COMPACTNESS PROPERTIES OF THE DG AND CG TIME STEPPING SCHEMES FOR PARABOLIC EQUATIONS

NOEL J. WALKINGTON*

Abstract. It is shown that for a broad class of equations that numerical solutions computed using the discontinuous Galerkin or the continuous Galerkin time stepping schemes of arbitrary order will inherit the compactness properties of the underlying equation. Convergence of numerical schemes for a phase field approximation of the flow of two fluids with surface tension is presented to illustrate these results.

Key words. discontinuous Galerkin, continuous Galerkin, parabolic equations.

1. Introduction. We consider the numerical approximation of solutions, $u : [0, T) \to U$, of evolution equations: $u(0) = u_0$, and

$$(1.1) \qquad u_t + A(u) = f(u).$$

Here U is a Banach space and each term of the equation takes values in U'. We allow both $A(u) = A(t, u)$ and $f = f(t, u)$ to be nonlinear and to depend explicitly upon t but do not express the later dependence in the notation. Existence of solutions of such equations is typically established by first proving a-priori bounds. These bounds, and the structure of the equation, are then used to establish sufficient compactness to control the nonlinear terms. In this paper we show that numerical schemes which use discontinuous Galerkin (DG) or continuous Galerkin (CG) time stepping schemes of arbitrary order will inherit the stability and compactness properties of a broad class of evolution equations. Under these circumstances Lax's meta-theorem

> "A (linear) numerical scheme converges if and only if it is stable and consistent"

is applicable. Of course the nonlinear analog of this old adage requires a compactness hypotheses to guarantee convergence of the nonlinear terms, and this is the main focus of this work.

Stability: Prototypically, a-priori bounds on solutions of equation (1.1) are established by either multiplying the equation by u, or multiplying by u_t. In these two situation we show that the following "rule of thumb" will frequently give rise to stable and convergent numerical schemes:

- If multiplication by u gives a-priori bounds, use a DG time stepping scheme.
- If multiplication by u_t gives a-priori bounds, use a CG time stepping scheme.

Recall that the lowest order DG and CG schemes are the (first order) implicit Euler and (second order) Crank Nicolson schemes respectively. Since the amount of work required for each scheme is the same this would suggest that CG time stepping schemes may be more desirable; however, their stability is more delicate. Below we show that the CG scheme can often be used in the first situation provided the nonlinear terms are approximated judiciously.

If the spatial operator is coercive in the sense that $(A(u), u) \geq c\|u\|_U^p$ for some $c > 0$ and $1 \leq p < \infty$, then multiplying equation (1.1) by u and integrating shows

$$(1.2) \qquad (1/2)\|u(t)\|_H^2 + \int_0^t (A(u), u) \leq (1/2)\|u_0\|_H^2 + \int_0^t (f(u), u).$$

Here we assume a pivot space structure $U \hookrightarrow H \hookrightarrow U'$ where U is embedded in a Hilbert space H which is identified with its dual and $(.,.)$ denotes the duality pairing. Under suitable structural and/or continuity hypotheses on $f(.)$, Gronwall's inequality can then be use to show

$$\|u\|_{L^\infty[0,T;H]}^2 + c\|u\|_{L^p[0,T;U]}^p \leq C\left(\|u_0\|_H^2 + C(f)\right).$$

Alternatively, if $A = \Phi'$ is the gradient of a potential, $\Phi : U \to \mathbb{R}$ (we assume Φ is independent of time), then multiplying equation (1.1) by u_t and integrating gives

$$(1.3) \qquad \int_0^t \|u_t\|_H^2 + \Phi(u(t)) = \Phi(u_0) + \int_0^t (f(u), u_t).$$

If Φ is coercive in the sense that $\Phi(u) \geq c\|u\|_U^p$, and if f satisfies suitable structural and/or continuity hypotheses, it follows that

$$(1.4) \qquad \|u_t\|_{L^2[0,T;H]} + c\|u\|_{L^\infty[0,T;U]}^p \leq C\left(\Phi(u_0) + C(f)\right).$$

Below we show that solutions computed using the DG and CG time stepping schemes inherit discrete analogs of the stability estimates (1.2) and (1.3) respectively. In this situation stability of the schemes is "automatically" inherited from the structural properties of the underlying equation.

Compactness: Compactness of solutions to the evolution equation (1.1) is frequently established using the Lions-Aubin theorem.

THEOREM 1.1. *(Lions-Aubin) Let B_0, B and B_1 be Banach spaces satisfying $B_0 \hookrightarrow\!\!\!\to B \hookrightarrow B_1$ where the first inclusion is compact. If $0 < T < \infty$, $1 \leq p, q < \infty$, and*

$$W = W(0, T) = \{u \in L^p[0, T; B_0] \mid u' \in L^q[0, T; B_1]\},$$

then the inclusion $W \hookrightarrow\!\!\!\to L^p[0, T; B]$ is compact.

In the current context, a typical application of this theorem would set $(B_0, B, B_1) = (U, H, U')$. For parabolic equations the embedding $U \hookrightarrow\!\!\!\to H$ is typically compact and the a-priori estimates bound solutions in $L^p[0, T; U]$. The crucial hypotheses on u_t is typically found by writing the equation as

$$(1.5) \qquad \int_0^T (u_t, v) = \int_0^T (f(u), v) - (A(u), v).$$

The continuity and/or structural hypotheses of f, A, and the a-priori estimates on u, are then used to bound u_t in $L^2[0, T; U']$. The Lions-Aubin theorem then establishes compactness of solutions in $L^p[0, T; H]$. This line of argument fails for numerical solutions constructed using the DG time stepping scheme; indeed, the solutions are discontinuous so their time derivatives are not integrable. This difficulty is circumvented in Theorem 3.1 below where we show that solutions satisfying the discrete analog of equation (1.5) are compact in $L^p[0, T; H]$.

When $A = \Phi'$ is the derivative of potential, $U \hookrightarrow H$, and solutions satisfy the bounds in equation (1.4), compactness of solutions in $L^p[0, T; H]$ follows directly from the Lions-Aubin theorem with $(B_0, B, B_1) = (U, H, H)$. In Theorem 4.1 it is shown that this compactness compactness property is inherited by numerical approximations constructed using the CG scheme. However, typically more is true. Writing the equation as

$$A(u) = f(u) - u_t,$$

the bounds in (1.4) show $A(u)$ is bounded in $L^p[0, T; H]$. For parabolic equations the subspace $D(A) = \{u \in U \mid A(u) \in H\}$ is typically compact. Application of the Lions-Aubin theorem with $(B_0, B, B_1) = (D(A), U, H)$ then establishes compactness in $L^p[0, T; U]$. Typically numerical solutions u_h are in U, but are never in $D(A)$, so this argument is not applicable. When A is linear, Theorem 4.5 below circumvents this difficulty to establish compactness of solutions constructed using the CG scheme.

1.1. Flows of Incompressible Immiscible Fluids. We will illustrate the applicability of our results by proving convergence of a class of numerical schemes for a phase field approximation of the equations modeling the flow of two incompressible Newtonian fluids where the surface tension between them is significant. To eliminate some technical detail, we assume that the density ρ of each fluid to be the same. Using the ideas in [6] our analysis can be extended to include this case.

The strong form of the problem considers $\Omega = \Omega_1(t) \cup \Omega_2(t) \cup S(t)$, where $\Omega_1(t) \cap \Omega_2(t) = \emptyset$ partitions Ω into the two regions occupied by each fluid and $S(t)$ is the surface separating them. Adopting the usual notation, then equations of the balance of mass and momentum require $div(u) = 0$, and

$$\rho u_t + (\rho u.\nabla)u + \nabla p - div\,(\mu D(u)) = \rho f \qquad \text{in } \Omega_1(t) \cup \Omega_2(t),$$

and $[-pI + \mu D(u)]n = \gamma H n$ on $S(t)$ where H is the mean curvature of $S(.)$, n is a normal, and $[.]$ denotes the jump across the surface. We implicitly assume that the velocity is continuous across the surface, which is transported with the flow, and assume Dirichlet boundary data for the velocity on $\partial \Omega$.

A weak statement of the momentum equation is

$$(1.6) \qquad \int_\Omega \left(\rho(u_t + (u.\nabla)u).v - p\,div(v) + \mu D(u) : D(v)\right) + \int_{S(t)} \gamma H v.n = \int_\Omega \rho f.v.$$

This is a very singular problem in the sense that the surface is unlikely to remain smooth so that H, n, and the surface measure may fail to exist. We consider a regularization based upon a phase field approximation of these quantities. Let $\phi : (0, T) \times \Omega \to \mathbb{R}$ be a smooth approximation of the phase function

$$\phi_0(t, x) = \begin{cases} -1 & x \in \Omega_1(t) \\ 1 & x \in \Omega_2(t), \end{cases}$$

so that $S(t) \simeq \{x \in \Omega \mid \phi(t, x) = 0\}$. In Section 6 formal asymptotic expansions are presented which motivate the following regularized approximation of the two fluids problem.

$$(1.7) \qquad \int_\Omega \left\{\rho u_t.v + (\rho/2)\left((u.\nabla)u.v - (u.\nabla)v.u\right)\right.$$

$$\left. -p\,div(v) + \mu D(u) : D(v) + (\gamma/\lambda)\left(\phi_t + u.\nabla\phi\right)\nabla\phi.v\right\} = \int_\Omega \rho f.v.$$

3

In this equation the density is constant, $\rho \in \mathbb{R}^+$, and the viscosity is a function of ϕ; $\mu = \mu(\phi)$. The phase function ϕ satisfies the equation

(1.8)
$$\int_\Omega \left(\phi_t + u.\nabla\phi\right)\psi + \lambda\epsilon\nabla\phi.\nabla\psi + (\lambda/\epsilon)F'(\phi)\psi = 0,$$

where $F \in C^1(\mathbb{R})$ is a classical "double well potential" with quadratic growth at infinity,

$$F(\phi) = \left\{ \begin{array}{ll} (1/2)(\phi^2 - 1)^2 & |\phi| \leq \sqrt{3} \\ 2\phi^2 - 4 & |\phi| \geq \sqrt{3}, \end{array} \right.$$

and $\lambda > 0$ is a "relaxation" parameter.

Bounds for solutions of this system of equations are naturally obtained by setting the test functions $v = u$ and $\psi = (\gamma/\lambda)\phi_t$ in the weak statements of the momentum equation and phase equations,

$$\frac{d}{dt}\int_\Omega \left((\rho/2)|u|^2 + (\gamma\epsilon/2)|\nabla\phi|^2 + (\gamma/\epsilon)F(\phi)\right) + \int_\Omega (\gamma/\lambda)(\phi_t + u.\nabla\phi)^2 + \mu|D(u)|^2 = \int_\Omega \rho f.u.$$

In Section 6 we illustrate our results by showing that numerical approximations constructed using classical finite element spaces for the spatial discretizations, and the DG and CG time stepping schemes for the momentum and phase equations respectively, converge.

1.2. Related Results. Thomee's book [27] presents a detailed analysis of the DG scheme for the heat equation. A majority of papers that address convergence of numerical schemes for nonlinear parabolic equations focus exclusively upon the first order implicit Euler time stepping scheme, e.g. [2, 12, 13, 22], and occasionally the Crank Nicolson scheme [16, 26]. The dearth of higher order schemes is frequently justified by the supposition that the regularity is low; however, in practice solutions are frequently piecewise smooth and the implicit Euler time stepping error swamps the spatial error if, for example, classical P_2/P_1 Taylor Hood elements are used for the Navier Stokes equations. A more likely reason is that for many equations the implicit Euler scheme trivially inherits the stability, monotonicity, and compactness properties of the underlying equation [10, 9, 26].

Frequently higher order time stepping schemes are analyzed using finite difference methodology, e.g. [1, 26], which is not well adapted to the variational setting where the equations are naturally posed. This typically leads to dense technical calculations to establish discrete versions of the stability and compactness estimates enjoyed by the equation. This contrasts with the DG and CG schemes analyzed herein which, being Galerkin schemes, inherit much of the underlying variational structure and functional analytic properties.

1.3. Notation. Spaces of Bochner integrable functions from a time interval $[0, T]$ to a Banach space U will be denoted as $L^p[0, T; U]$, and $\mathcal{P}_\ell[0, T; X]$ will indicate functions of the form $u(t) = \sum_{i=0}^\ell p_i(t)u_i$ with $u_i \in U$ and $p_i \in \mathcal{P}_\ell(0, T)$, the space of polynomials with degree less than or equal to ℓ. The derivative of $u \in L^p[0, T; U]$ is denoted as u' or u_t. The notation $U \hookrightarrow H$ is used to indicate that U is continuously embedded in H and $U \hookrightarrow\!\!\!\!\rightarrow H$ denotes a compact embedding. A bilinear mapping $a : U \times U \to \mathbb{R}$ is coercive if there exists $c > 0$ such that $a(u, u) \geq c\|u\|_U^2$.

In the examples, $\Omega \subset \mathbb{R}^d$ denotes a bounded open set with Lipschitz boundary, and standard notation is employed for the Lebesgue spaces, $L^p(\Omega)$, and Sobolev spaces, $W^{1,p}(\Omega)$, $H^1(\Omega)$.

4

Given a partition $0 = t^0 < t^1 < \ldots < t^N = T$ of $[0, T]$, subspace $U_h \subset U$, and $u_h \in C[0, T; U_h]$, we we frequently write $u^n = u_h(t^n)$, and if u_h is piecewise continuous we write $u_\pm^n$ for the left and right limits. If necessary, notation of the form u_h^0, or u_{h-}^0, is used for the sequence of initial values $u_h(0)$ or $u_h(0_-) \in U_h$. The duality pairing between $f \in U'$ and $u \in U$ is written as $f(u) = (f, u)$

The following form of the Arzela Ascolli theorem will be used in several places.

THEOREM 1.2. *Let X and Y be metric spaces and X be compact. Then $\mathcal{F} \subset C(X, Y)$ (with the sup metric) is precompact if and only if $\mathcal{F}_x \equiv \{f(x) : f \in \mathcal{F}\}$ is precompact in Y for each $x \in X$, and $\mathcal{F}$ is equicontinuous; i.e. for each $\epsilon > 0$ there exists $\delta > 0$ such that*

$$d_X(x_1, x_2) < \delta \quad \Rightarrow \quad d_Y(f(x_1), f(x_2)) < \epsilon, \ \forall f \in \mathcal{F}.$$

We will also make frequent use of the property that if X, Y and Z are Banach spaces, $X \hookrightarrow Y \hookrightarrow Z$ and $\|x\|_Y \leq C\|x\|_X^\theta \|x\|_Z^{1-\theta}$ for some $\theta \in (0, 1)$ and all $x \in X$, then bounded subsets of X which are compact in Z are also compact in Y.

1.4. Outline. In the next section we state the DG and CG schemes for parabolic equations in a general setting. The stability properties for a particular equation will depend upon the structure of the specific equation. However, typically estimates are found by either multiplying the equation by the solution or its time derivative, and we briefly recall the bounds that result in the discrete setting. In Sections 3 and 4 we establish compactness properties of the discrete DG and CG solutions respectively. When the spatial operator $A : U \to U'$ is linear, self adjoint, and elliptic, we show that the CG scheme inherits additional regularity properties in the second half of Section 4. In these sections various bounds are assumed upon the solution in order to decouple stability issues from compactness properties. In this setting, once bounds are established for a particular equation one can then refer to the appropriate theorem for the compactness properties.

In Section 5 we prove two lemmas used in Sections 3 and 4 to establish equicontinuity for solutions of the DG and CG schemes respectively. The arguments in this section exploit the structure of the piecewise polynomial spaces used to construct the DG and CG schemes, and are the key ingredients of the compactness proofs.

In the final section we illustrate the utility of our results by showing that numerical schemes for the two fluids problem will converge. The equations for the two fluid problem have the prototypical structure of a "complex fluid" where the momentum equation is coupled to an equation governing the evolution of the "microstructure". Indeed, this work was motivated by the authors interest in the Ericksen Leslie equations for the flow of nematic liquid crystals [17] which have this structure.

2. DG and CG Schemes. In this section review the DG and CG time stepping schemes and their basic properties. Throughout we assume that $U \hookrightarrow H \hookrightarrow U'$ are dense embeddings of Banach spaces and that the pivot space, H, is a Hilbert space. The numerical schemes we consider approximate the weak statement of equation (1.1):

$$(2.1) \qquad\qquad (u_t, v) + a(u, v) = f(v), \qquad v \in U,$$

where $a : U \times U \to \mathbb{R}$ is characterized by $a(u, v) = (A(u), v)$. For each subspace $U_h \subset U$ and partition $0 = t^0 < t^1 < \ldots < t^N = T$ of $[0, T]$, the DG and CG schemes construct on each interval (t^{n-1}, t^n) a function in $\mathcal{P}_\ell[t^{n-1}, t^n; U_h]$ which approximates a discrete version of (2.1).

2.1. DG Scheme. Given $u_-^0 \in U_h$, the DG scheme seeks $u_h \in L^2[0,T;U_h]$ satisfying, on each interval (t^{n-1}, t^n), $u_h \in \mathcal{P}_\ell[t^{n-1}, t^n; U_h]$ and

$$(2.2) \qquad \int_{t^{n-1}}^{t^n} \left((u_{ht}, v_h) + a(u_h, v_h) \right) + (u_+^{n-1} - u_-^{n-1}, v_+^{n-1}) = \int_{t^{n-1}}^{t^n} (f(u_h), v_h),$$

for all $v_h \in \mathcal{P}_\ell[t^{n-1}, t^n; U_h]$.

The distinguishing property of the DG scheme is that the test and trial spaces are the same so it is possible to set $v_h = u_h$. With this choice we find

$$(1/2)\|u_-^n\|_H^2 + (1/2)\|[u_-^{n-1}]\|_H^2 + \int_{t^{n-1}}^{t^n} a(u_h, u_h) = (1/2)\|u_-^{n-1}\|_H^2 + \int_{t^{n-1}}^{t^n} (f(u_h), u_h).$$

When $a(u,u) \geq c\|u\|_U^p$, summing this equation shows

$$(2.3) \qquad \|u_-^n\|_H^2 + \sum_{m=0}^{n-1} \|[u^m]\|_H^2 + c \int_0^{t^n} \|u_h\|_U^p \leq \|u_-^0\|_H^2 + \int_0^{t^n} |(f(u_h), u_h)|.$$

If f satisfies appropriate growth and/or continuity hypotheses the discrete solutions computed with the DG scheme will satisfy

$$\max_{1 \leq n \leq N} \|u_-^n\|_H^2 + \sum_{n=0}^{N-1} \|[u^n]\|_H^2 + c\|u_h\|_{L^p[0,T;U]}^p \leq C \left(\|u_-^0\|_H^2 + C(f) \right).$$

In Section 3 we show that this bound is sufficient for the discrete solutions to be compact in $L^r[0,T;H]$ for $1 \leq r < 2p$.

EXAMPLE 2.1. *The (skew symmetrized) nonlinear term in the Navier Stokes equations takes the form*

$$f(u)(v) = (1/2) \int_\Omega (u.\nabla v).u - (u.\nabla u).v.$$

Since $f(u)(u) = 0$, numerical solutions computed using DG time stepping and classical Taylor Hood finite elements is space will satisfy

$$\max_{1 \leq n \leq N} \|u_-^n\|_{L^2(\Omega)}^2 + \sum_{n=0}^{N-1} \|[u^n]\|_{H^1(\Omega)}^2 + c\|u_h\|_{L^2[0,T;H^1(\Omega)]}^2 \leq C\|u_-^0\|_{L^2(\Omega)}^2.$$

Compactness of solutions $\{u_h\}_{h>0}$ in $L^2[0,T;L^2(\Omega)]$, and the bound in $L^2[0,T;H^1(\Omega)]$, are sufficient to establish convergence of this class of numerical schemes.

One flaw of the higher order ($\ell > 1$) DG schemes is that equation (2.3) only bounds $\|u_h\|_H$ at the partition points. In certain problems further estimates are needed in order to apply Gronwall's inequality to bound the solution [7, 8].

2.2. CG Scheme. Given $u^0 \in U_h$, the CG scheme seeks $u_h \in C[0,T;U_h]$ satisfying, on each interval (t^{n-1}, t^n), $u_h \in \mathcal{P}_\ell[t^{n-1}, t^n; U_h]$ and

$$(2.4) \qquad \int_{t^{n-1}}^{t^n} (u_{ht}, v_h)_H + a(u_h, v_h) = \int_{t^{n-1}}^{t^n} (f(u_h), v_h)_H, \qquad v_h \in \mathcal{P}_{\ell-1}[t^{n-1}, t^n; U_h].$$

Since $u_{ht} \in \mathcal{P}_{\ell-1}[t^{n-1}, t^n; U_h]$ when $u_h \in \mathcal{P}_\ell[t^{n-1}, t^n; U_h]$ we may set $v_h = u_{ht}$. If $A(u) = \Phi'(u)$, where $\Phi : U \to \mathbb{R}$ is independent of time, this choice of test function gives

$$\int_{t^{n-1}}^{t^n} \|u_t\|_H^2 + \Phi(u^n) = \Phi(u^{n-1}) + \int_{t^{n-1}}^{t^n} (f(u_h), u_{ht}).$$

Summing this equation shows

(2.5)
$$\int_0^{t^n} \|u_{ht}\|_U^2 + \Phi(u^n) = \Phi(u^0) + \int_0^{t^n} (f(u_h), u_{ht}).$$

If $\Phi(u) \geq c\|u\|_U^p$ and f satisfies appropriate growth and/or continuity hypotheses the discrete solutions will satisfy

$$\|u_{ht}\|_{L^2[0,T;H]}^2 + c \max_{1 \leq n \leq N} \|u^n\|_U^p \leq C \left(\Phi(u_h^0) + C(f) \right).$$

In Section 4 it is shown that this bound is sufficient for the discrete solutions to be compact in $C[0,T;H]$.

One problem with higher order CG schemes ($\ell > 1$) is that equation (2.5) only bounds $\|u_h\|_U$ at the partition points. This can be problematic if, for example, $\|f(u)\|_H \leq C\|u\|_U$ since Gronwall's inequality will no longer be applicable. In this situation one may wish to set $v = u$ to first obtain an estimate on $L^p[0,T;U]$; however, this is not possible with the CG scheme. One may also wish to set $v = u$ if the CG scheme is used for an equation for which $a(.,.)$ is coercive, but not the gradient of a potential.

These issues with stability can sometimes be circumvented upon introducing the projection of u_h onto $\mathcal{P}_{\ell-1}[t^{n-1}, t^n, U_h]$.

DEFINITION 2.2. *Let $\ell > 0$ be an integer, $0 = t^0 < t^1 < \ldots < t^N = T$ be a partition of $[0,T]$, and H a Hilbert space with $U_h \subset H$ a closed subspace. Let $u \mapsto \bar{u}$ denote the projection of $L^1[0,T;H]$ onto $\{v_h \in L^1[0,T;U_h] \mid u_h|_{(t^{n-1},t^n)} \in \mathcal{P}_{\ell-1}[t^{n-1}, t^n; U_h]\}$ characterized by*

$$\int_{t^{n-1}}^{t^n} (\bar{u}, \bar{v})_H = \int_{t^{n-1}}^{t^n} (u, \bar{v})_H, \qquad \bar{v}_h \in \mathcal{P}_{\ell-1}[t^{n-1}, t^n; U_h].$$

Note that this projection depends upon the partition of $[0,T]$ and subspace U_h; however, this is not expressed in the notation.

When restricted to $L^2[0,T;U_h]$ the projection is algebraic in the sense that $\bar{u}$ is independent of the choice inner product on H. Indeed, if $u(t) = \sum_{i=0}^M u_i(t)e_i \in L^2[0,T;U_h]$, where $\{e_i\}_{i=1}^M$ is a basis of U_h and $1 \leq M = dim(U_h) \leq \infty$, then $\bar{u}(t) = \sum_{i=0}^M \bar{u}_i(t)e_i$, where, on each interval, $\bar{u}_i(t)$ is the $L^2(t^{n-1}, t^n)$ projection onto $\mathcal{P}_{\ell-1}(t^{n-1}, t^n)$ of $u_i(t)$. To verify this, let $\bar{v} \in \mathcal{P}_{\ell-1}[t^{n-1}, t^m; U_h]$ and compute

$$\int_{t^{n-1}}^{t^n} (\bar{u}, \bar{v})_H = \sum_{i,j=1}^M \int_{t^{n-1}}^{t^n} \bar{u}_i(t)\bar{v}_j(t)\, dt \, (e_i, e_j)_H$$

$$= \sum_{i,j=1}^M \int_{t^{n-1}}^{t^n} u_i(t)\bar{v}_j(t)\, dt \, (e_i, e_j)_H$$

$$= \int_{t^{n-1}}^{t^n} (u, \bar{v})_H.$$

7

If $b : U_h \times U_h \to \mathbb{R}$ is bilinear and $u \in L^2[0, T; U_h]$, a similar calculation shows

$$\int_{t^{n-1}}^{t^n} b(u, \bar{v}) = \int_{t^{n-1}}^{t^n} b(\bar{u}, \bar{v}),$$

for all $\bar{v} \in \mathcal{P}_{\ell-1}[t^{n-1}, t^n, U_h]$.

EXAMPLE 2.3. *For the numerical schemes $u_h|_{(t^{n-1}, t^n)} \in \mathcal{P}_\ell[t^{n-1}, t^n; U_h]$ and in this situation the projections are easily computed.*

If $\ell = 1$, the projection is piecewise constant and takes the value $\bar{u}_h = (u^{n-1} + u^n)/2$ on (t^{n-1}, t^n). Recall that the Crank Nicolson scheme takes the form

$$\frac{u^n - u^{n-1}}{t^n - t^{n-1}} + A \left(\frac{u^n + u^{n-1}}{2} \right) = f(t^{n+1/2}),$$

and the spatial term is precisely $A(\bar{u}_h)$.

Similarly, if $\ell = 2$ and

$$u_h(t) = (1/2)(s-1)su^{n-1} + (1 - s^2)u^{n+1/2} + (1/2)(s+1)su^n,$$

where $t = (1/2)(t^n + t^{n-1}) + (s/2)(t^n - t^{n-1})$ with $s \in (-1, 1)$, then

$$\bar{u}_h(t) = (1/2)(1/3 - s)u^{n-1} + (2/3)u^{n+1/2} + (1/2)(1/3 + s)u^n.$$

Similar formula are easily computed for higher order approximations.

When $a(.,.)$ is bilinear, and independent of time, the CG scheme may be written as

$$\int_{t^{n-1}}^{t^n} (u_{ht}, v_h)_H + a(\bar{u}_h, v_h) = \int_{t^{n-1}}^{t^n} (f(u_h), v_h), \qquad v_h \in \mathcal{P}_{\ell-1}(t^{n-1}, t^n; U_h).$$

If $a(.,.)$ is coercive, setting $v_h = \bar{u}_h$ and summing shows

(2.6) $$\qquad (1/2)\|u^n\|_H^2 + c \int_0^{t^n} \|\bar{u}_h\|_U^2 \leq (1/2)\|u^0\|_H^2 + \int_0^{t^n} (f(u_h), \bar{u}_h).$$

Under suitable hypotheses on f, this gives a bound of the form

$$\max_{1 \leq n \leq N} \|u^n\|_H^2 + \|\bar{u}_h\|_{L^2[0,T;U]} \leq C \left(\|u^0\|_H^2 + C(f) \right).$$

In Section 4 we show that this estimate is sufficient for the projections $\{\bar{u}_h\}_{h>0}$ to be compact in $L^r[0, T; H]$ for $1 \leq r < 4$. In general it may be necessary to judiciously approximate $f(u)$ by an expression of the form $f(u_h) \simeq \hat{f}(u_h, \bar{u}_h)$ in order to obtain the desired bound.

EXAMPLE 2.4. *The (skew symmetrized) nonlinear term in the Navier Stokes is*

$$(f(u), v) = (1/2) \int_\Omega (u.\nabla v).u - (u.\nabla u).v,$$

and satisfies $(f(u), u) = 0$. An approximation of this term satisfying $(f(u_h), \bar{u}_h) = 0$ is

$$(f(u_h), v) = (1/2) \int_\Omega (\bar{u}_h.\nabla \bar{u}_h).v - (\bar{u}_h.\nabla v).\bar{u}_h.$$

8

Compactness of the projections $\{\bar{u}_h\}_{h>0}$ *in* $L^2[0,T;L^2(\Omega)]$, *and the bound in* $L^2[0,T;H^1(\Omega)]$, *suffice to prove convergence of this term.*

When $a(.,.)$ is symmetric, bilinear, coercive, and independent of time, solutions of the CG scheme will satisfy the bounds in equations (2.5) and (2.6). This is not so for nonlinear operators. Solutions of the CG scheme obtained upon replacing $a(u_h, v_h)$ with $a(\bar{u}_h, v_h)$ will satisfy equation (2.6) but not equation (2.5).

3. Compactness Properties of the DG Scheme. When solutions of the evolution equation $u_t + A(u) = f(u)$ are bounded, estimates on the time derivative may be obtained by writing $u_t = f(u) - A(u) \equiv F(u)$. This frequently provides sufficient temporal regularity to show that solutions lie in a compact set of $L^2[0,T;H]$. The next theorem shows that the same is true for discrete solutions computed using a DG time stepping scheme. The theorem does not explicity require a bound on the initial data; this is subsumed by the bounds assumed upon the solution.

THEOREM 3.1. *Let H be a Hilbert space, U be a Banach space and $U \hookrightarrow\!\!\!\!\rightarrow H \hookrightarrow U'$ be dense compact embeddings. Fix $\ell \geq 0$ to be an integer, and $1 \leq p,q < \infty$. Let $h > 0$ be a (mesh) parameter and for each h let $\{t_h^i\}_{i=0}^{N_h}$ be a uniform partition of $[0,T]$ and $U_h \subset U$ be a subspace. Assume that*

1. *For each $h > 0$, $u_h \in \{u_h \in L^p[0,T;U] \mid u_h|_{(t_h^{n-1}, t_h^n)} \in \mathcal{P}_\ell(t_h^{n-1}, t_h^n; U_h)\}$ and on each interval satisfies*

$$\int_{t_h^{n-1}}^{t_h^n} (u_{ht}, v_h)_H + (u_+^{n-1} - u_-^{n-1}, v_+^{n-1})_H = \int_{t_h^{n-1}}^{t_h^n} (F_h, v_h),$$

for each $v_h \in \mathcal{P}_\ell(t^{n-1}, t^n; U_h)$.
2. *$\{u_h\}_{h>0}$ is bounded in $L^p[0,T;U]$ and $\{\|F_h\|_{L^q[0,T;U_h']}\}_{h>0}$ is bounded.*

Then,

1. *If $p > 1$ then $\{u_h\}_{h>0}$ is compact in $L^r[0,T;H]$ for $1 \leq r < 2p$.*
2. *If $1 \leq 1/p + 1/q < 2$ and $\sum_{n=1}^{N_h} \|[u_h^n]\|_H^2 < C$ is bounded independently of h, then $\{u_h\}_{h>0}$ is compact in $L^r[0,T;H]$ for $1 \leq r < 2/(1/p + 1/q - 1)$.*

Notice that the condition $1/p + 1/q < 2$ is just the requirement that one of p or q is strictly larger than one. The restriction to uniform partitions of $[0,T]$ was made to simplify the proof and can be relaxed to "quasi uniform" in the sense that there exists $\theta \in (0,1]$ such that

(3.1) $$\theta \max_{1 \leq n \leq N} (t_h^n - t_h^{n-1}) \leq \min_{1 \leq n \leq N} (t_h^n - t_h^{n-1}), \qquad h > 0.$$

Note too that for linear equations with $F(t,v) = f(t) - a(u,v)$ where $a : U \times U \to \mathbb{R}$ bilinear, continuous, and coercive, then $p = q = 2$ so $u_h \in L^r[0,T;H]$ for any $1 \leq r < \infty$ (c.f. the discussion at the end of Section 2.1).

We break the proof of this theorem into the following four steps.

1. A mild generalization of the Lions Aubin theorem which replaces the differentiability hypothesis with an equicontinuity hypothesis is first established.
2. The key step is to show that approximations computed using the DG time stepping scheme are equicontinuous. Lemma 3.3 states a little more; namely, they are uniformly equicontinuous.

3. Uniform equicontinuity implies that the solutions are bounded in a Besov space. The additional integrability in time follows from embedding results for these spaces.
4. The theorem follows upon combining the above.

The following theorem generalizes the Lions–Aubin Theorem and is similar in spirit to those found in [18, 19, 24] .This theorem does not require continuity in time, and is closely related to Kolmogorov's characterization of compact subsets of $L^p(\Omega)$ [23, 29].

THEOREM 3.2. *Let B_0 and B be a Banach spaces, and $B_0 \hookrightarrow\!\!\!\rightarrow B$ be a compact embedding. Let $\mathcal{F} \subset L^1[0,T;B_0]$ be bounded and suppose for some $1 \le p < \infty$ that $\mathcal{F}$ is equicontinuous in $L^p[0,T;B]$ in the sense that for all $\epsilon > 0$ there exists $\delta > 0$ such that*

$$\int_{\delta'}^T \|u(t) - u(t - \delta')\|_B^p \, dt \le \epsilon, \qquad u \in \mathcal{F}, \ \delta' < \delta.$$

Then for all $0 < \theta < T/2$ the set $\mathcal{F}|_{(\theta, T-\theta)} \equiv \{u|_{\theta, T-\theta} \mid u \in \mathcal{F}\}$ is precompact in $L^p[\theta, T - \theta; B]$.

Remark: Necessary and sufficient conditions for compactness in $L^p[0,T;B]$ are equicontinuity and uniform integrability: for all $\epsilon > 0$ there exists $\theta > 0$ such that

$$\int_0^\theta \|u(t)\|^p \, dt + \int_{T-\theta}^T \|u(t)\|^p \, dt \le \epsilon.$$

Uniform integrability is immediate if $\{u_h\}_{h>0}$ is bounded in $L^r[0,T;B]$ for some $r > p$.

Proof. Let $\phi : \mathbb{R} \to \mathbb{R}$ to be smooth, non-negative, with support in $(-1, 1)$, and have unit integral. For $\delta > 0$ let $\mathcal{F}_\delta = \{\phi_\delta * u \mid u \in \mathcal{F}\}$ be the mollifications, where u is extended by zero off $[0,T]$ and $\phi_\delta(s) = (1/\delta)\phi(s/\delta)$.

Since $\mathcal{F}$ is bounded in $L^1[0,T;B_0]$ it follows that $\mathcal{F}_\delta$ is a bounded subset of $C[0,T;B_0]$; in particular, $\mathcal{F}_{\delta t} \equiv \{u_\delta(t) \mid u_\delta \in \mathcal{F}_\delta\}$ is a bounded subset of B_0 for each $t \in [0,T]$, and hence precompact in B. Also

$$\|u_\delta(t_2) - u_\delta(t_1)\|_B \le \int_{|s|<\delta} |\phi_\delta(t_2 - s) - \phi_\delta(t_1 - s)| \, \|u(s)\|_B \, ds \le |t_2 - t_1| \, \|\phi_\delta\|_{Lip} \, \|u\|_{L^1[0,T;B]},$$

so $\mathcal{F}_\delta$ is uniformly Lipschitz. By the Arzela Ascolli theorem $\mathcal{F}_\delta$ is relatively compact in $C[0,T;B]$, so can be covered by an ϵ net for any $\epsilon > 0$. Moreover, since $C[0,T;B] \hookrightarrow L^p[0,T;B]$, it follows that for each $\delta > 0$ there is an $\epsilon/2$ net covering $\mathcal{F}_\delta$ with the $L^p[0,T;B]$ metric.

Next, write

$$u(t) - u_\delta(t) = \int_{|s|<\delta} \phi_\delta(s)(u(t) - u(t - s)) \, ds,$$

and use Holder's inequality (with $\phi_\delta = \phi_\delta^{1/p} \phi_\delta^{1/p'}$) to get

$$\|u(t) - u_\delta(t)\|_B^p \le \int_{|s|<\delta} \phi_\delta(s) \, \|u(t) - u(t - s)\|_B^p \, ds.$$

If $\delta < \theta$ it follows that

$$\int_\theta^{T-\theta} \|u(t) - u_\delta(t)\|_B^p \, dt \le \int_\theta^{T-\theta} \int_{|s|<\delta} \phi_\delta(s) \|u(t) - u(t - s)\|_B^p \, ds \, dt \le \sup_{\delta'<\delta} \int_\delta^T \|u(t) - u(t - \delta')\|_B^p \, dt.$$

10

Thus for any $\epsilon > 0$, equicontinuity of $\mathcal{F}$ in $L^p[0, T; B]$ enables us to select δ sufficiently small to guarantee

$$\|u - u_\delta\|_{L^p[\theta, T-\theta, B]} \leq \epsilon/2.$$

Since $\mathcal{F}_\delta$ can be covered by an $\epsilon/2$ net, it follows that $\mathcal{F}|_{(\theta, T-\theta)}$ is contained in the corresponding ϵ net, and the theorem follows. □

The key hypothesis is equicontinuity of the translates. The next lemma shows that translates of solutions computed using the DG time stepping scheme are bounded by a power of δ.

LEMMA 3.3. *Let H be a Hilbert space, U a Banach space, and $U \hookrightarrow H \hookrightarrow U'$ be dense embeddings. Let $0 = t^0 < t^1 < \ldots < t^N = T$ be a uniform partition of $[0, T]$, $U_h \subset U$ be a subspace, and $\ell \geq 0$. Let $1 \leq p, q < \infty$ with $1/p + 1/q \geq 1$ and assume that $u_h \in \{u_h \in L^p[0, T; U] \mid u_h|_{(t^{n-1}, t^n)} \in \mathcal{P}_\ell(t^{n-1}, t^n; U_h)\}$ and on each interval satisfies*

$$\int_{t^{n-1}}^{t^n} (u_{ht}, v_h)_H + (u_+^{n-1} - u_-^{n-1}, v_+^{n-1})_H = \int_{t^{n-1}}^{t^n} (F, v_h)$$

for each $v_h \in \mathcal{P}_\ell(t^{n-1}, t^n; U_h)$ where $F \in L^q[0, T; U_h']$.

Then for all $0 \leq \delta \leq T$ there exists a constant $C > 0$ of the form $C(\ell)\|F\|_{L^q[0, T; U_h']}\|u_h\|_{L^p[0, T; U]}$ such that

$$\int_\delta^T \|u_h(t) - u_h(t-\delta)\|_H^2 \, dt \leq C \max(\tau, \delta)^{1/q'} \delta^{1/p'},$$

where $\tau = T/N$, and

$$\int_\delta^T \|u_h(t) - u_h(t-\delta)\|_H^2 \, dt \leq \delta \sum_{n=1}^{N-1} \|[u^n]\|_H^2 + C\delta^{1/p'+1/q'},$$

where p' and q' are the dual exponents to p and q respectively.

The proof of this lemma is rather long so we postpone it until Section 5 below. The final ingredient needed for the proof of Theorem 3.1 states that estimates on the modulus of continuity provides additional integrability.

LEMMA 3.4. *Let W be a Banach space and $u \in L^p[0, T; W]$ for some $1 \leq p < \infty$. If $0 \leq \alpha < 1$ and*

$$\int_\delta^T \|u(t) - u(t-\delta)\|_W^p \, dt \leq C\delta^\alpha, \qquad 0 < \delta < T,$$

then $u \in L^q[0, T; W]$ for any $1 \leq q < p/(1-\alpha)$ with norm depending only upon $L^p[0, T; W]$ and C.

Notice that it suffices to prove this result for $W = \mathbb{R}$ since if $g(t) = \|u(t)\|_W$ then $\|u\|_{L^q[0, T; W]} = \|g\|_{L^q(0,T)}$, and

$$|g(t) - g(t-\delta)| = \big|\|u(t)\|_W - \|u(t-\delta)\|_W\big| \leq \|u(t) - u(t-\delta)\|_W.$$

The hypothesis in the lemma characterize functions in the Besov space $B_\infty^{\alpha/p, p}[0, T; W]$. Since $B_\infty^{\alpha/p, p}(\mathbb{R}) \subset W^{s, p}(\mathbb{R})$ for any $s < \alpha/p$ [14, 25], the Sobolev embedding theorem shows $B_\infty^{\alpha/p, p}(\mathbb{R}) \hookrightarrow L^q(\mathbb{R})$ for any $q < p/(1 - sp) = p/(1 - \alpha)$. We can now prove Theorem 3.1.

11

Proof. (of Theorem 3.1) By hypotheses $\mathcal{F} \equiv \{u_h\}_{h>0}$ is bounded in $L^p[0,T;U]$, and Lemma 3.3 shows that $\mathcal{F}$ is equicontinuous in $L^2[0,T;H]$. It follows that Theorem 3.2 is applicable with $B_0 = U$ and $B_1 = H$ which establishes compactness of $\mathcal{F}$ in $L^2[\theta, T - \theta; H]$ for all $0 < \theta < T/2$.

Lemma 3.3 establishes the modulus of continuity hypothesis of Lemma 3.4 so $\mathcal{F}$ is bounded in $L^r[0,T;H]$ for any $1 \leq r < r^*$ where $r^* = 2p$ in the first case and $r^* = 2/(1/p + 1/q - 1)$ in the second. In either case $r^* > 2$, so $\mathcal{F}$ is compact in $L^2[0,T;H]$. If $r < r^*$, let $r < s < r^*$ and write $r = 2\theta + (1 - \theta)s$ for some $\theta \in (0,1)$. Then

$$\|u\|_{L^r[0,T;H]} \leq \|u\|_{L^2[0,T;H]}^{\theta} \|u\|_{L^s[0,T;H]}^{1-\theta},$$

so $\mathcal{F}$ is compact in $L^r[0,T;H]$. $\square$

4. Compactness for the CG Scheme. When $A(u) = \Phi'(u)$ is the gradient of a coercive potential the natural bound satisfied by solutions of the CG scheme is

$$\|u_{ht}\|_{L^2[0,T;H]} + \max_{1 \leq i \leq N} \|u^n\|_U \leq C \left(\|u^0\|_U + C(f) \right).$$

We begin by showing this is sufficient for compactness in $C[0,T;H]$.

THEOREM 4.1. *Let $U \hookrightarrow\hookrightarrow H$ be a compact embedding of Banach spaces and $\mathcal{U} \equiv \{u_h\}_{h>0} \subset C[0,T;H]$ have derivatives bounded in $L^q[0,T;H]$ for some $1 \leq q \leq \infty$. Assume that there exists $C > 0$ such that for each $h > 0$ there exists a partition $0 = t_h^0 < t_h^1 < \ldots t_h^{N_h} = T$ with*

$$\max_{0 \leq n \leq N_h} \|u_h(t_h^n)\|_U \leq C \qquad and \qquad \lim_{h \to 0} \max_{1 \leq n \leq N_h} |t_h^n - t_h^{n-1}| = 0.$$

Then $\mathcal{U}$ is bounded in $C[0,T;H]$, and (i) if $q > 1$ then $\mathcal{U}$ is precompact in $C[0,T;H]$, and (ii) if $q = 1$ then $\mathcal{U}$ is precompact in $L^r[0,T;H]$ for any $1 \leq r < \infty$.

Proof. For each $h > 0$ define $\hat{u}_h \in C[0,T;U]$ to be the piecewise linear interpolant of $\{u_h(t_h^n)\}_{n=0}^{N_h}$. If $\tau_h^n = t_h^n - t_h^{n-1}$ and $t \in (t_h^{n-1}, t_h^n)$, then

$$\|\hat{u}_{ht}(t)\|_H = (1/\tau_h^n)\|u_h(t_h^n) - u_h(t_h^{n-1})\|_H \leq (1/\tau_h^n) \int_{t_h^{n-1}}^{t_h^n} \|u_{ht}(s)\|_H \, ds,$$

so $\|\hat{u}_{ht}\|_{L^q[0,T;H]} \leq \|u_{ht}\|_{L^q[0,T;H]}$ is bounded. It follows that $\{\hat{u}_h\}_{h>0}$ is equicontinuous in $C[0,T;H]$ and since $\{\hat{u}_h\}_{h>0}$ is bounded in $C[0,T;U]$ the sections $\hat{\mathcal{U}}_t \equiv \{\hat{u}_h(t)\}_{h>0}$ are bounded in U, so precompact in H, for each $t \in [0,T]$. The Arzela Ascolli theorem then shows $\{\hat{u}_h\}_{h>0}$ is precompact in $C[0,T;H]$.

To show compactness of $\{u_h\}_{h>0}$, fix $t \in (t_h^{n-1}, t_h^n)$ and compute

$$\|u_h(t) - \hat{u}_h(t)\|_H \leq \|u_h(t) - u_h(t_h^{n-1})\|_H + \|\hat{u}_h(t) - u_h(t_h^{n-1})\|_H \leq 2 \int_{t_h^{n-1}}^{t_h^n} \|u_{ht}(s)\| \, ds.$$

If $q > 1$ it follows that

$$\|u_h - \hat{u}_h\|_{C[0,T;H]} \leq 2 \max_{1 \leq n < N_h} |t_h^n - t_h^{n-1}|^{1/q'} \|u_{ht}\|_{L^q[0,T;H]} \to 0,$$

12

so $\mathcal{U}$ is also compact in $C[0,T;H]$. If $q = 1$ then

$$\|u_h - \hat{u}_h\|_{L^1[0,T;H]} \leq 2 \max_{1 \leq n < N_h} |t_h^n - t_h^{n-1}| \|u_{ht}\|_{L^1[0,T;H]} \to 0.$$

Since both $\{u_h\}$ and $\{\hat{u}_h\}_{h>0}$ are bounded in $C[0,T;H]$ it follows that $\|u_h - \hat{u}_h\|_{L^r[0,T;H]} \to 0$ for $1 \leq r < \infty$. $\square$

Setting the test function in the CG scheme to be the projection of the solution onto $\mathcal{P}_{\ell-1}[0,T;U_h]$ gives estimates of the form

$$\max_{1 \leq n \leq N} \|u^n\|_H^2 + \|\bar{u}_h\|_{L^p[0,T;U]}^p \leq C \left(\|u_h^0\|_H + C(f) \right).$$

The next theorem shows that this suffices to establish compactness of the projections $\{\bar{u}_h\}_{h>0}$ in $L^r[0,T;H]$ for $1 \leq r < 2p$ under mild restrictions on the meshes.

ASSUMPTION 4.2. *The orthogonal projection $P_h : H \to U_h$ is stable when restricted to U; that is, there exists $C_P > 0$ such that $\|P_h u\|_U \leq C_P \|u\|_U$ for all $u \in U$ and $h > 0$.*

When $U = H^1(\Omega) \hookrightarrow L^2(\Omega) = H$ and $U_h \subset U$ is a classical finite element space constructed over a triangulation of Ω the orthogonal projections $P_h : H \to U_h$ are stable when restricted to U for a broad class of meshes [3, 5, 11].

THEOREM 4.3. *Let U and H be a Hilbert spaces, and let $U \hookrightarrow H \hookrightarrow U'$ be dense compact embeddings. Fix $\ell \geq 0$ to be an integer, and $1 \leq p, q < \infty$. Let $h > 0$ be a (mesh) parameter and for each h let $U_h \subset U$ be a finite dimensional subspace and $\{t_h^i\}_{i=0}^{N_h}$ be a partition of $[0,T]$, and assume that $\lim_{h \to 0} \max_{1 \leq n \leq N_h} |t_h^{n+1} - t_h^n| = 0$. Assume that the orthogonal projections $P_h : H \to U_h$ satisfy Assumption 4.2. If*

1. *For each $h > 0$, $u_h \in \{u_h \in C[0,T;U] \mid u_h|_{(t_h^{n-1}, t_h^n)} \in \mathcal{P}_\ell[t_h^{n-1}, t_h^n; U_h]\}$ and on each interval satisfies*

$$\int_{t_h^{n-1}}^{t_h^n} (u_{ht}, v_h)_H = \int_{t_h^{n-1}}^{t_h^n} (F_h, v_h),$$

   for each $v_h \in \mathcal{P}_{\ell-1}[t^{n-1}, t^n; U_h]$ where $F_h \in L^q[0,T;U_h']$.
2. *The projections $\{\bar{u}_h\}_{h>0}$ onto $\mathcal{P}_{\ell-1}[0,T;U_h]$ characterized in Definition 2.2 are bounded in $L^p[0,T;U]$, and $\{\|F_h\|_{L^q[0,T;U_h']}\}_{h>0}$ is bounded.*
3. *The initial data $\{u_h^0\}_{h>0}$ are bounded in H,*

then $\{\bar{u}_h\}_{h>0}$ is precompact in $L^r[0,T;H]$ for $1 \leq r < 2p$.

The proof of theorem 4.3 uses properties of the projections summarized in the following lemma.

LEMMA 4.4. *Let $\ell > 0$ be an integer, U_h be a finite dimensional subspace of the Hilbert space H, and let $P_h : H \to U_h$ denote the orthogonal projection. Let $(.,.)_W$ be a (semi) inner product on U_h. Then there exists a constant $C_\ell > 0$ depending only upon ℓ, such that the projection $u \mapsto \bar{u}$ of Definition 2.2 satisfies*

$$\|\bar{u}\|_{L^p[t^{n-1},t^n;W]} \leq C_\ell \|P_h u\|_{L^p[t^{n-1},t^n;W]},$$

for all $1 \leq p \leq \infty$.

Proof. For $u_h \in U_h$ let $L_h u_h \in U_h$ be the "discrete Laplacian", that is,

$$(L_h u_h, v_h)_H = (u_h, v_h)_W, \qquad v_h \in U_h.$$

13

The key step is to observe that for $u \in U$ and $\bar{u} \in U_h$ that

$$(L_h \bar{u}, u)_H = (L_h \bar{u}, P_h u)_H = (\bar{u}, P_h u)_W.$$

Since L_h is linear it maps $\mathcal{P}_{\ell-1}[t^{n-1}, t^n; U_h]$ to itself, so

$$\|\bar{u}\|^2_{L^2[t^{n-1}, t^n; W]} = \int_{t^{n-1}}^{t^n} (L_h \bar{u}, \bar{u})_H = \int_{t^{n-1}}^{t^n} (L_h \bar{u}, u)_H \leq \|\bar{u}\|_{L^{p'}[t^{n-1}, t^n; W]} \|P_h u\|_{L^p[t^{n-1}, t^n; W]}.$$

The finite dimensionality of $\mathcal{P}_\ell(t^{n-1}, t^n)$, and a scaling argument, shows there exists a constant $C_\ell > 0$ depending only upon ℓ, such that

$$\|\bar{u}\|_{L^p[t^{n-1}, t^n; W]} \|\bar{u}\|_{L^{p'}[t^{n-1}, t^n; W]} \leq C_\ell \|\bar{u}\|^2_{L^2[t^{n-1}, t^n; W]}, \qquad \bar{u} \in \mathcal{P}_\ell[t^{n-1}, t^n; U_h].$$

Combining the estimates establishes the lemma. □

Proof. (of Theorem 4.3) We claim that it suffices to show that $\bar{\mathcal{U}} \equiv \{\bar{u}_h\}_{h>0}$ is precompact in $L^s[0, T; U']$ for all $1 \leq s < \infty$. Granted this, note that $\|u\|^2_H \leq \|u\|_U \|u\|_{U'}$ for any $u \in U$, so

$$\|\bar{u}_h\|_{L^r[0, T; H]} \leq \|\bar{u}_h\|^{1/2}_{L^{r/(2\theta)}[0, T; U]} \|\bar{u}_h\|^{1/2}_{L^{r/(2(1-\theta))}[0, T; U']}, \qquad \theta \in [0, 1].$$

Setting $1 \leq r < 2p$ and $\theta = r/2p$ shows

$$\|\bar{u}_h\|_{L^r[0, T; H]} \leq \|\bar{u}_h\|^{1/2}_{L^p[0, T; U]} \|\bar{u}_h\|^{1/2}_{L^{rp/(2p-r)}[0, T; U']},$$

from which we conclude that $\bar{\mathcal{U}}$ is compact in $L^r[0, T; H]$.

If $\mathcal{U} \equiv \{u_h\}_{h>}$ is compact in $L^s[0, T; U']$ then the previous lemma shows

$$\|\bar{u}_h\|_{L^s[t^{n-1}, t^n; U']} \leq C_\ell \|P_h u\|_{L^s[t^{n-1}, t^n; U']}.$$

Since

$$\|P_h u\|_{U'} = \sup_{v \in U} \frac{(P_h u, v)_H}{\|v\|_U} = \sup_{v \in U} \frac{(u, P_h v)_H}{\|v\|_U} \leq \sup_{v \in U} \frac{\|u\|_{U'} \|P_h v\|_U}{\|v\|_U} \leq C_P \|u\|_{U'},$$

it follows that

$$\|\bar{u}_h\|_{L^s[t^{n-1}, t^n; U']} \leq C_\ell C_P \|u\|_{L^s[t^{n-1}, t^n; U']}.$$

Then compactness of $\bar{\mathcal{U}}$ in $L^s[0, T; U']$ will follow from compactness of $\mathcal{U}$ in $L^s[0, T; U']$.

Compactness of $\mathcal{U}$ in $L^s[0, T; U']$ follows from Theorem 4.1 with spaces $H \hookrightarrow\!\!\!\rightarrow U'$. The only hypotheses to be verified is the differentiability. Given $v \in L^{q'}[t^{n-1}, t^n, U]$, we compute

$$\begin{aligned}
\int_{t^{n-1}}^{t^n} (u_{ht}, v)_H &= \int_{t^{n-1}}^{t^n} (u_{ht}, \bar{v})_H \\
&= \int_{t^{n-1}}^{t^n} (F_h, \bar{v}) \\
&\leq \|F_h\|_{L^q[t^{n-1}, t^n; U']} \|\bar{v}\|_{L^{q'}[t^{n-1}, t^n; U]} \\
&\leq \|F_h\|_{L^q[t^{n-1}, t^n; U'_h]} C_\ell C_P \|v\|_{L^{q'}[t^{n-1}, t^n; U]},
\end{aligned}$$

where the previous lemma was used in the last line. It follows that $\{u_{ht}\}_{h>0}$ is bounded in $L^q[0, T; U']$ so $\mathcal{U}$ is precompact in $L^s[0, T; U']$ for any $1 \leq s < \infty$. □

14

4.1. Compactness in $L^2[0, T; U]$. When u_t and $f(u)$ are in $L^p[0, T; H]$ we may write equation (1.1) as as $Au = f - u_t$ to conclude $Au \in L^p[0, T; H]$. Elliptic regularity may then be used to show that solutions are compact in $L^p[0, T; U]$. In this section analogous estimates are obtained for projections of solutions to the CG scheme when $A : U \to U'$ is linear.

THEOREM 4.5. *Let U be a Banach space, H be a Hilbert space and $U \hookrightarrow\!\!\!\to H$ be a compact embedding and $a : U \times U \to \mathbb{R}$ be bilinear, symmetric, continuous, and coercive. Fix $\ell \geq 0$ to be an integer, $1 \leq q < \infty$, $1 < p < \infty$, and for each $h > 0$ let $\{t_h^i\}_{i=0}^{N_h}$ be a uniform partition of $[0, T]$ and $U_h \subset U$ be a subspace. Assume that $\{U_h\}_{h>0} \subset U$ is dense in the sense that for all $v \in U$ there exists $v_h \in U_h$ such that $\lim_{h \to 0} \|v - v_h\|_U \to 0$.*

For each $h > 0$ let $u_h \subset C[0, T; U_h]$ satisfy $u_h|_{(t_h^{n-1}, t_h^n)} \in \mathcal{P}_\ell(t_h^{n-1}, t_h^n; U_h)$ and

$$\int_{t_h^{n-1}}^{t_h^n} a(\bar{u}_h, v_h) = \int_{t_h^{n-1}}^{t_h^n} (F_h, v_h), \qquad v_h \in \mathcal{P}_{\ell-1}[t^{n-1}, t^n; U_h],$$

where $\bar{u}_h$ is the projection of u_h onto $\{\bar{u} \in L^1[0, T; U_h] \mid \bar{u}|_{(t_h^{n-1}, t_h^n)} \in \mathcal{P}_{\ell-1}[t_h^{n-1}, t_h^n; U_h]\}$ of Definition 2.2. Assume that (i) $\{u_{ht}\}_{h>0}$ is bounded in $L^q[0, T; H]$, (ii) $\{F_h\}_{h>0}$ is bounded in $L^p[0, T; H]$, and (iii) $\{u_h^0\}_{h>0}$ is bounded in H. Then $\bar{\mathcal{U}} \equiv \{\bar{u}_h\}_{h>0}$ is bounded in $L^\infty[0, T; H]$ and precompact in $L^2[0, T; U]$.

The restriction to uniform partitions of $[0, T]$ was made to simplify the proof, and can be relaxed to the "quasi uniform" hypotheses given in equation (3.1). In the proof it will be convenient to work with the strong form of the equation, $u_t + Au = F$, which involves the operator $A : D(A) \to H$.

DEFINITION 4.6. *Let U be a Banach space, H be a Hilbert space and the embeddings $U \hookrightarrow H \hookrightarrow U'$ be dense. If $a : U \times U \to \mathbb{R}$ linear in the second argument, let*

$$D(A) = \{u \in U \mid \text{ there exists } C_u > 0 \text{ such that } |a(u, v)| \leq C_u \|v\|_H \text{ for all } v \in U\}.$$

Then $A : D(A) \to H$ is the map with $Au \in H$ characterized by $(Au, v)_H = a(u, v)$ for all $v \in U$.

In general the projections $\bar{u}$ introduced in Definition 2.2 will not be continuous in time. The following lemma shows that if u is differentiable, then it is possible to bound the translates of $\bar{u}$ needed to establish equicontinuity.

LEMMA 4.7. *Let $\ell \geq 1$ be an integer and $0 = t^0 < t^1 < \ldots t^N = T$ be a uniform partition of $[0, T]$. Let W be a (semi) inner product space, $1 \leq q \leq \infty$, and suppose that*

$$w \in \{C[0, T; W] \mid w|_{(t^{n-1}, t^n)} \in \mathcal{P}_\ell(t^{n-1}, t^n; W) \; 1 \leq n \leq N\},$$

and that $w' \in L^q[0, T; W]$. Then there exists a constant $C = C(\ell)$ depending only upon ℓ such that

$$\left(\int_\delta^T \|\bar{w}(t) - \bar{w}(t - \delta)\|_W^q \, dt\right)^{1/q} \leq C\|w'\|_{L^q[0, T; W]} \max(\delta, \tau)^{1/q'} \delta^{1/q},$$

where $\tau = T/N$ and $\bar{w}$ is the projection of w introduced in Definition 2.2.

We postpone the proof of this lemma until Section 5 and focus instead upon its application to the CG scheme.

Proof. (of Theorem 4.5) We first verify that $\bar{\mathcal{U}} \equiv \{\bar{u}_h\}_{h>0}$ is compact in $L^s[0, T; H]$ for any $1 \leq s < \infty$. The coercivity of $a(., .)$ and the bound on $\{F_h\}_{h>0}$ imply $\{\bar{u}_h\}_{h>0}$ is bounded in

15

$L^p[0, T; U]$. Then Lemma 4.7 (with space $W = H$) establishes the equicontinuity hypotheses of Theorem 3.2 from which we conclude (pre)compactness of $\bar{\mathcal{U}}$ in $L^p[\theta, T - \theta; H]$ for any $0 < \theta < T$. Since $\{u_{ht}\}$ is bounded in $L^q[0, T; H]$, and $\{u_h^0\}_{h>0}$ is bounded in H, it follows that $\mathcal{U} \equiv \{u_h\}_{h>0}$ is bounded in $C[0, T; H]$. Lemma 4.4 then shows $\bar{\mathcal{U}}$ is bounded in $L^\infty[0, T; H]$ so $\bar{\mathcal{U}}$ is precompact in $L^s[0, T; H]$ for any $1 \le s < \infty$.

Let $A_h : U_h \to U_h$ be the discrete approximation of A characterized by

$$(A_h w_h, z_h)_H = a(w_h, z_h) \qquad \text{for all } z_h \in U_h.$$

Then $A_h \bar{u}_h \in \mathcal{P}_{\ell-1}[t_h^{n-1}, t_h^n; U_h]$ on each interval of the partition, and by hypothesis $\{A_h \bar{u}_h\}_{h>0}$ is bounded in $L^p[0, T; H]$. We may then pass to a subsequence for which $\bar{u}_h \rightharpoonup u$ in $L^p[0, T; U]$, $u_h \to u$ in $L^{p'}[0, T; H]$ and $A_h u_h \rightharpoonup \alpha$ in $L^p[0, T; H]$.

By hypothesis $a(.,.)$ is an inner product equivalent to $(.,.)_U$ so it suffices to show that

$$\int_0^T \|\bar{u}_h\|_U^2 = \int_0^T a(\bar{u}_h, \bar{u}_h) \to \int_0^T a(u, u) = \int_0^T \|u\|_U^2.$$

For $v \in C[0, T; U]$ fixed, the approximation properties assumed for the discrete spaces guarantee the existence of $\{\bar{v}_h\}_{h>0}$ with $\bar{v}_h \in \{\bar{v}_h \in C[0, T; U_h] \mid \bar{v}_h|_{(t_h^{n-1}, t_h^n)} \in \mathcal{P}_{\ell-1}[t_h^{n-1}, t_h^n; U_h]\}$ for which $\bar{v}_h \to v$ in $C[0, T; U]$. Then

$$\int_0^T (\alpha, v)_H = \lim_{h \to 0} \int_0^T (A_h \bar{u}_h, \bar{v}_h)_H = \lim_{h \to 0} \int_0^T a(\bar{u}_h, \bar{v}_h) = \int_0^T a(u, v) = \int_0^T (Au, v)_H.$$

It follows that $\alpha = Au$. Then

$$\int_0^T a(\bar{u}_h, \bar{u}_h) = \int_0^T (A_h \bar{u}_h, \bar{u}_h)_H \to \int_0^T (Au, u)_H = \int_0^T a(u, u).$$

It follows that $\|\bar{u}_h\|_{L^2[0, T; U]} \to \|u\|_{L^2[0, T; U]}$ so that $\bar{u}_h \to u$ in $L^2[0, T; U]$. □

Remarks: The linearity and symmetry assumptions were used to give a concise statement and proof of this theorem. In many situations these assumptions can be relaxed.

- Symmetry of $a(.,.)$ was used in an essential way to obtain the apriori bounds; however, it is not essential in order to obtain strong convergence in $L^p[0, T; U]$ under the hypotheses of the theorem. This may be useful when this equation is part of a system for which bounds are available independently. Briefly, with $\alpha = Au \in L^p[0, T; H]$ as above, let $\Pi_h u(t) \in U_h$ be the solution of $a(\Pi_h u, v_h) = a(u, v_h)$ for all $v_h \in U_h$. Classical finite element theory shows $\|u - \Pi_h u\|_U \le C\|u - v_h\|_U$ for all $v_h \in U_h$. The density assumption then implies $\Pi_h u \to u$ in $L^2[0, T; U] \cap C[0, T; H]$. Then

  $$c\|\bar{u}_h - \Pi_h u\|_{L^2[0, T; U]}^2 \le \int_0^T a(\bar{u}_h - \Pi_h u, \bar{u}_h - \Pi_h u) = \int_0^T (A_h \bar{u}_h - Au, \bar{u}_h - \Pi_h u)_H \to 0.$$

- This line of argument also works for many nonlinear operators $A : U \to U'$. For example, for many monotone operators the map $u \mapsto (Au, u)$ is non-negative and strictly convex. Convergence of such functions often implies strong convergence in U [4, 28].

4.1.1. Higher Integrability in Time. The natural energy estimate (2.5) of the CG scheme only provides a discrete $L^\infty[0, T; U]$ bound of the form

$$\max_{1 \leq i \leq n} \|u^n\|_U \leq C.$$

Under the hypotheses of Theorem 4.5 we show that some higher integrability of u_h in time is available. Observe that

$$c\|\bar{u}_h\|_U^2 \leq a(\bar{u}_h, \bar{u}_h) = (A_h \bar{u}_h, \bar{u}_h)_H \leq \|A_h \bar{u}_h\|_H \|\bar{u}_h\|_H.$$

Since $\bar{\mathcal{U}} = \{\bar{u}_h\}_{h>0}$ is bounded in $L^\infty[0, T; H]$ and $\{A_h \bar{u}_h\}_{h>0}$ is bounded in $L^p[0, T; H]$ it follows that

(4.1) $$\sqrt{c}\|\bar{u}_h\|_{L^{2p}[0,T;U]} \leq \|\bar{u}_h\|_{L^\infty[0,T;H]}^{1/2} \|A_h \bar{u}_h\|_{L^p[0,T;H]}^{1/2}.$$

Under the hypotheses of Theorem 4.5 this shows $\bar{\mathcal{U}}$ is precompact in $L^q[0, T; U]$ for any $1 \leq q < 2p$.

4.1.2. Higher Spatial Regularity. When $a(.,.)$ is the bilinear form corresponding to an elliptic operator, functions in $D(A) \subset U$ typically exhibit higher spatial regularity. For classical finite element spaces $U_h \cap D(A)$ consists of the constant functions, and bounds on $A_h u_h$ in H do not immediately imply higher regularity. In this section we recall how to recover some additional regularity and the resulting bounds available for the evolution problems.

EXAMPLE 4.8. *In the classical situation with $U = H_0^1(\Omega)$ and $a(u, v) = (\nabla u, \nabla v)_{L^2(\Omega)}$, the domain of the operator is $D(A) = H^2(\Omega) \cap H_0^1(\Omega)$ provided $\Omega \subset \mathbb{R}^d$ is sufficiently regular. In three dimensions*

$$\|u\|_{W^{1,6}(\Omega)} \leq \|u\|_{H^2(\Omega)} \simeq \|Au\|_H,$$

and

$$\|u\|_{L^\infty(\Omega)} \leq C\|u\|_{H^1(\Omega)}^{1/2} \|u\|_{H^2(\Omega)}^{1/2}. \simeq C\|u\|_U^{1/2} \|Au\|_H^{1/2}.$$

Classical finite element spaces satisfy $U_h \subset L^\infty(\Omega) \cap W^{1,6}(\Omega)$ and analogous bounds often hold when $A_h u_h$ is substituted for Au.

Given $u_h \in U_h$ we write $\tilde{u}_h \in D(A)$ for the solution of $A\tilde{u}_h = A_h \bar{u}_h$; that is,

$$\tilde{u}_h \in U, \qquad a(\tilde{u}_h, v) = (A_h \bar{u}_h, v)_H, \qquad v \in U.$$

Restricting v to U_h shows

$$a(\bar{u}_h, v_h) = a(\tilde{u}_h, v_h), \qquad v_h \in U_h,$$

so $\bar{u}_h$ is the classical finite element approximation of $\tilde{u}_h$. In particular, when $a(.,.)$ is continuous and coercive

$$\|\bar{u}_h - w_h\|_U \leq C \inf_{w_h \in U_h} \|\tilde{u}_h - w_h\|_U.$$

Also,

$$c\|\tilde{u}_h\|_U^2 \leq a(\tilde{u}_h, \tilde{u}_h) = (A_h \bar{u}_h, \tilde{u}_h)_H = (A_h \bar{u}_h, P_h \tilde{u}_h)_H = a(\bar{u}_h, P_h \tilde{u}_h) \leq C\|\bar{u}_h\|_U \|P_h \tilde{u}_h\|_U.$$

Thus if the orthogonal projection $P_h : H \to U_h$ is stable when restricted to U it follows that $\|\tilde{u}_h\|_U$ is bounded by $\|\bar{u}_h\|_U$

17

By interpolating between U and $D(A)$ we can obtain estimates on $\tilde{u}_h$ in other spaces, (such as $W^{1,6}(\Omega)$ in the previous example), and approximation and inverse estimates can often be used to bound $\bar{u}_h$ in the same spaces.

LEMMA 4.9. *Let U be a Banach space, $a : U \times U \to \mathbb{R}$ be bilinear, continuous and coercive, and let $U_h \subset U$ be finite dimensional. Let W be a Banach space for which $U_h \subset W$ and $\|v\|_W \leq \|v\|_U^{1-\theta}\|Av\|_H^\theta$ for $v \in D(A)$ and some $\theta \in [0,1]$ fixed.*

Let $I_h : D(A) \to U_h$ be an interpolation operator satisfying

1. *$\|I_h v\|_W \leq C\|v\|_W$ for $v \in D(A)$, and*
2. *$\|v - I_h v\|_U \leq Ch^{\theta_1}\|v\|_U^{1-\theta}\|Av\|_H^\theta$ for some $\theta_1 \geq 0$,*

and let the following inverse inequality hold,

$$\sup_{0 \neq v_h \in U_h} \frac{\|v_h\|_W}{\|v_h\|_U} \leq C/h^{\theta_1}.$$

Then $\|u_h\|_W \leq C\|\tilde{u}\|_U^{1-\theta}\|A\tilde{u}\|_H^\theta$ for all $u_h \in U_h$ where $\tilde{u} \in D(A)$ is the solution of $A\tilde{u} = A_h u_h$.

In particular, if Assumption 4.2 holds and $\{\bar{u}_h\}_{h>0}$ is bounded in $L^q[0,T;U]$ and $\{A_h\bar{u}_h\}_{h>0}$ is bounded in $L^p[0,T;H]$, then $\{\bar{u}_h\}_{h>0}$ is bounded in $L^r[0,T;W]$ where $1/r = (1-\theta)/q + \theta/p$;

$$\|\bar{u}_h\|_{L^q[0,T;W]} \leq C\|\bar{u}_h\|_{L^q[0,T;U]}^{1-\theta}\|A_h\bar{u}_h\|_{L^p[0,T;H]}^\theta.$$

Proof. The coercivity and continuity of $a(.,.)$ on U guarantee $\|u_h - w_h\|_U \leq C\|\tilde{u} - w_h\|_U$ for any $w_h \in U_h$. We then compute

$$\|u_h\|_W \leq \|u_h - I_h\tilde{u}\|_W + \|I_h\tilde{u}\|_W \leq (C/h^{\theta_1})\|u_h - I_h\tilde{u}\|_U + \|I_h\tilde{u}\|_W \leq C\|\tilde{u}\|_U^{1-\theta}\|A\tilde{u}\|_H^\theta + C\|\tilde{u}\|_W,$$

and the conclusion follows. $\square$

EXAMPLE 4.10. *(after Heywood and Ranacher [15]) In the classical situation with $H = L^2(\Omega) \hookrightarrow H_0^1(\Omega) = U$ and $a(u,v) = (\nabla u, \nabla v)_{L^2(\Omega)}$, the domain of the operator is $D(A) = H^2(\Omega) \cap H_0^1(\Omega)$ when $\Omega \subset \mathbb{R}^d$ is sufficiently regular.*

Let $\{\bar{u}_h\}_{h>0}$ be (projections of) the solutions of the CG scheme bounded in $L^\infty[0,T;H]$ with $\{A_h\bar{u}_h\}_{h>0}$ bounded in $L^2[0,T;H]$. Then equation (4.1) shows

$$\|\bar{u}_h\|_{L^4[0,T;U]} \leq C\|\bar{u}_h\|_{L^\infty[0,T;H]}^{1/2}\|A_h\bar{u}_h\|_{L^2[0,T;H]}^{1/2}.$$

If $I_h : C(\bar{\Omega}) \to \mathbb{R}$ is the classical interpolation operator, then on regular meshes

$$\|v - I_h v\|_{H^1(\Omega)} \leq Ch^\theta\|v\|_{H^\theta(\Omega)} \leq Ch^\theta\|v\|_{H^1(\Omega)}^{1-\theta}\|v\|_{H^2(\Omega)}^\theta,$$

so $\theta_1 = \theta$. Moreover, $\|I_h u\|_W \leq C\|u\|_W$ for all the classical Lebesgue and Sobolev spaces. Note that if, for example, the domain has reentrant corners then $D(A) = H^{1+\mu}(\Omega)$ for some $0 < \mu < 1$ depending upon the reentrant angles. In this situation $\theta_1 = \mu\theta$.

(1) When $W = W^{1,s}(\Omega)$ and $\Omega \subset \mathbb{R}^d$ classical inverse inequalities for finite element function spaces constructed over quasi uniform meshes state

$$\|v_h\|_{W^{1,s}(\Omega)} \leq Ch^{\min(0,d/s-d/2)}\|v_h\|_{H^1(\Omega)}, \qquad v_h \in U_h.$$

18

In three dimensions $\|u\|_{W^{1,6}(\Omega)} \leq C\|u\|_{H^2(\Omega)}$ *and* $\|u_h\|_{W^{1,6}(\Omega)} \leq (C/h)\|u_h\|_{H^1(\Omega)}$ *so the theorem is applicable with* $\theta = 1$. *It follows that* $\|\bar{u}_h\|_{L^2[0,T;W^{1,6}(\Omega)]} \leq C\|A_h\bar{u}_h\|_{L^2[0,T;L^2(\Omega)]}$.

Similarly, $\|u\|_{W^{1,3}(\Omega)} \leq C\|u\|_{H^1(\Omega)}^{1/2}\|u\|_{H^2(\Omega)}^{1/2}$ *and* $\|u_h\|_{W^{1,3}(\Omega)} \leq (C/h^{1/2})\|u_h\|_{H^1(\Omega)}$, *so setting* $s = 3$ *and* $\theta = 1/2$ *shows*

$$\|\bar{u}_h\|_{L^{8/3}[0,T;W^{1,3}(\Omega)]} \leq C\|\bar{u}_h\|_{L^4[0,T;H^1(\Omega)]}^{1/2}\|A_h\bar{u}_h\|_{L^2[0,T;L^2(\Omega)]}^{1/2} \leq C\|\bar{u}_h\|_{L^\infty[0,T;L^2(\Omega)]}^{1/4}\|A_h\bar{u}_h\|_{L^2[0,T;L^2(\Omega)]}^{3/4}.$$

(2) When $W = L^s(\Omega)$ *the corresponding inverse estimate is*

$$\|v_h\|_{L^s(\Omega)} \leq Ch^{1+\min(0,d/s-d/2)}\|v_h\|_{H^1(\Omega)}.$$

In three dimensions $\|v_h\|_{L^\infty(\Omega)} \leq (C/h^{1/2})\|v_h\|_{H^1(\Omega)}$, *and we may apply the theorem with* $\theta = 1/2$ *to find*

$$\|\bar{u}_h\|_{L^{8/3}[0,T;L^\infty(\Omega)]} \leq C\|\bar{u}_h\|_{L^\infty[0,T;L^2(\Omega)]}^{1/4}\|A_h\bar{u}_h\|_{L^{p2}[0,T;L^2(\Omega)]}^{3/4}.$$

5. Proof of Equicontinuity. In this section we prove the crucial equicontinuity estimates stated in Lemmas 3.3 and 4.7. In the proofs we use the discrete Young inequality: if $1/p + 1/q = 1 + 1/r$ then then

$$\sum_{k,n} F_k G_n H_{k-n} \leq \|F\|_{\ell^q}\|G\|_{\ell^p}\|H\|_{\ell^{1/r'}} = \|F\|_{\ell^q}\|G\|_{\ell^p}\|H\|_{\ell^{1/p'+1/q'}}.$$

Precise limits on the sums will not be important since the sequences can always be extended by zero.

Proof. (of Lemma 3.3) By hypothesis

$$(5.1) \qquad \int_{t^{n-1}}^{t^n} (u_{ht}, v_h)_H + (u_+^{n-1} - u_-^{n-1}, v_+^{n-1})_H = \int_{t^{n-1}}^{t^n} \langle F_h(s), v_h \rangle\, ds.$$

with $\{F_h\}_{h>0}$ bounded in $L^q[0,T;U_h']$. Setting $v_h(s) = z_h \in U_h$ (independent of time) into the above and summing from m to n shows

$$(5.2) \qquad (u_-^n - u_-^m, z_h)_H = \int_{t^m}^{t^n} \langle F_h(s), z_h \rangle\, ds, \qquad 0 \leq m < n \leq N.$$

In order to estimate $u_h(t)$ at the intermediate times $t \in (t^{n-1}, t^n)$ we exploit certain properties of polynomials as in [7].

For $t \in (t^{n-1}, t^n)$ let $p = p(.;t) \in \mathcal{P}_\ell(t^{n-1}, t^n)$ satisfy

$$p(t^{n-1}) = 1, \qquad \int_{t^{n-1}}^{t^n} pq = \int_0^t q, \qquad q \in \mathcal{P}_{\ell-1}(t^{n-1}, t^n).$$

An explicit calculation shows

$$p(s;t) = 1 + \left(\frac{s - t^{n-1}}{\tau}\right)\sum_{k=0}^{\ell-1} c_k \hat{p}_k\left((s - t^{n-1})/\tau\right), \qquad \text{with} \qquad c_k = \int_{(t-t^{n-1})/\tau}^{1} \hat{p}_k(\xi)\, d\xi,$$

19

where $\tau = t^n - t^{n-1}$ and $\{\hat{p}_k\}_{k=0}^{\ell-1}$ is an orthonormal basis of $\mathcal{P}_{\ell-1}(0,1)$ with weighted inner product

$$(\hat{p}, \hat{q}) = \int_0^1 \xi \hat{p}(\xi) \hat{q}(\xi) \, d\xi.$$

For $t \in (t^{n-1}, t^n)$ it follows that

$$\|p\|_{L^\infty(t^{n-1}, t^n)} \le 1 + ((t^n - t)/\tau) \sum_{k=0}^{\ell-1} \|\hat{p}_k\|_{L^\infty(0,1)}^2 \le C_\ell,$$

so $p = p(.; t)$ is bounded independently of t.

Setting $v_h(s) = p(s; t) z_h$ into (5.1) shows

(5.3) $$\qquad (u_h(t) - u_-^{n-1}, z_h)_H = \int_{t^{n-1}}^{t^n} \langle F_h(\eta), z_h \rangle p(\eta; t) \, d\eta.$$

Here we used the fact that $u_{ht} \in \mathcal{P}_{\ell-1}(t^{n-1}, t^n; U)$ so the integral of (u_{ht}, v_h) is $(u_h(t) - u_+^{n-1}, z_h)$. If $s < t$, $s \in (t^{m-1}, t^m)$ and $t \in (t^{n-1}, t^n)$ then setting $t = s$ and $n = m$ into this identity shows

(5.4) $$\qquad (u_h(s) - u_-^{m-1}, z_h)_H = \int_{t^{m-1}}^{t^m} \langle F_h(\eta), z_h \rangle p(\eta; s) \, d\eta.$$

Combining equations (5.3), (5.2), and (5.4) shows

$$(u_h(t) - u_h(s), z_h) = \int_{t^{m-1}}^{t^n} \langle F_h(\eta), z_h \rangle p^{(t,s)}(\eta) \, d\eta,$$

for any $z_h \in U_h$, where

$$p^{(t,s)}(\eta) = \begin{cases} 1 - p(\eta; s) & \eta \in (t^{m-1}, t^m) \\ 1 & \eta \in (t^m, t^{n-1}) \\ p(\eta; t) & \eta \in (t^{n-1}, t^n) \end{cases}$$

By setting $z_h = u_h(t) - u_h(s)$ we find

(5.5) $$\quad \|u_h(t) - u_h(s)\|_H^2 \le C \int_{t^{m-1}}^{t^n} \|F_h(\eta)\|_{U_h'} \, d\eta \, \|u_h(t) - u_h(s)\|_U, \qquad 0 \le s \le t \le T,$$

where $\|p^{(s,t)}\|_{L^\infty} \le C$, a constant depending only upon ℓ.

Case 1, $\delta \ge \tau$: Let $m\tau \le \delta < (m+1)\tau$ and fix $t^* \in (t^{n-1}, t^n)$ so that $t - \delta \in (t^{n-m-1}, t^{n-m})$ when $t \in (t^*, t^n)$ and $t - \delta \in (t^{n-m-2}, t^{n-m-1})$ when $t \in (t^{n-1}, t^*)$ as illustrated in Figure 5.1. In the former case we set $s = t - \delta$ in equation (5.5) and integrate to obtain

$$\int_{t^*}^{t^n} \|u_h(t) - u_h(t - \delta)\|_H^2 \, dt \le C \int_{t^{n-m-1}}^{t^n} \|F_h(\eta)\|_{U_h'} \, d\eta \int_{t^*}^{t^n} \|u_h(t) - u_h(t - \delta)\| \, dt,$$

and similarly setting $s = t - \delta$ and $m \leftarrow m - 1$ in (5.5) gives

$$\int_{t^{n-1}}^{t^*} \|u_h(t) - u_h(t - \delta)\|_H^2 \, dt \le C \int_{t^{n-m-2}}^{t^n} \|F_h(\eta)\|_{U_h'} \, d\eta \int_{t^{n-1}}^{t^*} \|u_h(t) - u_h(t - \delta)\| \, dt.$$
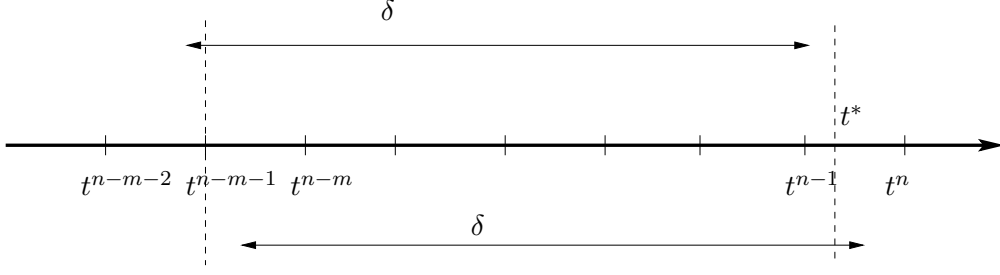
FIG. 5.1. *Two cases when $\delta \geq \tau$*

Combining the two pieces shows

$$(5.6) \quad \int_{t^{n-1}}^{t^n} \|u_h(t) - u_h(t-\delta)\|_H^2 \, dt \leq C \int_{t^{n-m-2}}^{t^n} \|F_h(\eta)\|_{U_h'} \, d\eta \int_{t^{n-1}}^{t^n} \|u_h(t) - u_h(t-\delta)\| \, dt.$$

Note: We only consider times $t > \delta$. If $t^{n-1} < \delta < t^n$ then $t^* = \delta$ and $n - m - 1 = 0$. In this situation an estimate for $u(t) - u(t-\delta)$ on (t^{n-1}, t^*) is neither needed nor defined. However, equation (5.6) is well defined if we adopt the convention that $u_h(t) - u_h(t-\delta) \equiv 0$ when $t \leq \delta$ and $F(\eta) \equiv 0$ when $\eta \leq 0$. This convention will be used in the remainder of the proof.

Write

$$\int_{t^{k-1}}^{t^k} \|F_h(s)\|_{U_h'} \, ds = F_k, \qquad \text{and} \qquad \int_{t^{n-1}}^{t^n} \|u_h(t) - u_h(t-\delta)\| \, dt = G_n.$$

Then the right hand side of equation (5.6) takes the from $\sum_{k=n-m-1}^{n} F_k G_n$ which we write as

$$\sum_{k=n-m-1}^{n} F_k G_n = \sum_{k=-m-1}^{N} F_k G_n H_{k-n},$$

where $H_\ell = 1$ if $\ell \in \{-m-1, \ldots 0\}$ and zero otherwise. Summing with respect to n and using Young's inequality shows

$$\sum_{n=0}^{N} \sum_{k=n-m-1}^{n} F_k G_n \leq \left(\sum_{k=1}^{N} F_k^q \right)^{1/q} \left(\sum_{n=1}^{N} G_n^p \right)^{1/p} (m+2)^{1/p'+1/q'},$$

(recall that $F_k \equiv 0$ for $k \leq 0$). Holder's inequality shows

$$\left(\sum_{k=0}^{N} F_k^q \right)^{1/q} = \left(\sum_{k=0}^{N} \left(\int_{t^{k-1}}^{t^k} \|F_h\|_{U_h'} \right)^q \right)^{1/q} \leq \left(\sum_{k=0}^{N} \int_{t^{k-1}}^{t^k} \|F_h\|_{U_h'}^q \tau^{q/q'} \right)^{1/q} = \|F_h\|_{L^q[0,T;U_h']} \tau^{1/q'},$$

and similarly $\left(\sum_n G_n^p \right)^{1/p} \leq \|u_h(.) - u_h(.-\delta)\|_{L^p[\delta,T;U]} \tau^{1/p'} \leq 2\|u_h\|_{L^p[0,T;U]} \tau^{1/p'}$. Then

$$\int_{\delta}^{T} \|u_h(t) - u_h(t-\delta)\|_H^2 \, dt \leq C\|F_h\|_{L^q[0,T;U_h']} \|u\|_{L^p[0,T;U]} ((m+2)\tau)^{1/p'+1/q'}$$

$$\leq C\|F_h\|_{L^q[0,T;U_h']} \|u\|_{L^p[0,T;U]} (3\delta)^{1/p'+1/q'}$$

where the conditions $m\tau < \delta$ and $m \geq 1$ were used for the final inequality.
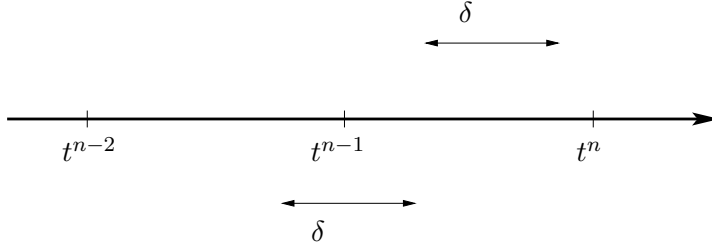
21

FIG. 5.2. *Two cases when $\delta < \tau$*

Case 2 $\delta < \tau$**:** Consider first the situation where $t^{n-1} < t - \delta < t \le t^n$ shown on the top of Figure 5.2. Equation (5.3) is now applicable for both times t and $t - \delta$, so

$$(5.7) \qquad (u_h(t) - u_h(t - \delta), z_h) = \int_{t^{n-1}}^{t^n} \langle F_h(\eta), z_h \rangle \left(p(\eta; t) - p(\eta; t - \delta) \right) ds.$$

With the notation introduced at the beginning of the proof, we have the explicit formula

$$p(\eta; t) - p(\eta; t - \delta)) = \left(\frac{\eta - t^{n-1}}{\tau} \right) \sum_{k=0}^{\ell-1} c_k \hat{p}_k \left((\eta - t^{n-1})/\tau \right), \quad \text{with} \quad c_k = \int_{(t-\delta-t^{n-1})/\tau}^{(t-t^{n-1})/\tau} \hat{p}_k(\xi) \, d\xi,$$

so

$$\|p(.; t) - p(.; t - \delta)\|_{L^\infty(t^{n-1}, t^n)} \le (\delta/\tau) \sum_{k=0}^{\ell-1} \|\hat{p}_k\|_{L^\infty(0,1)}^2 = C_\ell(\delta/\tau).$$

Setting $z_h = u_h(t) - u_h(t - \delta)$ in equation (5.7) and using this bound shows

$$\|u_h(t) - u_h(t - \delta)\|_H^2 \le (C\delta/\tau) \int_{t^{n-1}}^{t^n} \|F_h(\eta)\|_{U_h'} \, d\eta \, \|u_h(t) - u_h(t - \delta)\|_U.$$

Then

$$\int_{t^{n-1}+\delta}^{t^n} \|u_h(t) - u_h(t - \delta)\|_H^2 \, dt \le (C\delta/\tau) \int_{t^{n-1}}^{t^n} \|F_h(\eta)\|_{U_h'} \, d\eta \int_{t^{n-1}}^{t^n} \|u_h(t) - u_h(t - \delta)\|_U \, dt.$$

With the notation from Case 1 above we can write the right hand side as $\sum_k F_k G_n H_{k-n}$ where $H_i = \delta_{i0}$. Then summing and applying the Young's and Holder's inequalities as before shows

$$\sum_{n=1}^{N} \int_{t^{n-1}+\delta}^{t^n} \|u_h(t) - u_h(t - \delta)\|_H^2 \, dt \le (C\delta/\tau)\tau^{1/p'+1/q'} \|F_h\|_{L^q[0,T;U_h']} \|u\|_{L^p[0,T;U]}$$

$$(5.8) \qquad\qquad\qquad\qquad \le C\delta^{1/p'+1/q'} \|F_h\|_{L^q[0,T;U_h']} \|u\|_{L^p[0,T;U]},$$

where the conditions $\delta < \tau$ and $1/p + 1/q \ge 1$ (so $1 \ge 1/p' + 1/q'$) were used in the last step.

Consider next the situation where $t - \delta < t^{n-1} < t \le t^n$ shown in the bottom of Figure 5.2. Note that $t \in (\delta, T)$ so $n \ge 2$.

To establish the first statement of the theorem substitute $s = t - \delta$ and $m = n - 1$ into equation (5.5) and integrate over $t \in (t^{n-1}, t^{n-1} + \delta)$ to get

$$\int_{t^{n-1}}^{t^{n-1}+\delta} \|u_h(t) - u_h(t - \delta)\|_H^2 \, dt \le C \int_{t^{n-2}}^{t^n} \|F_h(\eta)\|_{U_h'} \, d\eta \int_{t^{n-1}}^{t^{n-1}+\delta} \|u_h(t) - u_h(t - \delta)\| \, dt.$$

22

In this situation

$$\left[\sum_{n=1}^{N}\left(\int_{t^{n-1}}^{t^{n-1}+\delta}\|u_h(t)-u_h(t-\delta)\|_U \, dt\right)^p\right]^{1/p} \le \|u_h(.)-u_h(.-\delta)\|_{L^p[0,T;U]}\delta^{1/p'} \le 2\|u_h\|_{L^p[0,T;U]}\delta^{1/p'},$$

and integrating F_h over the interval (t^{n-2},t^n) gives a factor of $\tau^{1/q'}$ instead of $\delta^{1/q'}$. It follows that application of the Young and Holder inequalities gives the estimate

$$(5.9) \qquad \sum_{n=1}^{N}\int_{t^{n-1}}^{t^{n-1}+\delta}\|u_h(t)-u_h(t-\delta)\|_H^2 \, dt \le C\|F_h\|_{L^q[0,T;U_h']}\|u\|_{L^p[0,T;U]}\tau^{1/q'}\delta^{1/p'}.$$

Combining this with the estimate in equation (5.8), and Case 1 above, establishes the first statement of the theorem.

To obtain the second statement of the theorem we need to sharpen the estimate in equation (5.9). Using the triangle inequality

$$\|u_h(t)-u_h(t-\delta)\|_H \le \|u_h(t)-u_h(t_+^{n-1})\|_H + \|[u^{n-1}]\|_H + \|u_h(t_-^{n-1})-u_h(t-\delta)\|_H,$$

so

$$\int_{t^{n-1}}^{t^{n-1}+\delta}\|u_h(t)-u_h(t-\delta)\|_H^2 \le 3\delta\|[u^{n-1}]\|_H^2 + 3\int_{t^{n-1}}^{t^{n-1}+\delta}\|u_h(t)-u_h(t_+^{n-1})\|_H^2 + \|u_h(t_-^{n-1})-u_h(t-\delta)\|_H^2.$$

Since $(t_+^{n-1},t) \subset (t^{n-1},t^n)$ and $(t-\delta,t_-^{n-1}) \subset (t^{n-2},t^{n-1})$, application of (5.3) shows

$$(u_h(t)-u_h(t_+^{n-1}),z_h)_H = \int_{t^{n-1}}^{t^n}\langle F_h(\eta),z_h\rangle\left(p(\eta;t)-p(\eta;t_+^{n-1})\right) \, ds$$

$$(u_h(t_-^{n-1})-u_h(t-\delta),z_h)_H = \int_{t^{n-2}}^{t^{n-1}}\langle F_h(\eta),z_h\rangle\left(p(\eta;t_-^{n-1})-p(\eta;t-\delta)\right) \, ds.$$

As in the calculations following equation (5.7), explicit formulae for the polynomials $p(.,.)$ show

$$\|p(.;t)-p(.;t_+^{n-1})\|_{L^\infty(t^{n-2},t^{n-1})} \le C(t-t^{n-1})/\tau \le C\delta/\tau,$$

and

$$\|p(.;t_-^{n-1})-p(.;t-\delta)\|_{L^\infty(t^{n-2},t^{n-1})} \le C(t^{n-1}-(t-\delta))/\tau \le C\delta/\tau.$$

Proceeding as before, we now obtain an additional factor of $(\delta/\tau)^{1/q'}$ on the right hand side of equation (5.9) which gives the second statement of the theorem. $\square$

The proof of equicontinuity for the projections of the CG scheme has a similar form. We begin with an elementary lemma which, on each interval, characterizes $\bar{p}'$ as a projection of p' where $\bar{p}$ is the projection defined in Definition 2.2

LEMMA 5.1. *Let $\ell \ge 1$ be an integer and $0 = t^0 < t^1 < \dots t^N = T$ be a partition of $[0,T]$. If $p \in \{C(0,T) \mid p|_{(t^{n-1},t^n)} \in \mathcal{P}_\ell(t^{n-1},t^n) \ 1 \le n \le N\}$, let $\bar{p}$ be the projection of p of Definition 2.2. Then there exists a constant $C = C(\ell) > 0$ depending only upon ℓ such that*

$$\|\bar{p}'\|_{L^r(t^{n-1},t^n)} \le C\|p'\|_{L^r(t^{n-1},t^n)} \qquad and \qquad |[\bar{p}^n]| \le C\|p'\|_{L^r(t^{n-1},t^{n+1})}\tau^{1/r'}$$

for any $1 \le r \le \infty$, where $[\bar{p}^n]$ denotes the jump in $\bar{p}$ at t^n. Similarly, if W is a (semi) inner product space and

$$w \in \{C[0,T;W] \mid w|_{(t^{n-1},t^n)} \in \mathcal{P}_\ell(t^{n-1},t^n;W) \ 1 \le n \le N\},$$

there exists $C = C(\ell) > 0$ depending only upon ℓ such that

$$\|\bar{w}'\|_{L^r[t^{n-1},t^n;W]} \le C\|w'\|_{L^r[t^{n-1},t^n;W]} \qquad and \qquad \|[\bar{w}^n]\|_W \le C\|w'\|_{L^r[t^{n-1},t^{n+1};W]}\tau^{1/r'}.$$

Proof. If $q \in \mathcal{P}_\ell(t^{n-1},t^n)$ then $q'(t) \in \mathcal{P}_{\ell-1}(t^{n-1},t^n)$ so using q as a test function in the definition of $\bar{p}$ and integrating by parts one obtains

$$\int_{t^{n-1}}^{t^n} -\bar{p}'q + \bar{p}q|_{t^{n-1}}^{t^n} = \int_{t^{n-1}}^{t^n} -p'q + pq|_{t^{n-1}}^{t^n} \qquad q \in \mathcal{P}_\ell(t^{n-1},t^n).$$

Selecting $q(t) = w(t)\hat{q}$ with $\hat{q} \in \mathcal{P}_{\ell-2}(t^{n-1},t^n)$ and $w(t) = (t-t^{n-1})(t^n-t)/\tau^2$ where $\tau = t^n - t^{n-1}$ shows

$$\bar{p}' \in \mathcal{P}_{\ell-2}(t^{n-1},t^n) \qquad \int_{t^{n-1}}^{t^n} \bar{p}'\hat{q}w = \int_{t^{n-1}}^{t^n} p'\hat{q}w, \qquad \hat{q} \in \mathcal{P}_{\ell-2}(t^{n-1},t^n).$$

Change of variables variables $t \mapsto (t-t^{n-1})/\tau$ and the equivalence of all norms on $\mathcal{P}_{\ell-2}(0,1)$ shows there exists a constant $C_\ell > 0$ such that $\|\bar{p}'\|_{L^r(t^{n-1},t^n)} \le C(\ell)\|p'\|_{L^r(t^{n-1},t^n)}$.

Next, let $q(t) = (t-t^{n-1})/(t^n-t^{n-1})$ on (t^{n-1},t^n) and $q(t) = (t^{n+1}-t)/(t^{n+1}-t^n)$ for $t \in (t^n, t^{n+1})$. Then $|q| \le 1$ and

$$\bar{p}^n_- - \bar{p}^n_+ = \int_{t^{n-1}}^{t^{n+1}} (p' - \bar{p}')q, \qquad \text{so} \qquad |[\bar{p}^n]| \le C\|p'\|_{L^r(t^{n-1},t^{n+1})}\tau^{1/r'}.$$

Next, if $w|_{(t^{n-1},t^n)} = \sum_{i=0}^\ell p_i(t)w_i \in \mathcal{P}_\ell[t^{n-1},t^n;W]$, we may assume without loss of generality that $\{w_i\}_{i=0}^\ell$ are orthonormal. If $r \le 2$ then

$$\|\bar{w}'(t)\|_W = \left(\sum_{i=0}^\ell |\bar{p}_i'(t)|^2\right)^{1/2} \le \left(\sum_{i=0}^\ell |\bar{p}_i'(t)|^r\right)^{1/r},$$

since $\|a\|_{\ell^s} \le \|a\|_{\ell^r}$ for $r \le s$. Then

$$\int_{t^{n-1}}^{t^n} \|\bar{w}'(t)\|_W^r \, dt \le \int_{t^{n-1}}^{t^n} \sum_{i=0}^\ell |\bar{p}_i'(t)|^r \, dt$$

$$\le C^r \int_{t^{n-1}}^{t^n} \sum_{i=0}^\ell |p_i'(t)|^r \, dt$$

$$\le C^r \int_{t^{n-1}}^{t^n} \left(\sum_{i=0}^\ell |p'(t)|^2\right)^{r/2} (\ell+1)^{1-r/2} \, dt$$

$$\le C^r \int_{t^{n-1}}^{t^n} \|w'(t)\|_W^r \, dt$$

24

Note that $(\ell + 1)^{1/r-1/2} \leq (\ell + 1)^{1/2}$ when $1 \leq r \leq 2$, so $\|\bar{w}'\|_{L^r[0,T;W]} \leq C\|w'\|_{L^r[0,T;W]}$ with constant independent of r. If $r \geq 2$ then Holder's inequality is used in the first step and the nesting of the ℓ^s norms in the second step.

The estimate for the jump then follows as for the polynomials. $\square$

Proof. (of Lemma 4.7) Let $0 \leq s < t \leq T$, and assume that $s \in (t^{n-m-1}, t^{n-m})$ and $t \in (t^{n-1}, t^n)$. Then, with a mild abuse of notation for the sums of integrals of $\bar{w}'$ on each interval we write

$$\bar{w}(t) - \bar{w}(s) = \int_s^t \bar{w}'(\xi)\, d\xi + \sum_{i=n-m}^{n-1} [\bar{w}^i].$$

Using Lemma 5.1 and Holder's inequality we obtain

$$\|\bar{w}(t) - \bar{w}(s)\|_W \leq \int_s^t \|\bar{w}'(\xi)\|_W\, d\xi + C \sum_{i=n-m}^{n-1} \int_{t^{i-1}}^{t^{i+1}} \|w'(\xi)\|_W\, d\xi \leq C \int_{t^{m-n-1}}^{t^n} \|w'(\xi)\|_W\, d\xi.$$

It follows that

(5.10) $$\|\bar{w}(t) - \bar{w}(s)\|_W^q = C \int_{t^{n-m-1}}^{t^n} \|w'(\xi)\|_W^q\, d\xi\, ((m+1)\tau)^{q/q'}$$

Case 1, $\delta \geq \tau$: Let $m\tau \leq \delta < (m+1)\tau$, fix $t^* \in (t^{n-1}, t^n)$ so that $t - \delta \in (t^{n-m-1}, t^{n-m})$ when $t \in (t^*, t^n)$ and $t - \delta \in (t^{n-m-2}, t^{n-m-1})$ when $t \in (t^{n-1}, t^*)$ as illustrated in Figure 5.1. In the first case equation (5.10) shows

$$\int_{t^*}^{t^n} \|\bar{w}_h(t) - \bar{w}_h(t - \delta)\|_W^q\, dt \leq C(t^n - t^*) \int_{t^{n-m-1}}^{t^n} \|w'(\xi)\|_W^q\, d\xi\, \delta^{q/q'}.$$

Combining this with a similar calculation for $t \in (t^{n-1}, t^*)$ shows

$$\int_{t^{n-1}}^{t^n} \|\bar{w}_h(t) - \bar{w}_h(t - \delta)\|_W^q\, dt \leq C\tau \int_{t^{n-m-2}}^{t^n} \|w'(\xi)\|_W^q\, d\xi\, \delta^{q/q'}.$$

Note: We only consider times $t > \delta$. If $t^{n-1} < \delta < t^n$ then $t^* = \delta$ and $n - m - 1 = 0$. In this situation an estimate for $\bar{w}(t) - \bar{w}(t - \delta)$ on (t^{n-1}, t^*) is neither needed nor defined. The estimate above is meaningful if we adopt the convention that $\bar{w}_h(t) - \bar{w}_h(t - \delta) \equiv 0$ when $t \leq \delta$ and $w'(\xi) \equiv 0$ when $\xi \leq 0$. This convention will be used in the remainder of the proof.

Summing the above we obtain

$$\int_\delta^T \|\bar{w}_h(t) - \bar{w}_h(t - \delta)\|_W^q\, dt \leq C\tau \sum_{n=1}^N \sum_{i=n-m-1}^n \int_{t^{i-1}}^{t^i} \|w'(\xi)\|_W^q\, d\xi\, dt\, \delta^{q/q'}$$

$$\leq C \int_0^T \|w'(\xi)\|_W^q\, d\xi (m+2)\tau \delta^{q/q'}.$$

Since $\delta \geq \tau$ we have $(m+2)\tau \leq 3\delta$ so

$$\left(\int_\delta^T \|\bar{w}_h(t) - \bar{w}_h(t - \delta)\|_W^q\, dt \right)^{1/q} \leq C\|w'\|_{L^q[0,T;W]}\delta.$$

25

Case 2 $\delta < \tau$**:** Consider first the situation where $t - \delta < t^{n-1} < t \le t^n$ shown in the bottom of Figure 5.2. Substituting $s = t - \delta$ and $m = 1$ into equation (5.10) shows

$$\|\bar{w}(t) - \bar{w}(t - \delta)\|_W^q \le C \int_{t^{n-2}}^{t^n} \|w'(\xi)\|_W^q \, d\xi \, \tau^{q/q'}$$

so

$$\int_{t^n}^{t^n + \delta} \|\bar{w}(t) - \bar{w}(t - \delta)\|^q \le C \int_{t^{n-2}}^{t^n} \|w'(\xi)\|_W^q \, d\xi \, \tau^{q/q'} \delta.$$

Consider next the situation where $t^{n-1} < t - \delta < t \le t^n$ shown on the top of Figure 5.2. Then

$$\|\bar{w}(t) - \bar{w}(t - \delta)\|_W^q \le \int_{t-\delta}^{t} \|\bar{w}'(\xi)\|_W^q \, d\xi \, \delta^{q/q'}.$$

Integrating over $t \in (t^{n-1} + \delta, t^n)$ and changing the order of integration on the right gives

$$\int_{t^{n-1}+\delta}^{t^n} \|\bar{w}(t) - \bar{w}(t - \delta)\|_W^q \, dt \le C \int_{t^{n-1}}^{t^n} \int_{\xi \vee (t^{n-1}+\delta)}^{(\xi+\delta)\wedge t^n} \|\bar{w}'(\xi)\|_W^q \, dt \, d\xi \, \delta^{q/q'}$$

$$\le C \int_{t^{n-1}}^{t^n} \|\bar{w}'(\xi)\|_W^q \, d\xi \, \delta^{1+q/q'}$$

$$\le C \int_{t^{n-1}}^{t^n} \|w'(\xi)\|_W^q \, d\xi \, \delta^q.$$

Combining the above and summing shows that, in the case $\delta < \tau$,

$$\left(\int_{\delta}^{T} \|\bar{w}_h(t) - \bar{w}_h(t - \delta)\|_W^q \, dt \right)^{1/q} \le C \|w'\|_{L^q[0,T;W]} \tau^{1/q'} \delta^{1/q}.$$

□

6. Application to the Model Problem. We recall certain asymptotic expansions that motivate the phase field approximation of the two fluids problem introduced in Section 1.1. If $\phi \simeq \chi_{\Omega_2} - \chi_{\Omega_1}$ is the phase function introduced in Section 1.1 and ψ is a smooth test function, then formal asymptotic expansions show

(6.1)
$$\int_{\Omega} \left(\epsilon \Delta \phi - (1/\epsilon) F'(\phi) \right) \psi \simeq \epsilon \int_{S} 2H\psi,$$

where $S = \bar{\Omega}_2 \cap \bar{\Omega}_1$ is the surface separating the two fluids and H is the mean curvature of S. Similarly, if v is a smooth vector valued test function then

(6.2)
$$\int_{\Omega} \left(\epsilon \Delta \phi - (1/\epsilon) F'(\phi) \right) \nabla \phi . v \simeq \int_{S} (8/3) H n . v,$$

where n is the normal to S.

To derive a skew symmetrized weak statement of the momentum equation we first formulate incompressibility condition as $\int_{\Omega} -u.\nabla \xi = 0$, for all test functions, ξ, vanishing on the boundary.

26

Setting $\xi = (\rho/2)u.v$ and adding this to the weak statement of the momentum equation given in equation (1.6) gives

$$\int_\Omega \left\{ \rho u_t.v + (\rho/2)\left((u.\nabla)u.v - (u.\nabla)v.u\right) \right.$$
$$\left. -p\,div(v) + \mu D(u) : D(v) \right\} + \int_{S(t)} \gamma H v.n = \int_\Omega \rho f.v.$$

A regularization of this equation is obtained upon using equation (6.2) to approximate the surface integral and absorbing the constant of 8/3 into the definition of γ,

(6.3)
$$\int_\Omega \left\{ \rho u_t.v + (\rho/2)\left((u.\nabla)u.v - (u.\nabla)v.u\right) \right.$$
$$\left. -p\,div(v) + \mu D(u) : D(v) + \gamma\left(\epsilon\Delta\phi - (1/\epsilon)F'(\phi)\right)\nabla\phi.v \right\} = \int_\Omega \rho f.v.$$

This approximation of the momentum equation contains the Laplacian of the phase function so a conforming finite element approximation would require Hermite elements for ϕ [20].

Since the surface separating the fluid is transported by the flow, the phase function should satisfy $\phi_t + u.\nabla\phi = 0$. However, solutions of this equation will not have sufficient regularity for the phase field approximation to be meaningful, so we admit approximations of the form $\phi_t + u.\nabla\phi = O(\epsilon)$. The perturbation we utilize is the one given in equation (6.1),

(6.4)
$$\phi_t + u.\nabla\phi = \lambda\left(\epsilon\Delta\phi - (1/\epsilon)F'(\phi)\right),$$

where $\lambda > 0$ is a "relaxation" constant. Equation (1.8) is the natural weak statement of this equation, and equation (1.7) is obtained upon using equation (6.4) to eliminate $\Delta\phi$ from the weak statement of the momentum equation (6.3). This regularization of the two fluids problem was introduced in [21].

6.1. Finite Element Approximation. We consider discrete solutions of the model problem which use finite elements in space and DG time stepping for the momentum equation (1.7) and CG time stepping for the equation for the phase (1.8). We assume that $\Omega \subset \mathbb{R}^d$, $d = 2$ or 3, is a bounded Lipschitz domain, and for each $h > 0$ that $\{t_h^n\}_{n=0}^{N_h}$ is a uniform partition of $[0,T]$ and $\mathcal{T}_h$ a quasi-uniform triangulation of Ω with maximal element diameter h and $N_h \to \infty$. For definiteness, assume $(U_h, P_h) \subset H_0^1(\Omega)^d \times L^2(\Omega)/\mathbb{R}$ is a classical Taylor Hood subspace constructed over $\mathcal{T}_h$, and let $D_h \subset \{\phi \in H^1(\Omega) \mid \phi|_{\partial\Omega} = 1\}$ be a classical Lagrange finite element subspace constructed over $\mathcal{T}_h$. Then set

$$\mathbb{U}_h = \{u_h \in L^2[0,T;U_h] \mid u_h|_{(t_h^{n-1},t_h^n)} \in \mathcal{P}_\ell[t_h^{n-1},t_h^n;U_h]\},$$

$$\mathbb{P}_h = \{p_h \in L^2[0,T;P_h] \mid p_h|_{(t_h^{n-1},t_h^n)} \in \mathcal{P}_\ell[t_h^{n-1},t_h^n;P_h]\},$$

$$\mathbb{D}_h = \{\phi_h \in C[0,T;D_h] \mid (\phi_h - 1)|_{(t_h^{n-1},t_h^n)} \in \mathcal{P}_\ell[t_h^{n-1},t_h^n;D_h]\},$$

$$\mathbb{E}_h = \{\psi_h \in L^2[0,T;D_h] \mid \psi_h|_{(t_h^{n-1},t_h^n)} \in \mathcal{P}_{\ell-1}[t_h^{n-1},t_h^n;D_h]\}.$$

On each time interval the discrete solutions satisfy $(u_h, p_h, \phi_h) \in \mathbb{U}_h \times \mathbb{P}_h \times \mathbb{D}_h$ and

$$\int_{t_h^{n-1}}^{t_h^n} \Big\{ (\rho\, u_{ht}, v_h) + (\rho/2)\left((u_h.\nabla)u_h, v_h\right) - (\rho/2)\left(u_h, (u_h.\nabla)v_h\right) - (p_h, div(v_h)) $$

$$ +(\mu_h D(u_h), D(v_h)) + (\gamma/\lambda)(\dot\phi_h, \nabla\bar\phi_h.v_h) \Big\} + \left(\rho(u_+^{n-1} - u_-^{n-1}), v_+^{n-1}\right) = \int_{t_h^{n-1}}^{t_h^n} (\rho f, v_h), $$

$$ \int_{t_h^{n-1}}^{t_h^n} (div(u_h), q_h) = 0, $$

$$ \int_{t_h^{n-1}}^{t_h^n} (\dot\phi_h, \psi_h) + (\lambda\epsilon/2)(\nabla\phi_h, \nabla\psi_h) + (\lambda/\epsilon)(F'(\phi_h), \psi_h) = 0. $$

for all $(v_h, q_h, \psi_h) \in \mathbb{U}_h \times \mathbb{P}_h \times \mathbb{E}_h$. In the above $\mu_h = \mu(\phi_h)$, $\dot\phi_h = \phi_{ht} + u_h.\nabla\bar\phi_h$, and $(.,.)$ denotes the natural $L^2(\Omega)$ pairing. For the analysis below it is important that the projection of the phase variable, $\bar\phi_h$ be used in the terms coupling the phase and momentum equations.

6.2. Estimates. Selecting $v_h = u_h$, $q_h = p_h$ and $\psi_h = (\gamma/\lambda)\phi_{ht}$ gives the discrete energy estimate

$$ \int_\Omega (\rho/2)\|u_-^n\|_{L^2(\Omega)}^2 + (\gamma\epsilon/2)\|\nabla\phi^n\|_{L^2(\Omega)}^2 + (\gamma/\epsilon)\|F(\phi^n)\|_{L^1(\Omega)} \Big|_{t^{n-1}}^{t^n} + (\rho/2)\|[u^{n-1}]\|_{L^2(\Omega)}^2 $$

$$ + \int_{t_h^{n-1}}^{t_h^n} \|\sqrt{\mu_h}D(u_h)\|_{L^2(\Omega)}^2 + (\gamma/\lambda)\|\dot\phi_h\|_{L^2(\Omega)}^2 = \int_{t_h^{n-1}}^{t_h^n} (\rho f, u_h). $$

Bounding the right hand side as $\|f\|_{H^{-1}(\Omega)}\|u_h\|_{H^1(\Omega)}$ we find

$$ \max_{1 \le n \le N} E(u_-^n, \phi^n) \le E(u_-^0, \phi^0) + C\int_0^T \|f\|_{H^{-1}(\Omega)}^2, $$

and

(6.5) $\quad \|\sqrt{\mu_h}\nabla u_h\|_{L^2[0,T;L^2(\Omega)]}^2 + (\gamma/\lambda)\|\dot\phi_h\|_{L^2[0,T;L^2(\Omega)]}^2 \le E(u_-^0, \phi^0) + C\int_0^T \|f\|_{H^{-1}(\Omega)}^2,$

where

$$ E(u, \phi) = (\rho/2)\|u\|_{L^2(\Omega)}^2 + (\gamma\epsilon/2)\|\nabla\phi\|_{L^2(\Omega)}^2 + (\gamma/\epsilon)\|F(\phi)\|_{L^1(\Omega)}, $$

and C depends upon the Korn and Poincare constants for Ω. In the above we have have assumed that $\mu_h = \mu(\phi_h)$ is bounded above and below by a positive constants. Below C will denote a constant which may depend upon the constants λ, ϵ, γ etc. and the initial data and right hand side f, but not upon the mesh parameter h.

Beginning with the energy estimate, it is possible to bootstrap higher order estimates and sufficient compactness to establish convergence.

1. Using the equation for ϕ we compute

$$ \int_{t_h^{n-1}}^{t_h^n} (\epsilon/2)(\nabla\bar\phi_h, \nabla\psi_h) + (4/\epsilon)(\bar\phi_h, \psi_h) = \int_{t_h^{n-1}}^{t_h^n} (\epsilon/2)(\nabla\phi_h, \nabla\psi_h) + (4/\epsilon)(\phi_h, \psi_h) $$

$$ = \int_{t_h^{n-1}}^{t_h^n} \Big((-1/\lambda)\dot\phi_h + (1/\epsilon)(4\phi_h - F'(\phi_h)), \psi_h \Big). $$

28

Note that $4\phi - F'(\phi) = 2\phi(3 - \phi^2)$ if $|\phi| \leq \sqrt{3}$ and zero otherwise, so $|4\phi - F'(\phi)| \leq 4$. Thus if $A_h\phi_h \in D_h$ is characterized by

$$(A_h\bar{\phi}_h, \psi_h) = (\epsilon/2)(\nabla\bar{\phi}_h, \nabla\psi_h) + (4/\epsilon)(\bar{\phi}_h, \psi_h) \qquad \psi_h \in D_h,$$

then $A_h\bar{\phi}_h \in L^2[0, T; L^2(\Omega)]$; specifically,

$$\|A_h\bar{\phi}_h\|_{L^2[0,T;L^2(\Omega)]} \leq (1/\lambda)\|\dot{\phi}_h\|_{L^2[0,T;L^2(\Omega)]} + (4/\epsilon)\sqrt{|\Omega|T} \leq C(f, u_{h-}^0, \phi^0, T),$$

and since $(A_h\bar{\phi}_h, \bar{\phi}_h) = \epsilon\|\nabla\bar{\phi}_h\|_{L^2(\Omega)}^2 + (1/\epsilon)\|\bar{\phi}\|_{L^2(\Omega)}^2$ it follows that

$$\|\nabla\bar{\phi}_h\|_{L^2[0,T;L^2(\Omega)]} + \|\bar{\phi}\|_{L^2[0,T;L^2(\Omega)]} \leq C(f, u_h^0, \phi^0, T).$$

In order to establish compactness of the solutions it is necessary to bound $\phi_{ht} = \dot{\phi}_h - u_h.\nabla\bar{\phi}_h$ in $L^2(\Omega)$. Higher order spatial estimates on u_h and $\bar{\phi}_h$ are needed for this.

2. Consider next the momentum equation. To verify the hypotheses of Theorem 3.1 the pressure is eliminated from the momentum equation. Define the discretely divergence free subspace Z_h by

$$Z_h = \{u_h \in U_h \mid (div(u_h), q_h) = 0 \text{ for all } q_h \in P_h\}.$$

Then on each interval, (t_h^{n-1}, t_h^n), the DG approximation of the momentum equation has $u_h \in \mathcal{P}_\ell(t^{n-1}, t^n; Z_h)$ and

(6.6)
$$\int_{t_h^{n-1}}^{t_h^n} (\rho\, u_{ht}.v_h) + \left(\rho(u_+^{n-1} - u_-^{n-1}), v_+^{n-1}\right)$$

$$= \int_{t_h^{n-1}}^{t_h^n} \left\{ (\rho f, v_h) - (\rho/2)\left(((u_h.\nabla)u_h, v_h) - (u_h, (u_h.\nabla)v_h)\right)\right.$$

$$\left. -(\mu_h D(u_h), D(v_h)) - (\gamma/\lambda)(\dot{\phi}_h, \nabla\bar{\phi}_h.v_h)\right\}$$

for each $v_h \in \mathcal{P}_\ell(t_h^{n-1}, t_h^n; Z_h)$. Selecting the subspace in Theorem 3.1 to be Z_h and recalling that $u_h \in L^2[0, T; H_0^1(\Omega)]$, the remaining hypothesis is to verify that the right hand side of (6.6) is in $L^1[0, T; Z_h']$ which is follows from the energy estimate (6.5) and the bound upon $\|\bar{\phi}\|_{L^2[0,T;H^1(\Omega)]}$. Theorem 3.1 (with $p = 2$) then establishes u_h is compact in $L^r[0, T; L^2(\Omega)]$ for $1 \leq r < 4$.

3. We next estimate for ϕ_{ht} using the identity $\phi_{ht} = \dot{\phi}_h - u_h.\nabla\bar{\phi}_h$. As in Example 4.10 we have $\|\bar{\phi}_h\|_{L^2[0,T;W^{1,6}(\Omega)]}$ bounded, so

$$\|u_h.\nabla\bar{\phi}_h\|_{L^2(\Omega)} \leq \|u_h\|_{L^3(\Omega)}\|\bar{\phi}_h\|_{W^{1,6}(\Omega)}$$

$$\leq \|u_h\|_{L^2(\Omega)}^{1/2}\|u_h\|_{L^6(\Omega)}^{1/2}\|\bar{\phi}_h\|_{W^{1,6}(\Omega)}$$

$$\leq C\|u_h\|_{L^2(\Omega)}^{1/2}\|u_h\|_{H^1(\Omega)}^{1/2}\|\bar{\phi}_h\|_{W^{1,6}(\Omega)}.$$

If $1 \leq q < 8/7$, then $1/q > 1/8 + 1/4 + 1/2$ and Holder's inequality shows $\|u_h.\nabla\bar{\phi}_h\|_{L^q[0,T;L^2(\Omega)]}$, and hence $\|\phi_{ht}\|_{L^q[0,T;L^2(\Omega)]}$, is bounded. It follows that ϕ_h, and hence $\bar{\phi}_h$, is bounded in $L^\infty[0, T; L^2(\Omega)]$.

4. Direct application of Theorem 4.5 and Lemma 4.9 then show $\{\bar{\phi}_h\}_{h>0}$ is compact in $L^q[0,T;H^1(\Omega)]$ for any $1 \le q < 4$ and $\{\bar{\phi}_h\}_{h>0}$ is bounded in $L^{8/3}[0,T;W^{1,3}(\Omega)]$.

5. We now have sharper estimates for the nonlinear terms in the momentum equation.

$$|(\dot{\phi}_h, v_h.\nabla\bar{\phi}_h)| \le \|\dot{\phi}_h\|_{L^2(\Omega)}\|\bar{\phi}_h\|_{W^{1,3}(\Omega)}\|v_h\|_{L^6(\Omega)} \le C\|\dot{\phi}_h\|_{L^2(\Omega)}\|\bar{\phi}_h\|_{W^{1,3}(\Omega)}\|v_h\|_{H^1(\Omega)}.$$

If $q = 8/7$ then $1/q = 1/2 + 1/(8/3)$, so

$$v_h \mapsto (\dot{\phi}_h, v_h.\nabla\bar{\phi}_h) \in L^{8/7}[0,T;H^{-1}(\Omega)].$$

The convection terms are bounded as

$$|(u_h.\nabla u_h, v_h)| \le \|u_h\|_{L^3(\Omega)}\|u_h\|_{H^1(\Omega)}\|v_h\|_{L^6(\Omega)} \le \|u_h\|_{L^2(\Omega)}^{1/2}\|u_h\|_{H^1(\Omega)}^{3/2}\|v_h\|_{H^1(\Omega)},$$

and

$$|(u_h.\nabla v_h, u_h)| \le \|u_h\|_{L^4(\Omega)}^2\|v_h\|_{H^1(\Omega)} \le \|u_h\|_{L^2(\Omega)}^{1/2}\|u_h\|_{H^1(\Omega)}^{3/2}\|v_h\|_{H^1(\Omega)}.$$

If $q < 8/7$ then $1/q > 1/8 + 1/(4/3)$ so

$$v_h \mapsto ((u_h.\nabla)u_h, v_h) - (u_h, (u_h.\nabla)v_h) \in L^q[0,T;H^{-1}(\Omega)].$$

We can then apply the second statement of Theorem 3.1 to deduce that $u_h \in L^r[0,T;L^2(\Omega)]$ for any $1 \le r < 2/(1/2 + 7/8 - 1) = 16/3$. Using this estimate in to improve the above bound on the convection term shows that we may set $q = 8/7$ to obtain $u_h \in L^{16/3}[0,T;L^2(\Omega)]$.

6.3. Passage to the Limit. Upon passing to a subsequence we establish convergence to a weak solution of the phase field approximation of the two fluids problem. Granted the bounds above, we may assume that

$$\begin{aligned}
\phi_h &\overset{*}{\rightharpoonup} \phi &&\text{in } L^\infty[0,T;L^2(\Omega)] \\
\phi_{ht} &\rightharpoonup \phi_t &&\text{in } L^{8/7}[0,T;L^2(\Omega)] \\
\dot{\phi}_h &\rightharpoonup \eta &&\text{in } L^2[0,T;L^2(\Omega)] \\
\bar{\phi}_h &\to \bar{\phi} &&\text{in } L^2[0,T;H^1(\Omega)] \cap L^s[0,T;L^2(\Omega)] &&1 \le s < \infty \\
u_h &\rightharpoonup u &&\text{in } L^2[0,T;H^1(\Omega)] \\
u_h &\to u &&\text{in } L^2[0,T;L^2(\Omega)] \cap L^s[0,T;L^2(\Omega)] &&1 \le s < 16/3
\end{aligned}$$

To show that $\bar{\phi} = \phi$ let $\psi \in \mathcal{D}((0,T) \times \Omega)$ and let $\bar{\psi}_h \in C[0,T;D_h] \cap \mathbb{E}_h$ converge to ψ in $W^{1,\infty}[0,T;W^{1,\infty}(\Omega)]$. Then using the definition of the projection $\phi_h \mapsto \bar{\phi}_h$ we find

$$\int_0^T (\phi, \psi) = \lim_{h \to 0} \int_0^T (\phi_h, \bar{\psi}_h) = \lim_{h \to 0} \int_0^T (\bar{\phi}_h, \bar{\psi}_h) = \int_0^T (\bar{\phi}, \psi).$$

Next, to show that $\eta = \dot{\phi} \equiv \phi_t + u.\nabla\phi$, recall that $\dot{\phi}_h = \phi_{ht} + u_h.\nabla\bar{\phi}_h$ and compute

$$\begin{aligned}
\int_0^T (\phi_t + u.\nabla\phi, \psi) &= \int_0^T (\phi_t, \psi) - (\phi, div(\psi u)) \\
&= \lim_{h \to 0} \int_0^T (\phi_{ht}, \psi) - (\bar{\phi}_h, div(\psi u_h)) \\
&= \lim_{h \to 0} \int_0^T (\dot{\phi}_{ht}, \psi) \\
&= \int_0^T (\eta, \psi),
\end{aligned}$$

30

so $\eta = \dot{\phi}$.

If $v_h \in C[0,T;Z_h] \cap \mathbb{U}_h$ and $v_h(T) = 0$ the discrete momentum equation shows

$$\int_0^T \Big\{ -(\rho\,u_h, v_{ht}) + (\rho/2)\left(((u_h.\nabla)u_h, v_h) - (u_h, (u_h.\nabla)v_h)\right)$$

$$+ (\mu_h D(u_h), D(v_h)) + \gamma(\dot{\phi}_h, \nabla\bar{\phi}_h.v_h) \Big\} = (\rho u_-^0, v_h(0)) + \int_0^T (\rho f, v_h),$$

Since the pair (U_h, P_h) satisfy the inf-sup condition it follows that for any $v \in \mathcal{D}([0,T) \times \Omega)$ there exists a sequence $v_h \in C[0,T;Z_h] \cap \mathbb{U}_h$ with $v_h(T) = 0$ such that $v_h \to v$ in $W^{1,\infty}[0,T;W^{1,\infty}(\Omega)]$. Upon passing to a subsequence we may assume

$$u_h \to u \text{ in } L^2[0,T;L^2(\Omega)]$$
$$u_h \rightharpoonup u \text{ in } L^2[0,T;H^1(\Omega)]$$
$$\dot{\phi}_h \rightharpoonup \dot{\phi} \text{ in } L^2[0,T;L^2(\Omega)]$$
$$\bar{\phi}_h \to \phi \text{ in } L^2[0,T;H^1(\Omega)].$$

The viscosity, $\mu_h = \mu(\phi_h)$, converges in $L^2[0,T;L^2(\Omega)]$ since $\mu : \mathbb{R} \to \mathbb{R}$ is continuous. It follows that we may pass to the limit in the momentum equation term-by-term.

Similarly, if $\psi \in \mathcal{D}((0,T) \times \Omega)$ there exists $\bar{\psi}_h \in C[0,T;D_h] \cap \mathbb{E}_h$ converging to ψ in $W^{1,\infty}[0,T;W^{1,\infty}(\Omega)]$. Since $F' : \mathbb{R} \to \mathbb{R}$ is continuous and has linear growth, and since Theorem 4.1 shows $\{\phi_h\}_{h>0}$ is compact in $C[0,T;L^2(\Omega)]$, it follows that $F'(\phi_h) \to F'(\phi)$ in $C[0,T;L^2(\Omega)]$. We may then pass to the limit term-by-term in the equation

$$\int_0^T (\dot{\phi}_h, \psi_h) + (\epsilon/2)(\nabla\phi_h, \nabla\psi_h) + (1/\epsilon)(F'(\phi_h), \psi_h) = 0.$$

to conclude that (u, ϕ) are a weak solution of the phase field approximation of the two fluids problem.

REFERENCES

[1] G. A. BAKER, V. A. DOUGALIS, AND O. A. KARAKASHIAN, *On a higher order accurate fully discrete Galerkin approximation to the Navier-Stokes equations*, Math. Comp., 39 (1982), pp. 339–375.

[2] J. W. BARRETT, X. FENG, AND A. PROHL, *Convergence of a fully discrete finite element method for a degenerate parabolic system modelling nematic liquid crystals with variable degree of orientation*, M2AN Math. Model. Numer. Anal., 40 (2006), pp. 175–199.

[3] J. H. BRAMBLE, J. E. PASCIAK, AND O. STEINBACH, *On the stability of the L^2 projection in $H^1(\Omega)$*, Math. Comp., 71 (2002), pp. 147–156 (electronic).

[4] H. BREZIS, *Convergence in $\mathcal{D}'$ and in L^1 under strict convexity*, in Boundary value problems for partial differential equations and applications, vol. 29 of RMA Res. Notes Appl. Math., Masson, Paris, 1993, pp. 43–52.

[5] C. CARSTENSEN, *Merging the Bramble-Pasciak-Steinbach and the Crouzeix-Thomée criterion for H^1-stability of the L^2-projection onto finite element spaces*, Math. Comp., 71 (2002), pp. 157–163 (electronic).

[6] K. CHRYSAFINOS AND N. J. WALKINGTON, *Error estimates for discontinuous Galerkin approximations of implicit parabolic equations*, SIAM J. Numer. Anal., 43 (2006), pp. 2478–2499 (electronic).

[7] ———, *Error estimates for the discontinuous Galerkin methods for parabolic equations*, SIAM J. Numer. Anal., 44 (2006), pp. 349–366 (electronic).

[8] K. Chrysafinos and N. J. Walkington, *Discontinuous Galerkin approximations for the Stokes and Navier Stokes equations*, Math. Comp., submitted (2008).

[9] M. G. Crandall, *An introduction to evolution equations governed by accretive operators*, in Dynamical Systems, Volume 1, An International Symposium, Academic Press, 1976.

[10] M. G. Crandall and T. M. Liggett, *Generation of semigroups of nonlinear transformations on general Banach spaces*, Ameraican Journal of Mathematics, 93 (1971), pp. 265–293.

[11] M. Crouzeix and V. Thomee, *The stability in l^p and $w^{1,p}$ of the l^2 projection onto finite element function spaces*, Math. Comp., 48 (1987), pp. 521–532.

[12] L. Diening, C. Ebmeyer, and M. Ruzicka, *Optimal convergence for the implicit space-time discretization of parabolic systems with p-structure*, SIAM J. Numer. Anal., 45 (2007), pp. 457–472 (electronic).

[13] X. Feng, *Fully discrete finite element approximations of the Navier-Stokes-Cahn-Hilliard diffuse interface model for two-phase fluid flows*, SIAM J. Numer. Anal., 44 (2006), pp. 1049–1072 (electronic).

[14] M. Frazier, B. Jawerth, and G. Weiss, *Littlewood-Paley theory and the study of function spaces*, vol. 79 of CBMS Regional Conference Series in Mathematics, Published for the Conference Board of the Mathematical Sciences, Washington, DC, 1991.

[15] J. G. Heywood and R. Rannacher, *Finite element approximation of the nonstationary Navier-Stokes problem. I. Regularity of solutions and second-order error estimates for spatial discretization*, SIAM J. Numer. Anal., 19 (1982), pp. 275–311.

[16] ——, *Finite-element approximation of the nonstationary Navier-Stokes problem. IV. Error analysis for second-order time discretization*, SIAM J. Numer. Anal., 27 (1990), pp. 353–384.

[17] F. Leslie, *Theory of flow phenomenum in liquid crystals*, in The Theory of Liquid Crystals, W. Brown, ed., vol. 4, New York, 1979, Academic Press., pp. 1–81.

[18] J.-L. Lions, *On some questions in boundary value problems of mathematical physics*, in Contemporary developments in continuum mechanics and partial differential equations (Proc. Internat. Sympos., Inst. Mat., Univ. Fed. Rio de Janeiro, Rio de Janeiro, 1977), vol. 30 of North-Holland Math. Stud., North-Holland, Amsterdam, 1978, pp. 284–346.

[19] J.-L. Lions and E. Magenes, *Non-homogeneous boundary value problems and applications. Vol. I*, Springer-Verlag, New York, 1972. Translated from the French by P. Kenneth, Die Grundlehren der mathematischen Wissenschaften, Band 181.

[20] C. Liu and N. J. Walkington, *Approximation of liquid crystal flows*, SIAM Journal on Numerical Analysis, 37 (2000), pp. 725–741.

[21] C. Liu and N. J. Walkington, *An Eulerian description of fluids containing visco-elastic particles*, Arch. Ration. Mech. Anal., 159 (2001), pp. 229–252.

[22] J. Rulla and N. J. Walkington, *Optimal rates of convergence for degenerate parabolic problems in two dimensions*, SIAM Journal on Numerical Analysis, 33 (1996), pp. 56–67.

[23] R. E. Showalter, *Monotone operators in Banach space and nonlinear partial differential equations*, Available online at http://www.ams.org/online_bks/surv49/, American Mathematical Society, Providence, RI, 1997.

[24] L. Tartar, *An introduction to Navier-Stokes equation and oceanography*, vol. 1 of Lecture Notes of the Unione Matematica Italiana, Springer-Verlag, Berlin, 2006.

[25] ——, *An introduction to Sobolev spaces and interpolation spaces*, vol. 3 of Lecture Notes of the Unione Matematica Italiana, Springer, Berlin, 2007.

[26] R. Temam, *Navier-Stokes Equations*, North Holland, 1977.

[27] V. Thomee, *Galerkin Finite Element Methods for Parabolic Problems*, vol. 1054 of Lecture Notes in Mathematics, Springer, 1984.

[28] A. Visintin, *Strong convergence results related to strict convexity*, Comm. Partial Differential Equations, 9 (1984), pp. 439–466.

[29] K. Yosida, *Functional Analysis*, vol. 123 of Grundlehren der mathematischen Wissenschaften, Springer–Verlag, 1980.