambda_n\}\setminus H_n) \\ &\le C\lambda_n^p \mathcal{L}^N(\{M(|D_s^2 U_n|) \ge \lambda_n\} \\ &\le C\lambda_n^{p-1}|D_s^2 U_n|(\mathbb{R}^N) \to 0.\end{aligned}$$

Thus $\{|\nabla^2 w_n|^p\}$ is equi-integrable. We now show that $\{w_n\}$ converges weakly in $W^{2,p}(\Omega;\mathbb{R}^d)$ to u. By (3.7), (3.8) and (3.9) we have

$$\lim_{n\to\infty} \lambda_n \mathcal{L}^N \left(\{x \in \mathbb{R}^N : w_n(x) \neq U_n(x)\}\right) = 0, \tag{3.10}$$

and so

$$\int_\Omega |w_n|\, dx = \int_{\{w_n=u_n\}} |u_n|\, dx + \int_{\{w_n\neq u_n\}} |w_n|\, dx \leq \int_\Omega |u_n|\, dx + C\lambda_n \mathcal{L}^N(\Omega \cap \{u_n \neq w_n\}) \leq C,$$

and since $\nabla w_n = \nabla u_n$ $\mathcal{L}^N$ a.e. on $\Omega \cap \{w_n = u_n\}$, we deduce, in a similar manner, that also $\{\nabla w_n\}$ is bounded in $L^1(\Omega, \mathbb{R}^{d\times N})$. We then have that $\{w_n\}$ is bounded in $W^{2,p}(\Omega;\mathbb{R}^d)$ and therefore, up to a subsequence, it converges weakly in $W^{2,p}(\Omega;\mathbb{R}^d)$ to some function $w \in W^{2,p}(\Omega;\mathbb{R}^d)$. In view of (3.10) we conclude that $w = u$ $\mathcal{L}^N$ a.e. on Ω.

Finally, if $u \in W^{2,\infty}(\Omega;\mathbb{R}^d)$, in order to match the boundary conditions define

$$r_n := \max\left\{ \left(\|w_n - u\|_{W^{1,p}(\Omega;\mathbb{R}^{d\times N})}\right)^{\frac{1}{3}}, 1/\lambda_n \right\}$$

and note that $r_n \to 0$ as $n \to +\infty$. Let

$$A_n := \{x \in \Omega : \operatorname{dist}(x, \partial\Omega) \leq r_n\},$$

and

$$B_n := \{x \in \Omega \backslash A_n : \operatorname{dist}(x, \partial A_n) \leq r_n\}.$$

For every n, let $\phi_n \in C_c^\infty(\Omega)$, $0 \leq \phi_n \leq 1$, $\phi_n = 1$ on $\Omega\backslash(A_n \cup B_n)$, $\phi_n = 0$ in A_n,

$$\|\nabla\phi_n\|_{L^\infty(\Omega)} \leq \frac{2}{r_n} \text{ and } \|\nabla^2\phi_n\|_{L^\infty(\Omega)} \leq \frac{4}{r_n^2}.$$

Define

$$v_n := \phi_n w_n + (1 - \phi_n) u.$$

Clearly $v_n = u$ in a neighborhood of $\partial\Omega$, and for n large enough

$$\begin{aligned}
\|v_n\|_{L^\infty(\Omega)} &\leq \|w_n\|_{L^\infty(\Omega)} + \|u\|_{L^\infty(\Omega)} \leq C\lambda_n,\\
\|\nabla v_n\|_{L^\infty(\Omega)} &\leq \|\nabla\phi_n\|_{L^\infty(\Omega)}\|w_n - u\|_{L^\infty(\Omega)} + \|\nabla w_n - \nabla u\|_{L^\infty(\Omega)} \leq C\lambda_n^2 + C\lambda_n \leq C\lambda_n^2.
\end{aligned}$$

Moreover

$$\begin{aligned}
\|v_n - u\|_{L^p(\Omega)} &\leq \|w_n - u\|_{L^p(\Omega)} \to 0,\\
\|\nabla v_n - \nabla u\|_{L^p(\Omega)} &\leq \|\nabla\phi_n\|_{L^\infty(\Omega)}\|w_n - u\|_{L^p(\Omega)} + \|\nabla w_n - \nabla u\|_{L^p(\Omega)}\\
&\leq C\left(\|w_n - u\|_{L^p(\Omega)}\right)^{2/3} + \|\nabla w_n - \nabla u\|_{L^p(\Omega)} \to 0,\\
\|\nabla^2 v_n - \nabla^2 w_n\|_{L^p(\Omega)} &\leq \|\nabla^2\phi_n\|_{L^\infty(\Omega)}\|w_n - u\|_{L^p(\Omega)} + C\|\nabla\phi_n\|_{L^\infty(\Omega)}\|\nabla w_n - \nabla u\|_{L^p(\Omega)}\\
&\qquad + \left(\int_{A_n\cup B_n} |\nabla^2 w_n - \nabla^2 u|^p\, dx\right)^{1/p}\\
&\leq C\left(\|w_n - u\|_{W^{1,p}(\Omega)}\right)^{1/3} + \left(\int_{A_n\cup B_n} |\nabla^2 w_n - \nabla^2 u|^p\, dx\right)^{1/p} \to 0,
\end{aligned}$$

where we have used the equi-integrability of $\{|\nabla^2 w_n|^p\}$. Hence $\{v_n\}$ is the sequence we were looking for with $\Lambda_n := \lambda_n^2$. ■

4 Lower semicontinuity in BH^p, $p > 1$, for 2-quasiconvex integrands

In this Section we prove Theorem 1.2.

The following Proposition may be found in [10]. For the convenience of the reader we present its proof.

Proposition 4.1 *Let $f : \Omega \times \mathbb{R}^d \times \mathbb{R}^{d\times N} \times \mathbb{S}^{d\times N\times N} \to [0, +\infty)$ be a normal integrand. Then for each $\varepsilon_0 > 0$ there exists a compact set $K \subset \Omega$ such that $|\Omega \setminus K| \leq \varepsilon_0$ and $f\ : K \times \mathbb{R}^d \times \mathbb{R}^{d\times N} \times \mathbb{S}^{d\times N\times N} \to [0, +\infty)$ is lower semicontinuous. Moreover, if f is continuous in the variable Λ, then for each $(x_0, u_0, \xi_0) \in K \times \mathbb{R}^d \times \mathbb{R}^{d\times N}$, $\varepsilon > 0$, and $L > 0$ there exists $\delta = \delta(x_0, u_0, \xi_0, \varepsilon, K, L) \in (0, 1)$ such that*

$$f(x_0, u_0, \xi_0, \Lambda) \leq f(x, u, \xi, \Lambda) + \varepsilon$$

for all $(x, u, \xi) \in K \times \mathbb{R}^d \times \mathbb{R}^{d\times N}$, with $|x - x_0| + |u - u_0| + |\xi - \xi_0| \leq \delta$ and all $|\Lambda| \leq L$.

Proof. The first part of the Proposition follows from Theorem 1.1 in Chapter 8 of [22]. To prove the second claim assume, for contradiction, that there exist $(x_0, u_0, \xi_0) \in K \times \mathbb{R}^d \times \mathbb{R}^{d\times N}$, $L > 0$, $\bar{\varepsilon} > 0$, and a sequence

$$\{(x_n, u_n, \xi_n, \Lambda_n)\} \subset K \times \overline{B_d(u_0, 1)} \times \overline{B_{d\times N}(\xi_0, 1)} \times \overline{B_{d\times N\times N}(0, L)},$$

such that

$$\bar{\varepsilon} + f(x_n, u_n, \xi_n, \Lambda_n) < f(x_0, u_0, \xi_0, \Lambda_n) \tag{4.1}$$

and $(x_n, u_n, \xi_n, \Lambda_n) \to (x_0, u_0, \xi_0, \Lambda_0)$ as $n \to \infty$, for some $\Lambda_0 \in \overline{B_{d\times N\times N}(0, L)}$. Since the function $f(x_0, u_0, \xi_0, \cdot)$ is continuous and f restricted to $K \times \mathbb{R} \times \mathbb{R}^{d\times N} \times \mathbb{S}^{d\times N\times N}$ is lower semicontinuous, for any $\varepsilon < \frac{1}{2}\bar{\varepsilon}$ there exists $\delta > 0$ such that

$$|f(x_0, u_0, \xi_0, \Lambda) - f(x_0, u_0, \xi_0, \Lambda_0)| \leq \varepsilon, \qquad f(x_0, u_0, \xi_0, \Lambda_0) \leq f(x, u, \xi, \Lambda) + \varepsilon$$

for all $(x, u, \xi, \Lambda) \in K \times \overline{B_d(u_0, 1)} \times \overline{B_{d\times N}(\xi_0, 1)} \times \overline{B_{d\times N\times N}(0, L)}$ with

$$|x - x_0| + |u - u_0| + |\xi - \xi_0| + |\Lambda - \Lambda_0| \leq \delta.$$

Thus for all n sufficiently large, also by (4.1), we have

$$\bar{\varepsilon} + f(x_n, u_n, \xi_n, \Lambda_n) < f(x_0, u_0, \xi_0, \Lambda_n) \leq f(x_0, u_0, \xi_0, \Lambda_0) + \varepsilon \leq f(x_n, u_n, \xi_n, \Lambda_n) + 2\varepsilon$$

which is a contradiction. ■

Proof of Theorem 1.2. Without loss of generality we may assume that

$$\liminf_{n\to\infty} \int_\Omega f(x, u_n, \nabla u_n, \nabla^2 u_n)\, dx = \lim_{n\to\infty} \int_\Omega f(x, u_n, \nabla u_n, \nabla^2 u_n)\, dx < \infty.$$

Passing to a subsequence, if necessary, there exists a nonnegative Radon measure μ such that

$$f(x, u_n(x), \nabla u_n(x), \nabla^2 u_n(x))\, \mathcal{L}^N \lfloor \Omega \overset{*}{\rightharpoonup} \mu$$

as $n \to \infty$, weakly $\star$ in the sense of measures. We claim that

$$\frac{d\mu}{d\mathcal{L}^N}(x_0) \geq f(x_0, u(x_0), \nabla u(x_0), \nabla^2 u(x_0)) \qquad \text{for } \mathcal{L}^N \text{ a.e. } x_0 \in \Omega. \tag{4.2}$$

If (4.2) holds, then the conclusion of the theorem follows immediately. Indeed, let $\varphi \in C_c(\Omega;\mathbb{R})$, $0 \le \varphi \le 1$. We have

$$\begin{aligned}\liminf_{n\to\infty}\int_\Omega f(x,u_n,\nabla u_n,\nabla^2 u_n)\,dx &\ge \liminf_{n\to\infty}\int_\Omega \varphi\, f(x,u_n,\nabla u_n,\nabla^2 u_n)\,dx\\ &= \int_\Omega \varphi\,d\mu \ge \int_\Omega \varphi\,\frac{d\mu}{d\mathcal{L}^N}\,dx \ge \int_\Omega \varphi\, f(x,u,\nabla u,\nabla^2 u)\,dx.\end{aligned}$$

By letting $\varphi \to 1$, and using Lebesgue Monotone Convergence Theorem, we obtain the desired result. To show (4.2) for each $j \in \mathbb{N}$ we apply Proposition 4.1 to obtain a compact set $K_j \subset \Omega$, with $|\Omega \setminus K_j| \le 1/j$, such that

$$f\ : K_j \times \mathbb{R}^N \times \mathbb{R}^{d\times N} \times \mathbb{S}^{d\times N\times N} \to [0,+\infty)$$

is lower semicontinuous. Let K_j^* be the set of Lebesgue points of χ_{K_j}, and set

$$\omega := \bigcup_j^\infty (K_j \cap K_j^*). \tag{4.3}$$

Then

$$|\Omega\setminus\omega| \le |\Omega\setminus K_j| \le \frac{1}{j} \to 0 \quad \text{as } j\to\infty.$$

Fix $x_0 \in \omega$ such that $a\left(x_0,u\left(x_0\right),\nabla u\left(x_0\right)\right) < \infty$ and

$$\frac{d\mu}{d\mathcal{L}^N}(x_0) = \lim_{\varepsilon\to 0^+}\frac{\mu(Q(x_0,\varepsilon))}{\varepsilon^N} < \infty, \tag{4.4}$$

$$\lim_{\varepsilon\to0^+}\frac{1}{\varepsilon^{N+1}}\int_{Q(x_0,\varepsilon)}|\nabla u(x)-\nabla u(x_0)-\nabla^2u(x_0)(x-x_0)|dx = 0,$$

$$\lim_{\varepsilon\to0^+}\frac{1}{\varepsilon^{N+2}}\int_{Q(x_0,\varepsilon)}|u(x)-u(x_0)-\nabla u(x_0)(x-x_0)-\frac{1}{2}\nabla^2u(x_0)(x-x_0)\cdot(x-x_0)|dx = 0. \tag{4.5}$$

Choosing $\varepsilon_k \searrow 0$ such that $\mu(\partial Q(x_0,\varepsilon_k)) = 0$, then

$$\begin{aligned}\frac{d\mu}{d\mathcal{L}^N}(x_0) &= \lim_{k\to\infty}\frac{\mu(Q(x_0,\varepsilon_k))}{\varepsilon_k^N}\\ &= \lim_{k\to\infty}\lim_{n\to\infty}\frac{1}{\varepsilon_k^N}\int_{Q(x_0,\varepsilon_k)} f(x,u_n,\nabla u_n,\nabla^2 u_n)\,dx \qquad (4.6)\\ &= \lim_{k\to\infty}\lim_{n\to\infty}\int_Q f(x_0+\varepsilon_k y, u(x_0)+\varepsilon_k^2 w_{n,k}(y)+\varepsilon_k\nabla u(x_0)y, \nabla u(x_0)+\varepsilon_k\nabla w_{n,k}(y), \nabla^2 w_{n,k}(y))\,dy,\end{aligned}$$

where

$$w_{n,k}(y) := \frac{u_n(x_0+\varepsilon_k y)-u(x_0)-\varepsilon_k\nabla u(x_0)y}{\varepsilon_k^2}.$$

Clearly $w_{n,k} \in BH^p(Q;\mathbb{R}^d)$, and by (4.5), $\lim_{k\to\infty}\lim_{n\to\infty}||w_{n,k}-w_0||_{W^{1,1}(Q;\mathbb{R}^d)} = 0$, where

$$w_0(y) := \nabla^2 u(x_0)y\cdot y.$$

Moreover, by (1.5), (4.4), and (4.6) we have that

$$\limsup_{k\to\infty}\limsup_{n\to\infty}\int_Q |\nabla^2 w_{n,k}|^p\,dy<\infty,$$

while (1.6) ensures that

$$\lim_{k\to\infty}\lim_{n\to\infty}\left|D^2_s w_{n,k}\right|(\overline{Q})=0. \tag{4.7}$$

By a standard diagonalization argument, we may extract a subsequence $\{w_k\}$ with $w_k:=w_{n_k,k}$ which converges to w_0 in $W^{1,1}(Q;\mathbb{R}^d)$ and such that

$$\sup_k\int_Q |\nabla^2 w_k|^p\,dy<\infty,\quad \lim_{k\to\infty}\left|D^2_s w_k\right|(\overline{Q})=0,$$

and

$$\frac{d\mu}{d\mathcal{L}^N}(x_0)\geq\lim_{k\to\infty}\int_Q f(x_0+\varepsilon_k y,u(x_0)+\varepsilon_k^2 w_k(y)+\varepsilon_k\nabla u(x_0)y,\nabla u(x_0)+\varepsilon_k\nabla w_k(y),\nabla^2 w_k(y))\,dy.$$

By Theorem 3.6 we may find a subsequence (not relabeled) of $\{w_k\}$ and a sequence $\{v_k\}\subset W^{2,\infty}(Q;\mathbb{R}^d)$ weakly converging to w_0 in $W^{2,p}(Q;\mathbb{R}^d)$ such that $v_k=w_0$ on ∂Q,

$$|\{y\in Q:\ v_k(y)\neq w_k(y)\}|\to 0\quad\text{as } k\to\infty \tag{4.8}$$

and $\{|\nabla^2 v_k|^p\}$ is equi-integrable. Hence

$$\frac{d\mu}{d\mathcal{L}^N}(x_0)\geq\liminf_{k\to\infty}\int_{\{v_k=w_k\}} f(x_0+\varepsilon_k y,u(x_0)+\varepsilon_k^2 v_k(y)+\varepsilon_k\nabla u(x_0)y,\nabla u(x_0)+\varepsilon_k\nabla v_k(y),\nabla^2 v_k(y))\,dy,$$

where we have used the fact that $f\geq 0$. As $x_0\in\omega$ there exists an integer j_0 such that $x_0\in K_{j_0}\cap K^*_{j_0}$. By Proposition 4.1 for any fixed integer $j\in\mathbb{N}$ there exists $0<\rho_j<1$ such that

$$f(x_0,u(x_0),\nabla u(x_0),\Lambda)\leq f(x,u,\xi,\Lambda)+\frac{1}{j} \tag{4.9}$$

for all $(x,u,v)\in K_{j_0}\times\overline{B_d(u(x_0),1)}\times\overline{B_{d\times N}(\nabla u(x_0),1)}$, with $|x-x_0|+|u-u(x_0)|+|\xi-\nabla u(x_0)|\leq\rho_j$ and all $\Lambda\in\mathbb{S}^{d\times N\times N}$ with $|\Lambda|\leq j$. Set

$$E_{k,j}:=\left\{y\in Q: v_k(y)=w_k(y),\ |\varepsilon_k^2 v_k(y)|\leq\frac{\rho_j}{2},\ |\varepsilon_k\nabla v_k(y)|\leq\rho_j,\ |\nabla^2 v_k(y)|\leq j\right\}.$$

Since $\{v_k\}$ is bounded in $W^{2,1}(Q;\mathbb{R}^d)$, and in view of (4.8), we have that

$$\lim_{j\to\infty}\lim_{k\to\infty}|Q\setminus E_{k,j}|=0. \tag{4.10}$$

Thus

$$\begin{aligned}\frac{d\mu}{d\mathcal{L}^N}(x_0)&\geq\lim_{j\to\infty}\lim_{k\to\infty}\int_{E_{k,j}} f(x_0+\varepsilon_k y,u(x_0)+\varepsilon_k^2 v_k(y)+\varepsilon_k\nabla u(x_0)y,\\&\qquad\nabla u(x_0)+\varepsilon_k\nabla v_k(y),\nabla^2 v_k(y))\,dy,\\&=\lim_{j\to\infty}\lim_{k\to\infty}\frac{1}{\varepsilon_k^N}\int_{D_{k,j}} f\left(x,u(x_0)+\varepsilon_k^2 v_k\left(\frac{x-x_0}{\varepsilon_k}\right)+\nabla u(x_0)(x-x_0),\right.\\&\qquad\left.\nabla u(x_0)+\varepsilon_k\nabla v_k\left(\frac{x-x_0}{\varepsilon_k}\right),\nabla^2 v_k\left(\frac{x-x_0}{\varepsilon_k}\right)\right)\,dx,\end{aligned}$$

where $D_{k,j} := x_0 + \varepsilon_k E_{k,j}$. By (4.9)

$$\frac{d\mu}{d\mathcal{L}^N}(x_0) \geq \liminf_{j\to\infty} \liminf_{k\to\infty} \frac{1}{\varepsilon_k^N} \int_{K_{j_0}\cap D_{k,j}} f\left(x, u(x_0) + \varepsilon_k^2 v_k\left(\frac{x-x_0}{\varepsilon_k}\right) + \nabla u(x_0)(x-x_0),\right.$$
$$\left. \nabla u(x_0) + \varepsilon_k \nabla v_k\left(\frac{x-x_0}{\varepsilon_k}\right), \nabla^2 v_k\left(\frac{x-x_0}{\varepsilon_k}\right)\right) dx \quad (4.11)$$
$$\geq \liminf_{j\to\infty} \liminf_{k\to\infty} \frac{1}{\varepsilon_k^N} \int_{K_{j_0}\cap D_{k,j}} f\left(x_0, u(x_0), \nabla u(x_0), \nabla^2 v_k\left(\frac{x-x_0}{\varepsilon_k}\right)\right) dx,$$

where we used the fact that for fixed j and for k large enough $|\nabla u(x_0)(x-x_0)| < \rho_j/2$ for every $x \in D_{k,j}$. Since $|\nabla^2 v_k| \leq j$ in $E_{k,j}$, and by virtue of the growth condition on f, for each fixed j we have

$$\frac{1}{\varepsilon_k^N} \int_{D_{k,j}\setminus K_{j_0}} f\left(x_0, u(x_0), \nabla u(x_0), \nabla^2 v_k\left(\frac{x-x_0}{\varepsilon_k}\right)\right) dx$$
$$\leq a\left(x_0, u(x_0), \nabla u(x_0)\right)(1+j^p)\frac{|Q(x_0,\varepsilon_k)\setminus K_{j_0}|}{\varepsilon_k^N} \to 0$$

as $k \to \infty$, because x_0 is a Lebesgue point of $\chi_{K_{j_0}}$. Consequently, from (4.11) we get

$$\frac{d\mu}{d\mathcal{L}^N}(x_0) \geq \liminf_{j\to\infty} \liminf_{k\to\infty} \frac{1}{\varepsilon_k^N} \int_{D_{k,j}} f\left(x_0, u(x_0), \nabla u(x_0), \nabla^2 v_k\left(\frac{x-x_0}{\varepsilon_k}\right)\right) dx$$
$$= \lim_{j\to\infty} \lim_{k\to\infty} \int_{E_{k,j}} f(x_0, u(x_0), \nabla u(x_0), \nabla^2 v_k(y))\, dy$$
$$= \lim_{k\to\infty} \int_Q f(x_0, u(x_0), \nabla u(x_0), \nabla^2 v_k(y))\, dy,$$

where we have used the growth condition on f, the p-equi-integrability of $\{v_k\}$, and (4.10). Since $v_k = w_0$ on ∂Q it now follows from the 2-quasiconvexity of f that

$$\frac{d\mu}{d\mathcal{L}^N}(x_0) \geq f(x_0, u(x_0), \nabla u(x_0), \nabla^2 u(x_0)).$$

■

Note that the assumption $|D_s^2 u_n|(\Omega) \to 0$, as $n \to +\infty$, made in Theorem 1.2 was used only to derive (4.7). Indeed, Theorem 1.2 holds even if we require that $|D_s^2 u_n| \overset{*}{\rightharpoonup} \tau$ and $\tau \perp \mathcal{L}^N$, i.e., $d\tau/d\mathcal{L}^N = 0$, $\mathcal{L}^N$-a.e. in Ω instead of $|D_s^2 u_n|(\Omega) \to 0$. The small modifications needed in the previous proof are as follows: choose x_0 and ε_k satisfying the further requirement that

$$\frac{d\tau}{d\mathcal{L}^N}(x_0) = \lim_{k\to+\infty} \frac{\tau(Q(x_0,\varepsilon_k))}{\mathcal{L}^N(Q(x_0,\varepsilon_k))} = 0, \quad \tau(\partial Q(x_0,\varepsilon_k)) = 0.$$

Then

$$\lim_{k\to+\infty} \lim_{n\to+\infty} |D_s^2 v_{n,k}|(\bar{Q}) = \lim_{k\to+\infty} \lim_{n\to+\infty} \frac{|D_s^2 u_n|(\overline{Q(x_0,\varepsilon_k)})}{\varepsilon_k^N}$$
$$= \lim_{k\to+\infty} \frac{\tau(Q(x_0,\varepsilon_k))}{\varepsilon_k^N} = \frac{d\tau}{d\mathcal{L}^N}(x_0) = 0.$$

We record this result below.

Theorem 4.2 *Let $\Omega \subset \mathbb{R}^N$ be an open bounded set and let*

$$f : \Omega \times \mathbb{R}^N \times \mathbb{R}^{d\times N} \times \mathbb{S}^{d\times N\times N} \to [0, +\infty)$$

be a normal integrand, 2−quasiconvex in Λ, and such that (1.5) holds. Then for every $u \in W^{1,1}(\Omega;\mathbb{R}^d)$ and any sequence $\{u_n\} \subset BH(\Omega;\mathbb{R}^d)$ converging to u in $W^{1,1}(\Omega;\mathbb{R}^d)$ and such that

$$|D^2_s u_n| \overset{*}{\rightharpoonup} \tau \perp \mathcal{L}^N$$

we have

$$\int_\Omega f(x, u, \nabla u, \nabla^2 u)\, dx \le \liminf_{n\to\infty} \int_\Omega f(x, u_n, \nabla u_n, \nabla^2 u_n)\, dx.$$

Remark 4.3 If Ω is a Lipschitz, bounded, open set then the results of Theorems 1.2, 4.2, hold even if we replace the assumption $u_n \to u$ in $W^{1,1}(\Omega,\mathbb{R}^d)$ with $u_n \to u$ in $L^1(\Omega,\mathbb{R}^d)$. Indeed, by the interpolation inequality (see Theorem 2.2) we have for every $\varepsilon > 0$

$$\|\nabla u_n - \nabla u\|_{L^1(\Omega)} \le C(\varepsilon)\|u_n - u\|_{L^1(\Omega)} + \varepsilon|\nabla^2 u_n \mathcal{L}^N - D^2 u|(\Omega);$$

hence first letting $n \to +\infty$ and then letting $\varepsilon \to 0^+$ we obtain the convergence of the gradients.

5 Lower semicontinuity in BH for 2-quasiconvex integrands

In this Section we prove that when $p = 1$ we still have a lower semicontinuity result for integrands essentially of the type $f = f(x,\Lambda)$. Precisely, let $\Omega \subset \mathbb{R}^N$ be an open, bounded domain, and let

$$f : \Omega \times \mathbb{R}^d \times \mathbb{R}^{d\times N} \times \mathbb{S}^{d\times N\times N} \to [0, +\infty)$$

satisfy

(H1) f is Borel measurable;

(H2) $f(x, u, \xi, \cdot)$ is 2-quasiconvex;

(H3) there exists a positive constant C such that

$$0 \le f(x, u, \xi, \Lambda) \le C(1 + |\Lambda|)$$

for $\mathcal{L}^N$ a.e. $x \in \Omega$ and all $(u, \xi, \Lambda) \in \mathbb{R}^d \times \mathbb{R}^{d\times N} \times \mathbb{S}^{d\times N\times N}$;

(H4) for all $(x_0, u_0, \xi_0) \in \Omega \times \mathbb{R}^d \times \mathbb{R}^{d\times N}$ and $\varepsilon > 0$ there exist $\delta_0 > 0$ and a modulus of continuity ρ, with $\rho(s) \le C_0(1 + s)$ for $s > 0$ and for some $C_0 > 0$, such that

$$f(x_0, u_0, \xi_0, \Lambda) - f(x, u, \xi, \Lambda) \le \varepsilon(1 + f(x, u, \xi, \Lambda)) + \rho(|(u, \xi) - (u_0, \xi_0)|) \tag{5.1}$$

for all $x \in \Omega$ with $|x - x_0| \le \delta_0$, and for all $(u, \xi, \Lambda) \in \mathbb{R}^d \times \mathbb{R}^{d\times N} \times \mathbb{S}^{d\times N\times N}$.

The result below may be found in Fonseca, Leoni, Malý and Paroni [25].

Proposition 5.1 *Let $f : \mathbb{S}^{d\times N\times N} \to [0, \infty)$ be a 2-quasiconvex function such that*

$$0 \le f(\Lambda) \le C\,(1 + |\Lambda|)$$

for all $\Lambda \in \mathbb{S}^{d\times N\times N}$. If $\{u_n\}$ is a sequence of functions in $W^{2,1}(B;\mathbb{R}^d)$ converging to 0 in $W^{1,1}(Q;\mathbb{R}^d)$, and $\{\|\nabla^2 u_n\|_{L^1(\Omega,\mathbb{S}^{d\times N\times N})}\}$ is bounded, then

$$\omega_N f(0) \le \liminf_{n\to\infty} \int_B f(\nabla^2 u_n)\, dx.$$

The following lemma was proved by Guidorzi and Poggiolini [31] (see also [39])

Lemma 5.2 *Let $h : \Omega \to \mathbb{S}^{d\times N\times N}$ be a 2-quasiconvex function. If there exists a constant C such that*

$$|h(\Lambda)| \leq C(1 + |\Lambda|)$$

then

$$|h(\Lambda_1) - h(\Lambda_2)| \leq C_\infty|\Lambda_1 - \Lambda_2|$$

for some constant $C_\infty = C_\infty(N, d, C)$.

Theorem 5.3 *Assume that the conditions* (H1)-(H4) *hold. If $u_n \in BH(\Omega, \mathbb{R}^d)$, $u \in W^{1,1}(\Omega, \mathbb{R}^d)$, $\{\|\nabla^2 u_n\|_{L^1(\Omega)}\}$ is bounded, $|D^2_s u_n| \overset{*}{\rightharpoonup} \tau \perp \mathcal{L}^N$, and if $u_n \to u$ in $W^{1,1}(\Omega, \mathbb{R}^d)$, then*

$$\int_\Omega f(x, u(x), \nabla u(x), \nabla^2 u(x))\, dx \leq \liminf_{n\to+\infty} \int_\Omega f(x, u_n(x), \nabla u_n(x), \nabla^2 u_n(x))\, dx.$$

Proof. We proceed as in the proof of Theorem 1.2 until the diagonalization argument after (4.7), precisely until we find a sequence $\{v_k\}$ of functions with bounded Hessian such that

$$\lim_{k\to+\infty} \|v_k - v\|_{W^{1,1}(Q)} = 0, \qquad \sup_k \|\nabla^2 v_k\|_{L^1(Q)} < +\infty, \qquad \lim_{k\to+\infty} |D^2_s v_k|(\overline{Q}) = 0$$

and

$$\frac{d\mu}{d\mathcal{L}^N}(x_0) = \lim_{k\to+\infty} \int_Q f(x_0 + \varepsilon_k y, u(x_0) + \varepsilon_k^2 v_k(y) + \varepsilon_k \nabla u(x_0) y, \nabla u(x_0) + \varepsilon_k \nabla v_k(y), \nabla^2 v_k(y))\, dy,$$

where $v(y) := \frac{1}{2}\nabla^2 u(x_0) y \cdot y$. Using (5.1), for all $\varepsilon > 0$ we find

$$\begin{aligned}(1+\varepsilon)\frac{d\mu}{d\mathcal{L}^N}(x_0) + \varepsilon \quad &\geq \quad \liminf_{k\to\infty} \int_Q f(x_0, u(x_0), \nabla u(x_0), \nabla^2 v_k(y))\; dy \\ &\quad - \limsup_{k\to\infty} \int_Q \rho(|(\varepsilon_k \nabla u(x_0) y + \varepsilon_k^2 v_k(y), \varepsilon_k \nabla v_k(y))|)\, dy.\end{aligned}$$

But $(\varepsilon_k \nabla u(x_0) y + \varepsilon_k^2 v_k(y), \varepsilon_k \nabla v_k(y)) \to 0$ in $L^1(Q, \mathbb{R}^{d+d\times N})$ and hence by Lebesgue Dominated Convergence Theorem we arrive at

$$(1+\varepsilon)\frac{d\mu}{d\mathcal{L}^N}(x_0) + \varepsilon \geq \lim_{k\to\infty} \int_Q f(x_0, u(x_0), \nabla u(x_0), \nabla^2 v_k(y))\; dy.$$

Hereafter we denote by $w_k := v_k - v$ and by $h : \mathbb{S}^{d\times N\times N} \to \mathbb{R}$ the function defined by

$$h(\Lambda) := f(x_0, u(x_0), \nabla u(x_0), \Lambda + \nabla^2 u(x_0)) = f(x_0, u(x_0), \nabla u(x_0), \Lambda + \nabla^2 v(y)),$$

so that $w_k \to 0$ in $W^{1,1}(Q, \mathbb{R}^d)$, $\sup_k \|\nabla^2 w_k\|_{L^1(Q)} < +\infty$,

$$\lim_{k\to+\infty} |D^2_s w_k|(\overline{Q}) = 0 \tag{5.2}$$

and

$$(1+\varepsilon)\frac{d\mu}{d\mathcal{L}^N}(x_0) + \varepsilon \geq \lim_{k\to\infty} \int_Q h(\nabla^2 w_k(y))\; dy. \tag{5.3}$$

Let η be the standard symmetric mollifier and define $\eta_h(x) := \eta(x/h)/h^N$ and $w_{k,h} := \eta_h * w_k$. Then $\nabla^2 w_{k,h} = \eta_h * \nabla^2 w_k + \eta_h * D_s^2 w_k$, and by using Lemma 5.2 we find

$$\int_Q h(\nabla^2 w_{k,h})\,dx \le \int_Q h(\eta_h * \nabla^2 w_k)\,dx + C_\infty |D_s^2 w_k|(Q(0,1+h)),$$

and by the Lebesgue Dominated Convergence Theorem we obtain

$$\begin{aligned}\liminf_{h\to 0} \int_Q h(\nabla^2 w_{k,h})\,dx &\le \liminf_{h\to 0} \int_Q h(\eta_h * \nabla^2 w_k)\,dx + C_\infty |D_s^2 w_k|(\overline{Q}) \\ &\le \int_Q h(\nabla^2 w_k)\,dx + C_\infty |D_s^2 w_k|(\overline{Q}).\end{aligned}$$

This, together with (5.2) and (5.3) yields

$$(1+\varepsilon)\frac{d\mu}{d\mathcal{L}^N}(x_0) + \varepsilon \ge \lim_{k\to\infty} \int_Q h(\nabla^2 w_k(y))\ dy \ge \limsup_{k\to\infty} \liminf_{h\to 0} \int_Q h(\nabla^2 w_{k,h})\,dx.$$

Clearly $\lim_{k\to\infty} \lim_{h\to 0} \|w_{k,h}\|_{W^{1,1}(Q;\mathbb{R}^d)} = 0$, and

$$\limsup_{k\to\infty} \lim_{h\to 0} \|\nabla^2 w_{k,h}\|_{L^1(Q;\mathbb{S}^{d\times N\times N})} \le \limsup_{k\to\infty} \|\nabla^2 w_k\|_{L^1(Q;\mathbb{S}^{d\times N\times N})} + \limsup_{k\to\infty} |D_s^2 w_k|(\bar{Q}) < +\infty.$$

By a standard diagonalization argument we may then find a sequence $z_k \in C^\infty(Q;\mathbb{R}^d)$ such that $z_k \to 0$ in $W^{1,1}(Q,\mathbb{R}^d)$, $\sup_k \|\nabla^2 z_k\|_{L^1(Q)} < +\infty$, and

$$(1+\varepsilon)\frac{d\mu}{d\mathcal{L}^N}(x_0) + \varepsilon \ge \lim_{k\to\infty} \int_Q h(\nabla^2 z_k(y))\ dy.$$

Hence applying Proposition 5.1 we arrive at

$$(1+\varepsilon)\frac{d\mu}{d\mathcal{L}^N}(x_0) + \varepsilon \ge h(0)$$

and thence the theorem is proved by letting $\varepsilon \searrow 0$. ■

6 Characterizations of the 2-quasiconvex envelope

Throughout this section $f : \mathbb{S}^{d\times N\times N} \to \mathbb{R}$, denotes a continuous function satisfying

$$\frac{1}{C}|\Lambda|^p \le f(\Lambda) \le C(1+|\Lambda|^p)$$

for every $\Lambda \in \mathbb{S}^{d\times N\times N}$ and some constant $C > 0$ and $p \ge 1$. We define

$$Q_2 f(F) := \inf\left\{\int_Q f(F + \nabla^2\psi)\,dx : \psi \in W_0^{2,p}(Q;\mathbb{R}^d)\right\}$$

where $F \in \mathbb{S}^{d\times N\times N}$. It may be shown that the function $Q_2 f$ is 2-quasiconvex (see [29]) and that (see [12])

Proposition 6.1

$$Q_2 f(F) = \inf\left\{\liminf_{n\to+\infty} \int_Q f(\nabla^2 u_n)\,dx \quad : \quad u_n \in W^{2,p}(Q;\mathbb{R}^d) \right.$$
$$\left. \text{and}\, u_n \rightharpoonup \frac{1}{2}F(x,x)\,\text{in}\, W^{2,p}(Q;\mathbb{R}^d)\right\}.$$

The analog results for first order gradients with Qf in place of $Q_2 f$ are well known (see Dacorogna [16]). Moreover, in this case, using a result of Fonseca and Müller [28], Larsen [33] proved that if $g : \mathbb{R}^{d\times N} \to \mathbb{R}$ is a continuous function such that $1/C|\xi| \le g(\xi) \le C(1+|\xi|)$ for some $C > 0$, then

$$Qg(G) = \inf\left\{\liminf_{n\to+\infty}\int_Q g(\nabla u_n)\,dx \quad : \quad u_n \in BV(Q;\mathbb{R}^d), u_n \to Gx \text{ in } L^1(Q;\mathbb{R}^d) \right.$$
$$\left. \text{and } |D_s u_n|(Q) \to 0\right\} \tag{6.1}$$

for every $G \in \mathbb{R}^{d\times N}$.

Heuristically, representation (6.1) indicates that as the singular part of the gradient of u_n is getting closer and closer to zero, then u_n is getting closer and closer to a $W^{1,1}$-function, and we may allow competing infimizing sequences to have small singular gradients.

We now show that in order to recover the 2-quasiconvexification via BH functions, it is not necessary to require that the singular part of the Hessian of the infimizing sequence to disappear in the limit, but it suffices to guarantee that it will not diffuse into the bulk part.

Theorem 6.2

$$Q_2 f(F) = \inf\left\{\liminf_{n\to+\infty}\int_Q f(\nabla^2 u_n)\,dx \quad : \quad u_n \in BH(Q;\mathbb{R}^d), u_n \to 1/2F(x,x) \text{ in } W^{1,1}(Q;\mathbb{R}^d) \right.$$
$$\left. \text{and } |D_s^2 u_n| \overset{*}{\rightharpoonup} \tau \perp \mathcal{L}^N\right\}.$$

Proof. Denote by $\widetilde{Q}_2 f$ the right hand side of the equation above. By Proposition 6.1, $\widetilde{Q}_2 f \le Q_2 f$. Conversely, since $Q_2 f \le f$ and $Q_2 f$ is quasiconvex, by Theorems 4.2 and 5.3 we have that

$$Q_2 f(F) \le \liminf_{n\to+\infty}\int_Q Q_2 f(\nabla^2 u_n)\,dx \le \liminf_{n\to+\infty}\int_Q f(\nabla^2 u_n)\,dx,$$

for every sequence $u_n \in BH(Q;\mathbb{R}^d)$ such that $u_n \to 1/2F(x,x)$ in $W^{1,1}(Q;\mathbb{R}^d)$ and $|D_s^2 u_n| \overset{*}{\rightharpoonup} \tau \perp \mathcal{L}^N$. Thence $Q_2 f \le \widetilde{Q}_2 f$. ■

Since all the arguments we have used work, with obvious modifications, also in the case of first order gradients, we also have

Theorem 6.3 *If $g : \mathbb{R}^{d\times N} \to \mathbb{R}$ is such that $1/C|\xi|^p \le g(\xi) \le C(1+|\xi|^p)$ for some $C > 0$ and $p \ge 1$, then*

$$Qg(G) = \inf\left\{\liminf_{n\to+\infty}\int_Q g(\nabla u_n)\,dx \quad : \quad u_n \in BV(Q;\mathbb{R}^d), u_n \to Gx \text{ in } L^1(Q;\mathbb{R}^d) \right.$$
$$\left. \text{and } |D_s u_n| \overset{*}{\rightharpoonup} \tau \perp \mathcal{L}^N\right\}.$$

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