DISCONTINUOUS GALERKIN APPROXIMATIONS OF THE STOKES AND NAVIER-STOKES EQUATIONS

KONSTANTINOS CHRYSAFINOS[†] AND NOEL J. WALKINGTON[*]

Abstract. Numerical schemes to compute approximate solutions of the evolutionary Stokes and Navier-Stokes equations are studied. The schemes are discontinuous in time and conforming in space and of arbitrarily high order. Fully-discrete error estimates are derived and dependence of the viscosity constant is carefully tracked. It is shown that the errors are bounded by projection errors of the exact solution which exhibit optimal rates when the solutions are smooth.

1. Introduction. We consider the evolutionary Navier-Stokes problem; that is, we seek a velocity field $\mathbf{u}$ and a pressure p such that

$$\begin{cases} \mathbf{u}_t - \nu\Delta\mathbf{u} + \mathbf{u}\cdot\nabla\mathbf{u} + \nabla p &= \mathbf{f} \qquad \text{in } \Omega\times(0,T], \\ \nabla\cdot\mathbf{u} &= 0 \qquad \text{in } \Omega\times(0,T], \\ \mathbf{u}(0) &= \mathbf{u}_0 \qquad \text{in } \Omega, \end{cases} \tag{1.1}$$

given initial data $\mathbf{u}_0$ and forcing term $\mathbf{f}$. Here, Ω denotes a bounded domain in $\mathbb{R}^d$ with $d = 2, 3$.

There is an abundant literature concerning analytical and computational aspects of the Stokes and Navier-Stokes equations. Various schemes based on finite elements have been proposed for approximating solutions of (1.1). However, when it comes to the analysis of higher order schemes under minimal regularity assumptions the literature is limited. In this paper we analyze a class of schemes which are conforming in space and use the discontinuous Galerkin time stepping scheme. The discontinuous Galerkin technique provides an appropriate setting to analyze adaptive refinement and coarsening strategies, and Lagrangian or moving mesh schemes, all of which give rise to different meshes at each time step. While this complicates the analysis, our results show that this does not cause any degradation in the rates of convergence.

Our estimates are expressed in terms of projections of the exact solution onto the discrete spaces which exhibit optimal rates of convergence when the standard parabolic regularity holds. Our main results are "almost symmetric" in the sense that error in the solution is essentially bounded by the projection errors in the same norm. Estimates of this form show that that the error will be as good as the approximation and regularity theory allows, and can be viewed as a parabolic version of the classical Cea's Lemma [10]. Symmetric estimates have also been derived by Dupont, Liu and Liu, Dupont, Bank, Garcia and Santos [14, 26] to analyze moving mesh finite element methods, and by the authors in [7, 8] for a general class of implicit parabolic PDE's with time-dependent non-selfadjoint coefficients whose solutions may exhibit low regularity.

A central idea of our analysis is the development of a parabolic analog of the elliptic projection [31]. Stability of this projection does not require any relationship between the time and space

[*]Department of Mathematical Sciences, Carnegie Mellon University, Pittsburgh, PA 15213 USA. Email address:noelw@andrew.cmu.edu. Supported in part by National Science Foundation Grant DMS–0511309. This work was also supported by the NSF through the Center for Nonlinear Analysis.

[†]Department of Mathematics, School of Applied Mathematics and Physical Sciences, National Technical University of Athens, Zografou Campus, Athens 15780, Greece. Email address:chrysafinos@math.ntua.gr

[‡]AMS Classification: 65M12, 65M60; May 16, 2008

partitions. The dependence of the constants appearing in the projection estimates upon the viscosity ν is carefully tracked.

1.1. Related Results. Second order semi-discrete (in space) finite element error estimates for the evolutionary Navier-Stokes equations were studied by Heywood and Rannacher in [17], and semi-discrete estimates of arbitrary order were developed by the same authors in [18] under various regularity assumptions on the solution $(\mathbf{u}, p)$. More recently, high order error estimates for the spacial discretization of the time-dependent Navier-Stokes equations where developed by Bause [4], while Hou [20] obtained symmetric error estimates of arbitrary order under minimal regularity assumptions for the evolutionary Stokes problem. High order fully-discrete schemes for smooth solutions were studied by Baker, Dougalis and Karakashian in [2], and Heywood and Rannacher [19] established second-order (in time) estimates using the Crank-Nickolson time stepping scheme under weaker regularity assumptions. For some classical results regarding the stability of various fully-discrete Galerkin schemes one may consult the book of Temam [29] or the papers by Pironeau [27] and Süli in [28] who analyzed characteristic Galerkin time stepping schemes. Finally, an analysis of an implicit Runge-Kutta nonconforming scheme was provided by Katsaounis in [23].

The discontinuous Galerkin method was first introduced to model and simulate neutron transport by Lasaint and Raviart in [24]. Various aspects concerning applications of the DG scheme for hyperbolic problems were presented by Cockburn, Karnadiakis and Shu in [11] and by Johnson in [22] (see also the references within). Estimates for discontinuous solutions for the DG schemes for the transport equation were derived by Walkington in [32]. The DG method for ordinary differential equations was first analyzed by Delfour, Hager and Trochu in [13] who showed that the DG scheme was super convergent at the partition points (order $2k+2$ for polynomials of degree k). Jamet [21] established rates of order $\mathcal{O}(\tau^k)$ for linear parabolic problems and subsequently Eriksson, Johnson and Thomée [15] established $\mathcal{O}(\tau^{2k-1})$ estimates at the partition points for smooth solutions. An excellent exposition of their results and, more generally, the DG method for parabolic equations, can be found in Thomée's book [31]. Recently there has been a lot of work on the development and analysis of discontinuous Galerkin methods for elliptic problems. A comprehensive survey and comparison of this work was given by Arnold, Brezzi, Cockburn and Marini in [1] which contains many references related to this approach.

1.2. Outline. The standard abstract setting for the Navier-Stokes equations and the numerical schemes we consider, are introduced in the next section. One of the limitations of DG time stepping schemes is that the discrete energy estimate only bounds the solution at the partition points $\{t^n\}$. In order to bound the solution at times between these, fine properties of polynomials are needed. These are developed in Section 3.

To circumvent the technical difficulties associated with the discontinuities introduced at each time step we introduce a parabolic analog of the classical elliptic projection [31]. This parabolic projection is developed in Section 4 and, like its elliptic counterpart, is the discrete solution of an associated linear (Stokes) problem. Error estimates for the Navier-Stokes equations are then developed in Section 5.

1.3. Notation. It will be assumed that the equations are to be solved in a bounded connected Lipschitz domain $\Omega \subset \mathbb{R}^d$ with $d = 2$ or $d = 3$. Vector valued functions on Ω will be denoted with a bold face, $\mathbf{u}, \mathbf{v}, \mathbf{w}$. Standard notation is used for Sobolev spaces of scalar and

vector valued functions, with norms denoted $\|.\|_{H^s(\Omega)}$. We also denote solenoidal (divergence free) spaces by $\mathcal{V}(\Omega)$,$V(\Omega)$ and $W(\Omega)$ where,

$$\mathcal{V}(\Omega) = \{u \in C_0^\infty(\Omega) : \nabla \cdot \mathbf{u} = 0\}$$

$$V(\Omega) = \{u \in H_0^1(\Omega) : \nabla \cdot \mathbf{u} = 0\}$$

$$W(\Omega) = \{u \in L^2(\Omega) : \nabla \cdot \mathbf{u} = 0 \text{ and } \mathbf{u} \cdot \mathbf{n} = 0\}.$$

The spaces $V(\Omega)$ and $W(\Omega)$ are equipped with the $H^1(\Omega)$ and $L^2(\Omega)$ norms respectively. With Ω as above, $V(\Omega)$ and $W(\Omega)$ are closed subspaces of $H_0^1(\Omega)$ and $L^2(\Omega)$ respectively [16], and $C_0^\infty(\Omega) \cap V(\Omega)$ is dense in $V(\Omega)$ and $W(\Omega)$. We denote by $L_0^2(\Omega)$ the space

$$L_0^2(\Omega) = \big\{p \in L^2(\Omega) : \int_\Omega p = 0\big\},$$

equipped with the $\|.\|_{L^2(\Omega)}$ norm. The dual spaces of $H_0^1(\Omega)$ and $V(\Omega)$ are denoted by $H^{-1}(\Omega)$ and $V(\Omega)^*$ respectively. The later space is defined by

$$\|f\|_{V(\Omega)^*} = \sup_{v \in V(\Omega): \|v\|_{H^1(\Omega)} \leq 1} \langle f, v\rangle,$$

where $\langle ., .\rangle$ denotes the duality paring between $H_0^1(\Omega)$ and $H^{-1}(\Omega)$.

For any Banach space X let $L^p[0,T;X]$, $1 \leq p < \infty$, and $L^\infty[0,T;X]$ denote the spaces of p-integrable functions taking values in X with norm

$$\|f\|_{L^p[0,T;X]} = \Big(\int_0^T \|f(t)\|_X^p\Big)^{1/p}, \qquad \|f\|_{L^\infty[0,T;X]} = \text{esssup}_{t\in[0,T]}\|f\|_X < \infty,$$

and let $H^1[0,T;X]$ denote the space of functions with square integrable derivatives with norm

$$\|f\|_{H^1[0,T;X]} = \Big(\int_0^T \|f\|_X^2 dt + \int_0^T \|f_t\|_X^2 dt\Big)^{1/2}.$$

2. Weak Statements and Numerical Scheme. The numerical schemes will approximate the natural weak formulation of equation (1.1) which seeks

$$(\mathbf{u}, p) \in L^2[0,T;H^1(\Omega)] \cap H^1[0,T;H^{-1}(\Omega)] \times L^2[0,T;L_0^2(\Omega)]$$

such that for a.e $t \in (0,T]$,

$$\begin{cases} \langle \mathbf{u}_t, \mathbf{v}\rangle + a(\mathbf{u},\mathbf{v}) + b(\mathbf{v},p) + c(\mathbf{u},\mathbf{u},\mathbf{v}) &= \langle \mathbf{f}, \mathbf{v}\rangle & \forall\, \mathbf{v} \in H_0^1(\Omega), \\ b(\mathbf{u},q) &= 0 & \forall\, q \in L_0^2(\Omega), \\ (\mathbf{u}(0,x),\mathbf{v}) &= (\mathbf{u}_0,\mathbf{v}) & \forall\, \mathbf{v} \in W(\Omega). \end{cases} \tag{2.1}$$

The bilinear forms $a : H^1(\Omega) \times H^1(\Omega) \to \mathbb{R}$ and $b : H^1(\Omega) \times L^2(\Omega) \to \mathbb{R}$ are defined by

$$a(\mathbf{u},\mathbf{v}) = \nu \int_\Omega (\nabla \mathbf{u}) : (\nabla \mathbf{v}), \qquad \text{and} \qquad b(\mathbf{v},p) = \int_\Omega q(\nabla.\mathbf{v}),$$

where $(\nabla \mathbf{u}) : (\nabla \mathbf{v}) = \sum_{i,j=1}^{d} u_{i,j} v_{i,j}$ and the index after the comma indicates the derivative with respect to x_j. The nonlinear term $c(.,.,.)$ is trilinear on $H^1(\Omega)^d$ and takes the form

$$c(\mathbf{u},\mathbf{v},\mathbf{w}) = (1/2)\left(\hat{c}(\mathbf{u},\mathbf{v},\mathbf{w}) - \hat{c}(\mathbf{u},\mathbf{w},\mathbf{v})\right), \qquad \text{where} \qquad \hat{c}(\mathbf{u},\mathbf{v},\mathbf{w}) = \sum_{i,j=1}^{d} \int_\Omega \mathbf{u}_j \Big(\frac{\partial \mathbf{u}_i}{\partial x_j}\Big) \mathbf{w}_i.$$

The bilinear form $a(.,.)$ is continuous and satisfies the classical coercivity condition. In particular there exist constants $C, c > 0$, depending only on the domain Ω through the Poincaré and Korn inequalities, such that

$$\big|a(\mathbf{u},\mathbf{v})\big| \le \nu C \|\mathbf{u}\|_{H^1(\Omega)} \|\mathbf{v}\|_{H^1(\Omega)} \qquad \text{and} \qquad a(\mathbf{u},\mathbf{u}) \ge c\nu \|\mathbf{u}\|^2_{H^1(\Omega)}.$$

In the proof below typically take $c = 1$ which amounts to taking the norm on $H^1(\Omega)^d$ to be $(a(\mathbf{u},\mathbf{u})/\nu)^{1/2}$. The bilinear form $b(.,.)$ is also continuous and satisfies the classical inf-sup condition [5].

LEMMA 2.1. *There exist constants C, c depending only on Ω such that the bilinear form $b(.,.)$ satisfies the following properties:*

1. *Continuity:*
$$b(\mathbf{v}, q) \le C \|\mathbf{u}\|_{H^1(\Omega)} \|q\|_{L^2(\Omega)}.$$
2. *Inf-sup condition:*
$$\inf_{q \in L^2_0(\Omega)} \sup_{\mathbf{v} \in H^1(\Omega)} \frac{b(\mathbf{v}, q)}{\|\mathbf{v}\|_{H^1_0(\Omega)} \|q\|_{L^2_0(\Omega)}} \ge c > 0.$$

The trilinear forms satisfy the following continuity and skew symmetry properties [29].

LEMMA 2.2. *The trilinear forms satisfy*

$$c(\mathbf{u},\mathbf{w},\mathbf{v}) = \hat{c}(\mathbf{u},\mathbf{v},\mathbf{w}) = -\hat{c}(\mathbf{u},\mathbf{w},\mathbf{v}), \qquad \mathbf{u} \in V(\Omega), \quad \mathbf{v},\mathbf{w} \in H^1_0(\Omega),$$

and

$$c(\mathbf{u},\mathbf{v},\mathbf{w}) = -c(\mathbf{u},\mathbf{w},\mathbf{v}), \qquad \mathbf{u},\mathbf{v},\mathbf{w} \in H^1(\Omega),$$

and the following bounds,

$$\begin{aligned} |\hat{c}(\mathbf{u},\mathbf{v},\mathbf{w})| &\le \|\mathbf{u}\|_{L^p(\Omega)} \|\nabla \mathbf{v}\|_{L^2(\Omega)} \|\mathbf{w}\|_{L^{p'}(\Omega)}, \qquad 1/p + 1/p' = 1/2, \\ |c(\mathbf{u},\mathbf{v},\mathbf{w})| &\le C \|\mathbf{u}\|_{H^1(\Omega)} \|\mathbf{v}\|_{H^1(\Omega)} \|\mathbf{w}\|_{H^1(\Omega)}, \end{aligned}$$

where $C > 0$ depends upon the domain Ω.

In two dimensions ($d = 2$)

$$\begin{aligned} \Big|c(\mathbf{u},\mathbf{v},\mathbf{w})\Big| \le C \|\mathbf{u}\|^{1/2}_{L^2(\Omega)} \|\nabla \mathbf{u}\|^{1/2}_{L^2(\Omega)} \Big(&\|\mathbf{v}\|_{H^1(\Omega)} \|\mathbf{w}\|^{1/2}_{L^2(\Omega)} \|\nabla \mathbf{w}\|^{1/2}_{L^2(\Omega)} \\ &+ \|\mathbf{w}\|_{H^1(\Omega)} \|\mathbf{v}\|^{1/2}_{L^2(\Omega)} \|\nabla \mathbf{v}\|^{1/2}_{L^2(\Omega)} \Big), \end{aligned} \tag{2.2}$$

while in three dimensions ($d = 3$) [19],

$$\Big|c(\mathbf{u},\mathbf{v},\mathbf{w})\Big| \le C \|\mathbf{u}\|^{1/2}_{L^2(\Omega)} \|\mathbf{u}\|^{1/2}_{H^1(\Omega)} \|\mathbf{v}\|_{H^1(\Omega)} \|\mathbf{w}\|_{H^1(\Omega)}. \tag{2.3}$$

2.1. Fully discrete formulation. To approximate the solution of (2.1) we introduce a partition $0 = t_0 < t_1 < ... < t_N = T$ and for each interval (t^{n-1}, t^n) discrete spaces $U_h^n \subset H_0^1(\Omega)$ for the velocity and $Q_h^n \subset L_0^2(\Omega)$ for the pressure. We let $\mathcal{P}_k[t^{n-1}, t^n; U_h^n]$ and $\mathcal{P}_k[t^{n-1}, t^n; Q_h^n]$ denote the set of polynomials of degree k in time with values in U_h^n and Q_h^n respectively. On each partition (t^{n-1}, t^n) the Discontinuous Galerkin method constructs an approximate velocity

$$\mathbf{u}_h \in \mathcal{U}^h \equiv \left\{\mathbf{u}_h \in L^2[0,T; H_0^1(\Omega)] : \mathbf{u}_h|_{(t^{n-1},t^n]} \in \mathcal{P}_k[t^{n-1}, t^n; U_h^n]\right\}$$

and pressure

$$p_h \in \mathcal{P}^h \equiv \left\{p_h \in L^2[0,T; L_0^2(\Omega)] : p_h|_{(t^{n-1},t^n]} \in \mathcal{P}_k[t^{n-1}, t^n; Q_h^n]\right\}$$

satisfying

$$\begin{cases} (\mathbf{u}_-^n, \mathbf{v}_-^n) + \displaystyle\int_{t^{n-1}}^{t^n} \Big(- \langle \mathbf{u}, \mathbf{v}_{ht}\rangle + a(\mathbf{u}_h, \mathbf{v}_h) + c(\mathbf{u}_h, \mathbf{u}_h, \mathbf{v}_h) + b(\mathbf{v}_h, p_h)\Big) \\ \qquad = (\mathbf{u}_-^{n-1}, \mathbf{v}_+^{n-1}) + \displaystyle\int_{t^{n-1}}^{t^n} \langle \mathbf{f}, \mathbf{v}_h\rangle \quad \forall\, \mathbf{v}_h \in \mathcal{P}_k[t^{n-1}, t^n; U_h^n], \\ \displaystyle\int_{t^{n-1}}^{t^n} b(\mathbf{u}_h, q_h) = 0 \qquad \forall\, q_h \in \mathcal{P}_k[t^{n-1}, t^n; Q_h^n], \\ \mathbf{u}_h(0) = \mathbf{u}_-^0. \end{cases} \tag{2.4}$$

Here we write $\mathbf{u}_\pm^n = \mathbf{u}_h(t_\pm^n)$ and for the traces from above and below respectively. We assume that U_h^n, and Q_h^n satisfy the usual approximation properties and inf-sup condition [10, 16].

ASSUMPTION 1. *Let $\{\mathcal{T}_h^n \mid 0 \le n \le N_h,\ h > 0\}$ be a quasi-regular family of triangulations of $H_0^1(\Omega)$ and let $U_h^n \subset H_0^1(\Omega)$ and $Q_h^n \subset L_0^2(\Omega)$ be finite element subspaces constructed over $\mathcal{T}_h^n$ using piecewise polynomials satisfying the following approximation properties.*

There exists an integer $\ell \ge 1$ and constants $C, c > 0$ independent of h, $\mathbf{u}$, and p such that,

1. *If $\mathbf{u} \in H^{l+1}(\Omega) \cap H_0^1(\Omega)$*

$$\inf_{\mathbf{u}_h \in U_h^n} \|\mathbf{u} - \mathbf{u}_h\|_{H^s(\Omega)} \le C h^{l+1-s} \|\mathbf{u}\|_{H^{l+1}(\Omega)}, \quad 0 \le l \le \ell, \quad s = -1, 0, 1.$$

2. *If $p \in H^l(\Omega) \cap L_0^2(\Omega)$*

$$\inf_{q_h \in Q_h^n} \|p - q_h\|_{L^2(\Omega)} \le C h^l \|p\|_{H^l(\Omega)}, \quad 0 \le l \le \ell.$$

3. *The subspaces U_h^n, Q_h^n satisfy the classical inf-sup condition (2.5),*

$$\inf_{q_h \in Q_h^n} \sup_{\mathbf{v}_h \in U_h^n} \frac{b(\mathbf{v}_h, q_h)}{\|\mathbf{v}_h\|_{H^1(\Omega)} \|q_h\|_{L^2(\Omega)}} \ge c. \tag{2.5}$$

4. *If the triangulations are quasi-uniform then there exists $C_{inv} > 0$ independent of h such that*

$$\|\mathbf{u}_h\|_{H^1(\Omega)} \le (C_{inv}/h) \|\mathbf{u}_h\|_{L^2(\Omega)}, \qquad \forall\, \mathbf{u}_h \in U_h^n.$$

We next define the *discrete divergence free* subspace of $\mathcal{P}_k[t^{n-1}, t^n; U_h^n]$ to be

$$\mathcal{Z}_h^n \equiv \big\{ \mathbf{u}_h \in \mathcal{P}_k[t^{n-1}, t^n; U_h^n] \ \mid \ \int_{t^{n-1}}^{t^n} b(\mathbf{u}_h, q_h) = 0 \quad \forall\, q_h \in \mathcal{P}_k[t^{n-1}, t^n; Q_h^n] \big\}.$$

The following lemma characterizes the spaces $\mathcal{Z}_h^n$ and illustrates why it is natural to chose the time variation (i.e. degree k) for the approximate pressure and velocity to be the same.

LEMMA 2.3. *Let* $Z_h^n = \big\{ \mathbf{u}_h \in U_h^n : b(\mathbf{u}_h, q_h) = 0,\ \forall\, q_h \in Q_h^n \big\}$, *then* $\mathcal{Z}_h^n = \mathcal{P}_k[t^{n-1}, t^n; Z_h^n]$.

Proof. It is clear that $\mathcal{P}_k[t^{n-1}, t^n; Z_h^n] \subset \mathcal{Z}_h^n$. It suffices to show the reverse inclusion. Let $\{\eta_i\}_{i=0}^k \subset \mathcal{P}_k[t^{n-1}, t^n]$ be an orthonormal basis with the $L^2[t^{n-1}, t^n]$ inner product. Then write $\mathbf{u}_h \in \mathcal{P}_k[t^{n-1}, t^n; U_h^n]$ and $q_h \in \mathcal{P}_k[t^{n-1}, t^n; Q_h^n]$ as

$$\mathbf{u}_h = \sum_{i=0}^k \eta_i \mathbf{u}_i \qquad \text{and} \qquad q_h = \sum_{j=0}^k \eta_j q_j.$$

Since $b(.,.) : U_h^n \times Q_h^n \to \mathbb{R}$ is bilinear (and independent of time)

$$\int_{t^{n-1}}^{t^n} b(\mathbf{u}_h, q_h) = \sum_{i,j=0}^k \int_{t^{n-1}}^{t^n} \eta_i \eta_j b(\mathbf{u}_i, q_j) = \sum_{i=0}^k b(\mathbf{u}_i, q_i).$$

If $\mathbf{u}_h \in \mathcal{Z}_h^n$ the left hand expression vanishes for all $\{q_j\}_{j=0}^k \subset Q_h^n$. Therefore, $\mathbf{u}_i \in Z_h^n$ for each $0 \le i \le k$, and since Z_h^n is a linear space it follows that $\mathbf{u}_h \in \mathcal{P}_k[t^{n-1}, t^n; Z_h^n]$. □

This lemma allows us to write

$$\begin{aligned} \mathcal{Z}^h &\equiv \{ \mathbf{u}_h \in \mathcal{U}^h \ \mid \ \int_0^T b(\mathbf{u}_h, q_h) = 0,\ \forall\, q_h \in \mathcal{P}^h \} \\ &= \{ \mathbf{u}_h \in \mathcal{U}^h \ \mid \ \mathbf{u}_h|_{(t^{n-1}, t^n]} \in \mathcal{Z}_h^n \} \\ &= \{ \mathbf{u}_h \in \mathcal{U}^h \ \mid \ \mathbf{u}_h|_{(t^{n-1}, t^n]} \in \mathcal{P}_k[t^{n-1}, t^n; Z_h^n] \}. \end{aligned}$$

3. Discrete Characteristic Functions and Interpolants. In this section we present some elementary properties of polynomials required to establish stability and error estimates for the DG approximations of the Stokes and Navier-Stokes equations.

3.1. Discrete characteristic functions. Estimates for evolution equations are frequently obtained by multiplying the equation by $\mathbf{u}$ and integrating to an arbitrary time $t \in [0, T]$. This amounts to selecting the test function to be $\chi_{[0,t)}\mathbf{u}$, and this choice is not available in the discrete context unless the terminal time is one of the partition points. In the past this problem has been circumvented by deriving bounds for certain temporal derivatives of the solution, so that the solution between the partition points was effectively controlled by values at these points [31]. This approach is problematic for the (Navier-) Stokes equations since the time derivative of the pressure is usually not available, and it is difficult to get separate estimates for $\mathbf{u}_t$ and ∇p given a bound on $\mathbf{u}_t - \nabla p$ in $H^{-1}(\Omega)$.

To estimate the solution at times $t \in [t^{n-1}, t^n)$ we first recall the discrete characteristic functions as introduced in [7, Section 2.3], and then extend the construction in [7, 8] to accommodate

(discretely) divergence free subspaces. The discrete characteristic functions on each interval are invariant under translation so it is convenient to work on the interval $[0,\tau)$ with $\tau = t^n - t^{n-1}$. We begin by considering polynomials $r \in \mathcal{P}_k[0,\tau]$. A discrete approximation of $\chi_{[0,t)}r$ is the polynomial $\tilde{r} \in \{\tilde{r} \in \mathcal{P}_k[0,\tau] | \tilde{r}(0) = r(0)\}$ satisfying

$$\int_0^\tau \tilde{r}q = \int_0^t rq \qquad \forall\, q \in \mathcal{P}_{k-1}[0,\tau].$$

The above construction is motivated by the fact that we may put $q = r'$ to obtain $\int_0^\tau \tilde{r}r' = \int_0^t rr' = (1/2)(r^2(t) - r^2(0))$.

We next extend this elementary construction to approximate functions of the form $\chi_{[0,t)}\mathbf{v}$ for $\mathbf{v} \in \mathcal{P}_k[0,\tau;V]$ where V is any semi inner product space. If $\mathbf{v} \in \mathcal{P}_k[0,\tau;V]$ we can write $\mathbf{v} = \sum_{i=0}^k r_i(t)\mathbf{v}_i$ where $\{r_i\} \subset \mathcal{P}_k[0,\tau]$ and $\{\mathbf{v}_i\} \subset V$. If we define $\tilde{\mathbf{v}} = \sum_{i=0}^k \tilde{r}_i(t)\mathbf{v}_i$ it is clear that $\tilde{\mathbf{v}} \in \mathcal{P}_k[0,\tau;V]$ satisfies

$$\tilde{\mathbf{v}}(0) = \mathbf{v}(0), \text{ and } \int_0^\tau (\tilde{\mathbf{v}},\mathbf{w})_V = \int_0^t (\mathbf{v},\mathbf{w})_V \qquad \forall\, \mathbf{w} \in \mathcal{P}_{k-1}[0,\tau;V]. \tag{3.1}$$

The following elementary lemma from [7, Lemma 2.7] shows that the mapping $\mathbf{v} \mapsto \tilde{\mathbf{v}}$ is continuous on $\mathcal{P}_k[0,\tau,V]$.

Lemma 3.1. *Let V be a semi-inner product space, then the mapping $\mathbf{v} = \sum_{i=0}^k r_i(t)\mathbf{v}_i \mapsto \tilde{\mathbf{v}} = \sum_{i=0}^k \tilde{r}_i(t)\mathbf{v}_i$ on $\mathcal{P}_k[0,\tau;V]$ is continuous in $\|\cdot\|_{L^2[0,\tau;V]}$. In particular, there exists $C_k > 0$ depending only upon k such that*

$$\|\tilde{\mathbf{v}}\|_{L^2[0,\tau;V]} \le C_k\|\mathbf{v}\|_{L^2[0,\tau;V]}, \quad \textit{and} \quad \|\tilde{\mathbf{v}} - \chi_{[0,t)}\mathbf{v}\|_{L^2[0,\tau;V]} \le C_k\|\mathbf{v}\|_{L^2[0,\tau;V]}.$$

Notice that the above construction is purely algebraic in the sense that equation (3.1) holds for any choice of inner product on V. The following lemma gives an explicit formula, $\tilde{\mathbf{v}}(s) = \phi(s)\mathbf{z}$, when $\mathbf{v}(s) = \mathbf{z}$ is constant.

Lemma 3.2. *Fix $t \in [0,\tau]$ and let $p \in \mathcal{P}_k[0,\tau]$ be characterized by*

$$p(0) = 1, \qquad \int_0^\tau pq = \int_0^t q, \qquad q \in \mathcal{P}_{k-1}[0,\tau].$$

Then,

$$p(s) = 1 + (s/\tau)\sum_{i=0}^{k-1} c_i\hat{p}_i(s/\tau), \qquad c_i = \int_{t/\tau}^1 \hat{p}_i(\eta)\,d\eta,$$

where $\{\hat{p}_i\}_{i=0}^{k-1}$ is an orthonormal basis of $\mathcal{P}_{k-1}[0,1]$ in the (weighted) space $L^2_w[0,1]$ having inner product

$$(\hat{p},\hat{q}) = \int_0^1 \eta\hat{p}(\eta)\hat{q}(\eta)\,d\eta.$$

In particular,

$$\|p\|_{L^\infty[0,\tau]} \le 1 + (1 - t/\tau)\sum_{i=0}^{k-1} \|\hat{p}_i\|^2_{L^\infty[0,1]} \le C_k,$$

with C_k independent of $t \in [0,\tau]$.

3.2. An exponential interpolant. Bounds for DG approximations of the Stokes equations in $L^\infty[0,T;H^1(\Omega)]$ will be obtained using the polynomial interpolant of $e^{-\lambda(t-t^{n-1})}\mathbf{u}_h$ introduced in this section.

DEFINITION 3.3. *Let V be a linear space, and $\lambda > 0$ be given. If $\mathbf{v} = \sum_{i=0}^k r_i(t)\mathbf{v}_i \in \mathcal{P}_k[t^{n-1},t^n;V]$, where $r_i \in \mathcal{P}_k[t^{n-1},t^n], \mathbf{v}_i \in V$, we define the* exponential interpolant *of $\mathbf{v}$ by*

$$\bar{\mathbf{v}} = \sum_{i=0}^k \bar{r}_i(t)\mathbf{v}_i$$

where $\bar{r}_i \in \mathcal{P}_k[t^{n-1},t^n]$ is the approximation of $r_i(t)e^{-\lambda(t-t^{n-1})}$ satisfying $\bar{r}_i(t^{n-1}) = r_i(t^{n-1})$ and

$$\int_{t^{n-1}}^{t^n} \bar{r}_i(t)q(t)dt = \int_{t^{n-1}}^{t^n} r_i(t)q(t)e^{-\lambda(t-t^{n-1})}dt, \qquad q \in \mathcal{P}_{k-1}[t^{n-1},t^n].$$

Note that the map $\mathbf{v} \to \bar{\mathbf{v}}$ does not define a projection, since $\bar{\bar{\mathbf{v}}} \neq \bar{\mathbf{v}}$, so we will refer to it as an exponential interpolant. Unlike the polynomial approximations of $\chi_{[t^{n-1},t)}\mathbf{u}_h$ introduced above, the difference between the interpolant $\bar{\mathbf{v}}$ and $\mathbf{v}$ remains small.

LEMMA 3.4. *Let V and Q be linear spaces and $\mathbf{v} \mapsto \bar{\mathbf{v}}$ be the map constructed in Definition 3.3 with parameter $\lambda > 0$. If $L(.,.) : V \times Q \to \mathbb{R}$ is a bilinear mapping and $\mathbf{v} \in \mathcal{P}_k[t^{n-1},t^n;V]$, then*

$$\int_{t^{n-1}}^{t^n} L(\bar{\mathbf{v}}(t),q(t))dt = \int_{t^{n-1}}^{t^n} L(\mathbf{v}(t),q(t))e^{-\lambda(t-t^{n-1})}dt, \qquad q \in \mathcal{P}_{k-1}[t^{n-1},t^n;Q].$$

If $(.,.)_V$ is a (semi) inner product on V, then there exists a constant $C_k > 0$ independent of $\lambda \geq 0$ such that

$$\|\mathbf{v} - \bar{\mathbf{v}}\|_{L^2[t^{n-1},t^n;V]} \leq C_k(1 - e^{-\lambda(t^n-t^{n-1})})\|\mathbf{v}\|_{L^2[t^{n-1},t^n;V]} \leq C_k\lambda(t^n - t^{n-1})\|\mathbf{v}\|_{L^2[t^{n-1},t^n;V]};$$

in particular,

$$\|\bar{\mathbf{v}}\|_{L^2[t^{n-1},t^n;V]} \leq (1 + C_k)\|\mathbf{v}\|_{L^2[t^{n-1},t^n;V]}.$$

Proof. (*Sketch*) The proof follows along the lines of [7, Lemma 2.3,2.4]. The key step, is to prove that

$$\int_{t^{n-1}}^{t^n} (r_i - \bar{r}_i)^2 \leq C_k(1 - e^{-\lambda(t^n-t^{n-1})}) \int_{t^{n-1}}^{t^n} r_i^2.$$

For that purpose, note that since $r_i(t^{n-1}) = \bar{r}_i(t^{n-1})$ we may write $r_i - \bar{r}_i = (t - t^{n-1})p_i$, for some $p_i \in \mathcal{P}_{k-1}[t^{n-1},t^n]$. Using the definition of the interpolant we derive

$$\int_{t^{n-1}}^{t^n} (t - t^{n-1})p_i q = \int_{t^{n-1}}^{t^n} (1 - e^{-\lambda(t-t^{n-1})})r_i q \quad \forall\, q \in \mathcal{P}_{k-1}[t^{n-1},t^n].$$

Selecting $q = p_i$ and using the equivalence of norms on the $\mathcal{P}_k[t^{n-1},t^n]$ shows

$$\|p\|_{L^2[t^{n-1},t^n]} \leq C(k)(1 - e^{-\lambda(t-t^{n-1})})\|r_i\|_{L^2[t^{n-1},t^n]}.$$

The remaining calculations are elementary, see [7, Lemma 2.4]. ∎

A classical scaling argument and the finite dimensionality of $\mathcal{P}_k[t^{n-1}, t^n]$ implies the following lemma.

LEMMA 3.5. *Let V be a linear space and $(.,.)_V$ be a (semi) inner product on V and let $w(t) \geq 0$ be a nonzero element of $L^1[t^{n-1}, t^n]$ and set $\tau = t^n - t^{n-1}$. Then, there exists a constant $C_k > 0$ depending only upon k and w such that*

$$\|\mathbf{v}\|_{L^p[t^{n-1},t^n;V]} \leq \tau^{1/p-1/2}\Big(C_k \int_{t^{n-1}}^{t^n} w((t-t^{n-1})/\tau)\|\mathbf{v}(t)\|_V^2 dt\Big)^{1/2}$$

for all $\mathbf{v} \in \mathcal{P}_k[t^{n-1}, t^n; V]$ and $1 \leq p \leq \infty$. In particular, if $(1/p) + (1/p') = 1$,

$$\|\mathbf{v}\|_{L^p[t^{n-1},t^n;V]}\|\mathbf{v}\|_{L^{p'}[t^{n-1},t^n;V]} \leq C_k \int_{t^{n-1}}^{t^n} w((t-t^{n-1})/\tau)\|\mathbf{v}(t)\|_V^2 dt.$$

This inverse estimate can be used to bound $\mathbf{v} - \bar{\mathbf{v}}$ in $L^p[t^{n-1}, t^n; V]$ for any $p \geq 1$.

LEMMA 3.6. *Let V be a linear space and $(.,.)_V$ be a semi-inner product on V and let $\mathbf{v} \mapsto \bar{\mathbf{v}}$ denote the exponential interpolant on $\mathcal{P}_k[t^{n-1}, t^n; V]$ constructed in Definition 3.3. Set $\tau = t^n - t^{n-1}$, then there exists a constant $C_k > 0$ depending only upon k such that*

$$\|\mathbf{v} - \bar{\mathbf{v}}\|_{L^p[t^{n-1},t^n;V]} \leq C_k(1 - e^{\lambda\tau})\|\mathbf{v}\|_{L^p[t^{n-1},t^n;V]} \leq C_k\lambda\tau\|\mathbf{v}\|_{L^p[t^{n-1},t^n;V]},$$

for all $\mathbf{v} \in \mathcal{P}_k[t^{n-1}, t^n; V]$ and $1 \leq p \leq \infty$. In particular,

$$\|\bar{\mathbf{v}}\|_{L^p[t^{n-1},t^n;V]} \leq (1 + C_k)\|\mathbf{v}\|_{L^p[t^{n-1},t^n;V]}.$$

Proof. Using Lemma 3.5 with $w(t) \equiv 1$ and Lemma 3.4 we obtain,

$$\|\mathbf{v} - \bar{\mathbf{v}}\|_{L^p[t^{n-1},t^n;V]} \leq \tau^{1/p-1/2}C_k\|\mathbf{v} - \bar{\mathbf{v}}\|_{L^2[t^{n-1},t^n;V]} \leq \tau^{1/p-1/2}C_k(1 - e^{\lambda\tau})\|\mathbf{v}\|_{L^2[t^{n-1},t^n;V]}.$$

If $1/p + 1/p' = 1$, Hölder's inequality and another application of Lemma 3.5 shows

$$\|\mathbf{v}\|^2_{L^2[t^{n-1},t^n;V]} \leq \|\mathbf{v}\|_{L^p[t^{n-1},t^n;V]}\|\mathbf{v}\|_{L^{p'}[t^{n-1},t^n;V]} \leq \|\mathbf{v}\|_{L^p[t^{n-1},t^n;V]}C_k\tau^{1/p'-1/2}\|\mathbf{v}\|_{L^2[t^{n-1},t^n;V]},$$

so

$$\|\mathbf{v}\|_{L^2[t^{n-1},t^n;V]} \leq C_k\tau^{1/p'-1/2}\|\mathbf{v}\|_{L^p[t^{n-1},t^n;V]} = C_k\tau^{1/2-1/p}\|\mathbf{v}\|_{L^p[t^{n-1},t^n;V]},$$

and the lemma follows. □

4. DG scheme for an auxiliary PDE. In this section we introduce a "parabolic" projection ($\mathbb{P}_h$ below) that will be used to construct test functions for the DG scheme used to approximate the Navier-Stokes equations. It is designed to circumvent the difficulties associated with the use of different spaces for each time step and the jump terms that appear in the DG scheme. We consider DG approximations for the evolutionary Stokes equations; that is

$$(\mathbf{u}, r) \in L^2[0, T; H_0^1(\Omega)] \cap H^1[0, T; H^{-1}(\Omega)] \times L^2[0, T; L_0^2(\Omega)]$$

and

$$\begin{cases} \langle \mathbf{u}_t, \mathbf{v}\rangle + a(\mathbf{u},\mathbf{v}) + b(\mathbf{v}, r) &= \langle \mathbf{f}, \mathbf{v}\rangle \qquad \forall\, \mathbf{v} \in H_0^1(\Omega), \\ b(\mathbf{u}, q) &= 0 \qquad \forall\, q \in L_0^2(\Omega), \\ (\mathbf{u}(0), \mathbf{w}) &= (\mathbf{u}_0, \mathbf{w}) \qquad \forall\, \mathbf{w} \in W(\Omega). \end{cases} \tag{4.1}$$

The numerical schemes are based upon the weak formulation: $\mathbf{u} \in L^2[0,T; H_0^1(\Omega)] \cap H^1[0,T; H^{-1}(\Omega)] \hookrightarrow C[0,T; L^2(\Omega)]$, $r \in L^2[0,T; L_0^2(\Omega)]$ and

$$\begin{cases} (\mathbf{u}(T), \mathbf{v}(T)) + \displaystyle\int_0^T -\langle \mathbf{u}, \mathbf{v}_t\rangle + a(\mathbf{u},\mathbf{v}) + b(\mathbf{v}, r) \\ \qquad = (\mathbf{u}(0), \mathbf{v}(0)) + \displaystyle\int_0^T \langle \mathbf{f}, \mathbf{v}\rangle \quad \forall\, \mathbf{v} \in L^2[0,T; H_0^1(\Omega)] \cap H^1[0,T; H^{-1}(\Omega)], \\ \displaystyle\int_0^T b(\mathbf{u}, q) = 0 \qquad \forall\, q \in L^2[0,T; L_0^2(\Omega)]. \end{cases} \tag{4.2}$$

Given a partition $0 = t^0 < t^1 < ... < t^N = T$ of $[0,T]$ and a collection $\{U_h^n\}_{n=0}^N$, $\{Q_h^n\}_{n=0}^N$ of subspaces of $H_0^1(\Omega)$ and $L_0^2(\Omega)$ respectively the DG method constructs an approximate solution $u_h \in \mathcal{U}^h$, and a pressure $p_h \in \mathcal{P}^h$ satisfying

$$\begin{cases} (\mathbf{u}_-^n, \mathbf{v}_-^n) + \displaystyle\int_{t^{n-1}}^{t^n} \Big(-\langle \mathbf{u}_h, \mathbf{v}_{ht}\rangle + a(\mathbf{u}_h, \mathbf{v}_h) + b(\mathbf{v}_h, r_h)\Big) \\ \qquad -(\mathbf{u}_-^{n-1}, \mathbf{v}_+^{n-1}) = \displaystyle\int_{t^{n-1}}^{t^n} \langle \mathbf{f}, \mathbf{v}_h\rangle \qquad \forall\, \mathbf{v}_h \in \mathcal{P}_k[t^{n-1}, t^n; U_h^n], \\ \displaystyle\int_{t^{n-1}}^{t^n} b(\mathbf{u}_h, q_h) = 0 \qquad \forall q_h \in \mathcal{P}_k[t^{n-1}, t^n; Q_h^n]. \end{cases} \tag{4.3}$$

4.1. Projections. Projections onto the discretely divergence subspaces play an important role in the analysis of numerical schemes for both the Stokes and Navier-Stokes equations. Here we define our "parabolic" projection, $\mathbb{P}_h$, and compare it with the classical Stokes projection Π_h.

Definition 4.1.

1. *$P_n : L^2(\Omega) \to Z_h^n$ denotes the projection onto the discretely divergence free subspace $Z_h^n = \{\mathbf{u}_h \in U_h^n \mid b(\mathbf{u}_h, q_h) = 0,\ q_h \in Q_h^n\}$; that is*

$$P_n\mathbf{w} \in Z_h^n, \qquad (P_n\mathbf{w}, \mathbf{v}_h) = (\mathbf{w}, \mathbf{v}_h) \qquad \forall\, \mathbf{v}_h \in Z_h^n.$$

2. *The Stokes projection $\Pi_n : H_0^1(\Omega) \to U_h^n$ is the first component of the pair $(\mathbf{u}_h, r_h) \in U_h^n \times Q_h^n$ satisfying*

$$a(\mathbf{u}_h, \mathbf{v}_h) + b(\mathbf{v}_h, r_h) = a(\mathbf{u}, \mathbf{v}_h), \qquad b(\mathbf{u}_h, q_h) = 0,$$

for all $(\mathbf{v}_h, q_h) \in U_h^n \times Q_h^n$.

Formally $\Pi_n\mathbf{u}$ is the Galerkin approximation of the stationary Stokes problem with right hand side $\mathbf{f} = -\nu\Delta\mathbf{u}$. The parabolic version introduced next is the DG approximation of the evolutionary Stokes' problem with right hand side $\mathbf{f} = \mathbf{u}_t - \nu\Delta\mathbf{u}$.

Definition 4.2.

1. *The projection* $\mathbb{P}_n^{loc} : C[t^{n-1}, t^n; L^2(\Omega)] \to \mathcal{P}_k[t^{n-1}, t^n; Z_h^n]$ *satisfies* $(\mathbb{P}_n^{loc}\mathbf{u})^n = P_n\mathbf{u}(t^n)$, *and*
$$\int_{t^{n-1}}^{t^n} (\mathbf{u} - \mathbb{P}_n^{loc}\mathbf{u}, \mathbf{v}_h) = 0, \qquad \forall\, \mathbf{v}_h \in \mathcal{P}_{k-1}[t^{n-1}, t^n; Z_h^n].$$
Here we have used the convention $(\mathbb{P}_n^{loc}\mathbf{u})^n \equiv (\mathbb{P}_n^{loc}\mathbf{u})(t^n)$.
2. *The projection* $\mathbb{P}_h^{loc} : C[0, T; L^2(\Omega)] \to \mathcal{Z}^h$ *satisfies*
$$\mathbb{P}_h^{loc}\mathbf{u} \in \mathcal{Z}^h \qquad \text{and} \qquad (\mathbb{P}_h^{loc}\mathbf{u})|_{(t^{n-1},t^n]} = \mathbb{P}_n^{loc}(\mathbf{u}|_{[t^{n-1},t^n]}).$$
3. $\mathbb{P}_h : H^1[0, T; H^{-1}(\Omega)] \cap L^2[0, T; H_0^1(\Omega)] \to \mathcal{Z}^h$ *is the velocity component of the discontinuous Galerkin solution* $(\mathbf{u}_h, r_h)$ *of (4.3) with* $\mathbf{f} = \mathbf{u}_t - \nu\Delta\mathbf{u}$, *and initial data* $\mathbf{u}_-^0 = P_0(\mathbf{u}(0))$.

The projections $\mathbb{P}_n^{loc}$ are local in time, and the approximation error of the DG scheme for the Navier Stokes equations will ultimately be bounded by projection errors of the form $\mathbf{u} - \mathbb{P}_n^{loc}\mathbf{u}$. The following theorem, adapted from [9], bounds this error.

THEOREM 4.3. *Let* $Z_h^n \subset H_0^1(\Omega)$ *be a finite dimensional subspace and* $P_n : L^2(\Omega) \to Z_h^n$ *be the orthogonal projection. Let* $(.,.)_V$ *be a (semi) inner product on* $H_0^1(\Omega)$, *then there exists a constant* C_k, *depending only upon* $k \geq 1$, *such that*
$$\|\mathbf{u} - \mathbb{P}_n^{loc}\mathbf{u}\|_{L^p[t^{n-1},t^n;V]} \leq \|\mathbf{u} - P_n\mathbf{u}\|_{L^p[t^{n-1},t^n;V]} + C_k(t^n - t^{n-1})^{k+1}\|P_n\mathbf{u}^{(k+1)}\|_{L^p[t^{n-1},t^n;V]},$$
for all $1 \leq p \leq \infty$. *In particular, if the restriction of* P_n *to* $(V, \|.\|_V)$ *is stable, in the sense that* $\|P_n\mathbf{u}\|_V \leq C\|\mathbf{u}\|_V$, *then*
$$\|\mathbf{u} - \mathbb{P}_n^{loc}\mathbf{u}\|_{L^p[t^{n-1},t^n;V]} \leq C\|\mathbf{u} - \mathbf{w}_h\|_{L^p[t^{n-1},t^n;V]} + C(t^n - t^{n-1})^{k+1}\|\mathbf{u}^{(k+1)}\|_{L^p[t^{n-1},t^n;V]},$$
for any $\mathbf{w}_h \in L^p[t^{n-1}, t^n; Z_h^n]$. *Here* $\mathbf{u}^{(k+1)}$ *denotes the* $(k+1)^{th}$ *time-derivative of* $\mathbf{u}$.

Approximation properties of P_n depend crucially upon the inf-sup condition for the pair (U_h^n, Q_h^n). We briefly recall the development of the projection estimates for the Stokes projection since similar arguments will be developed in the parabolic setting.

LEMMA 4.4. *Let* $a : H_0^1(\Omega) \times H_0^1(\Omega) \to \mathbb{R}$ *and* $b : H_0^1(\Omega) \times L_0^2(\Omega) \to \mathbb{R}$ *be the bilinear forms introduced in Section 2, and let the subspaces* $U_h^n \subset H_0^1(\Omega)$ *and* $Q_h^n \subset L_0^2(\Omega)$ *satisfy the inf-sup condition. Define the Stokes projection* $\Pi_n : H_0^1(\Omega) \to U_h^n$ *to be the first component of the pair* $(\mathbf{u}_h, r_h) \in U_h^n \times Q_h^n$ *satisfying*
$$a(\mathbf{u}_h, \mathbf{v}_h) + b(\mathbf{v}_h, r_h) = a(\mathbf{u}, \mathbf{v}_h), \qquad b(\mathbf{u}_h, q_h) = 0,$$
for all $(\mathbf{v}_h, q_h) \in U_h^n \times Q_h^n$. *If* $\mathbf{u} \in V(\Omega)$ *(that is* $div(\mathbf{u}) = 0$*), then there is a constant* $C > 0$ *independent of* ν *such that*
$$\|\mathbf{u} - \Pi_n\mathbf{u}\|_{H^1(\Omega)} \leq C \inf_{\mathbf{v}_h \in U_h^n} \|\mathbf{u} - \mathbf{v}_h\|_{H^1(\Omega)}.$$
If additionally Ω *is sufficiently regular to admit* $H^2(\Omega) \times H^1(\Omega)$ *regularity of the Stokes operator and* (U_h^n, Q_h^n) *contain the piecewise linear functions over a regular mesh, then*
$$\|\mathbf{u} - \Pi_n\mathbf{u}\|_{L^2(\Omega)} \leq Ch \inf_{\mathbf{v}_h \in U_h^n} \|\mathbf{u} - \mathbf{v}_h\|_{H^1(\Omega)},$$

where h is the diameter of the largest element of the mesh.

Since $\|\mathbf{u} - P_n\mathbf{u}\|_{L^2(\Omega)} \le \|\mathbf{u} - \Pi_n\mathbf{u}\|_{L^2(\Omega)}$ it follows that the projection error $\mathbf{u} - P_n\mathbf{u}$ will exhibit optimal rates of convergence in $L^2(\Omega)$. We summarize this and other properties needed for the analysis below in the following corollary.

COROLLARY 4.5. *Under the hypotheses of the lemma (in particular, $\mathbf{u} \in V(\Omega)$),*

1. $\|\mathbf{u} - P_n\mathbf{u}\|_{L^2(\Omega)} \le Ch \inf_{\mathbf{v}_h \in U_h^n} \|\mathbf{u} - \mathbf{v}_h\|_{H^1(\Omega)}$.
2. *If the mesh is quasi-uniform so that the inverse estimate $\|\mathbf{u}_h\|_{H^1(\Omega)} \le (C/h)\|\mathbf{u}_h\|_{L^2(\Omega)}$ holds for $\mathbf{u}_h \in U_h^n$, then*

$$\|\mathbf{u} - P_n\mathbf{u}\|_{H^1(\Omega)} \le C \inf_{\mathbf{v}_h \in U_h^n} \|\mathbf{u} - \mathbf{v}_h\|_{H^1(\Omega)}.$$

3. *If (U_h^m, Q_h^m) and (U_h^n, Q_h^n) are two spaces satisfying the inf-sup condition over two meshes, then*

$$\|P_n(I - P_m)\mathbf{u}\|_{H^{-1}(\Omega)} \le Ch\|\mathbf{u} - P_m\mathbf{u}\|_{L^2(\Omega)} \le Ch^2 \inf_{\mathbf{v}_h \in U_h^m} \|\mathbf{u} - \mathbf{v}_h\|_{H^1(\Omega)},$$

where h is the maximal diameter of an element in the two meshes.

Proofs of Lemma 4.4 and Corollary 4.5 are sketched in Appendix B

4.2. Projection Error in $L^\infty[0,T;L^2(\Omega)] \cap L^2[0,T;H^1(\Omega)]$. The following theorem estimates the error at the partition points and is the analog of [7, Theorem 2.2].

THEOREM 4.6. *Let $\mathbf{u} \in H^1[0,T;H^{-1}(\Omega)] \cap L^2[0,T;H_0^1(\Omega)]$ and $\mathbf{u}_h = \mathbb{P}_h\mathbf{u}$. Then the error $\mathbf{e}_h = \mathbb{P}_h^{loc}\mathbf{u} - \mathbf{u}_h$ satisfies*

$$\begin{aligned}\|\mathbf{e}_{h-}^n\|_{L^2(\Omega)}^2 + \nu \int_0^{t^n} \|\mathbf{e}_h\|_{H^1(\Omega)}^2 + 1/2 \sum_{i=0}^{n-1} |[\mathbf{e}_h^i]|_{L^2(\Omega)}^2 \le C\Big(\int_0^{t^n} \nu \|(I - \mathbb{P}_h^{loc})\mathbf{u}\|_{H^1(\Omega)}^2 \\ + \sum_{i=0}^{n-1} \min\left(C_k/(\Delta t^{i+1}\nu) \left\|P_{i+1}(I - P_i)\mathbf{u}(t^i)\right\|_{H^{-1}(\Omega)}^2, \left\|(I - P_i)\mathbf{u}(t^i)\right\|_{L^2(\Omega)}^2\right)\Big),\end{aligned}$$

where C depends only upon Ω and C_k depends only upon k. Here $[\mathbf{e}_h^i] = \mathbf{e}_{h+}^i - \mathbf{e}_{h-}^i$ is the jump in $\mathbf{e}_h$ at t^i, $\Delta t^i \equiv t^i - t^{i-1}$, and $P_i : L^2(\Omega) \to Z_h^i$ is the orthogonal projection onto the discretely divergence free space.

Proof. Substituting $\mathbf{f}(\mathbf{v}) = \langle \mathbf{u}_t, \mathbf{v}\rangle + a(\mathbf{u}, \mathbf{v})$ into equation (4.3) gives the orthogonality relation for the total error $\mathbf{e} = \mathbf{u} - \mathbf{u}_h$,

$$\begin{cases} (\mathbf{e}_-^n, \mathbf{v}_-^n) + \displaystyle\int_{t^{n-1}}^{t^n} \Big(- \langle \mathbf{e}, \mathbf{v}_{ht}\rangle + a(\mathbf{e}, \mathbf{v}_h) - b(\mathbf{v}_h, r_h)\Big) - (\mathbf{e}_-^{n-1}, \mathbf{v}_+^{n-1}) = 0, \\ \displaystyle\int_{t^{n-1}}^{t^n} b(\mathbf{u}_h, q_h) = 0. \end{cases} \tag{4.4}$$

for all $(\mathbf{v}_h, q_h) \in \mathcal{P}_k[t^{n-1}, t^n; U_h^n] \times \mathcal{P}_k[t^{n-1}, t^n; Q_h^n]$.

Write $\mathbf{e} = (\mathbf{u} - \mathbb{P}_n^{loc}\mathbf{u}) + (\mathbb{P}_n^{loc}\mathbf{u} - \mathbf{u}_h) \equiv \mathbf{e}_p + \mathbf{e}_h$ and note that $\mathbf{e}_h \in \mathcal{P}_k[t^{n-1}, t^n; Z_h^n]$. Rearranging the orthogonality relation (4.4), and using properties of $\mathbb{P}_n^{loc}$, shows

$$(\mathbf{e}_{h-}^n, \mathbf{v}_-^n) - \int_{t^{n-1}}^{t^n} \Big(\langle \mathbf{e}_h, \mathbf{v}_{ht}\rangle + a(\mathbf{e}_h, \mathbf{v}_h)\Big) - (\mathbf{e}_{h-}^{n-1}, \mathbf{v}_+^{n-1}) = ((I - P_{n-1})\mathbf{u}(t^{n-1}), \mathbf{v}_+^{n-1}) + \int_{t^{n-1}}^{t^n} \Big((\mathbf{e}_p, \mathbf{v}_{ht}) - a(\mathbf{e}_p, \mathbf{v}_h) + b(\mathbf{v}_h, r_h)\Big), \tag{4.5}$$

provided $\mathbf{v}_-^n \in Z_h^n$. Setting $\mathbf{v}_h = \mathbf{e}_h$ then shows

$$(1/2)\|\mathbf{e}_{h-}^n\|^2 + \nu C \int_{t^{n-1}}^{t^n} \|\mathbf{e}_h\|_{H^1(\Omega)}^2 + (1/2)\|[\mathbf{e}_h^{n-1}]\|_{L^2(\Omega)}^2 \le (1/2)\|\mathbf{e}_{h-}^{n-1}\|_{L^2(\Omega)}^2 + ((I - P_{n-1})\mathbf{u}(t^{n-1}), \mathbf{e}_{h+}^{n-1}) - \int_{t^{n-1}}^{t^n} a(\mathbf{e}_p, \mathbf{e}_h). \tag{4.6}$$

Here we used the fact that $\mathbf{e}_h(t) \in Z_h^n$ so that $b(\mathbf{e}_h, r_h) = 0$, and $\mathbf{e}_{ht} \in \mathcal{P}_{k-1}[t^{n-1}, t^n, Z_h^n]$ so $\int_{t^{n-1}}^{t^n} (\mathbf{e}_p, \mathbf{e}_{ht}) = \int_{t^{n-1}}^{t^n} ((I - \mathbb{P}_n^{loc})\mathbf{u}, \mathbf{e}_{ht}) = 0$. The last term on the second line is bounded using the Cauchy-Schwarz inequality, and the first term is bounded in two different ways. Since $\mathbf{e}_{h-}^{n-1} \in U_h^{n-1}$ an estimate independent of ν is computed as

$$\begin{aligned}((I - P_{n-1})\mathbf{u}(t^{n-1}), \mathbf{e}_{h+}^{n-1}) &= ((I - P_{n-1})\mathbf{u}(t^{n-1}), \mathbf{e}_{h+}^{n-1} - \mathbf{e}_{h-}^{n-1}) \\ &\le \|(I - P_{n-1})\mathbf{u}(t^{n-1})\|_{L^2(\Omega)}^2 + (1/4)\|[\mathbf{e}_h^{n-1}]\|_{L^2(\Omega)}^2,\end{aligned}$$

Since $\mathbf{e}_{h+}^{n-1} \in U_h^n$, an alternative estimate is obtained upon writing

$$\begin{aligned}((I - P_{n-1})\mathbf{u}(t^{n-1}), \mathbf{e}_{h+}^{n-1}) &= (P_n(I - P_{n-1})\mathbf{u}(t^{n-1}), \mathbf{e}_{h+}^{n-1}) \\ &\le \|P_n(I - P_{n-1})\mathbf{u}(t^{n-1})\|_{H^{-1}(\Omega)} \|\mathbf{e}_{h+}^{n-1}\|_{H^1(\Omega)}\end{aligned}$$

To estimate $\|\mathbf{e}_{h+}^{n-1}\|_{H^1(\Omega)}$ we use the following trace estimate for functions in $\mathcal{P}_k[t^{n-1}, t^n, U_h^n]$,

$$\|\mathbf{e}_{h+}^{n-1}\|_{H^1(\Omega)}^2 \le (C_k/\Delta t^n) \int_{t^{n-1}}^{t^n} \|\mathbf{e}_h\|_{H^1(\Omega)}^2.$$

The finite dimensionality of $\mathcal{P}_k(t^{n-1}, t^n)$, and a scaling argument, shows that the constant C_k depends only upon k. Therefore,

$$((I - P_{n-1})\mathbf{u}(t^{n-1}), \mathbf{e}_{h+}^{n-1}) \le (C_k/\Delta t^n \nu)\|P_n(I - P_{n-1})\mathbf{u}(t^{n-1})\|_{H^{-1}(\Omega)}^2 + (\nu/4) \int_{t^{n-1}}^{t^n} \|\mathbf{e}_h\|_{H^1(\Omega)}^2.$$

Substituting these estimates into equation (4.6) and summing completes the proof. □

The previous theorem bounds the $L^2(\Omega)$ norm of the error at the partition points. A bootstrap argument is used next to obtain bounds at arbitrary times $t \in (0, T)$.

THEOREM 4.7. *Let* $\mathbf{u} \in H^1[0, T; H^{-1}(\Omega)] \cap L^2[0, T; H_0^1(\Omega)]$ *and* $\mathbf{u}_h = \mathbb{P}_h\mathbf{u}$. *Then the error* $\mathbf{e}_h = \mathbb{P}_h^{loc}\mathbf{u} - \mathbf{u}_h$ *satisfies*

$$\|\mathbf{e}_h(t)\|_{L^2(\Omega)}^2 + \nu \int_0^{t^n} \|\mathbf{e}_h\|_{H^1(\Omega)}^2 + \sum_{i=0}^{n-1} \|[\mathbf{e}_h^i]\|_{L^2(\Omega)} \le C\Big(\int_0^{t^n} \nu\|(I - \mathbb{P}_h^{loc})\mathbf{u}\|_{H^1(\Omega)}^2 + \sum_{i=0}^{n-1} \min\big(1/(\Delta t^{i+1}\nu)\left\|P_{i+1}(I - P_i)\mathbf{u}(t^i)\right\|_{H^{-1}(\Omega)}^2, \left\|(I - P_i)\mathbf{u}(t^i)\right\|_{L^2(\Omega)}^2\big)\Big),$$

where C is a constant depending only on k and the domain Ω. Here $[\mathbf{e}_h^i] = \mathbf{e}_{h+}^i - \mathbf{e}_{h-}^i$ is the jump in $\mathbf{e}_h$ at t^i, $\Delta t^i \equiv t^i - t^{i-1}$, and $P_i : L^2(\Omega) \to Z_h^i$ is the orthogonal projection onto the discretely divergence free space.

Proof. Given Theorem 4.6 it suffices to bound $\|\mathbf{e}_h(t)\|_{L^2(\Omega)}$ for $t \in (t^{n-1}, t^n]$. Adopting the notation in the previous proof, we begin with equality (4.5)

$$(\mathbf{e}_{h-}^n, \mathbf{v}_-^n) - \int_{t^{n-1}}^{t^n} \Big(\langle \mathbf{e}_h, \mathbf{v}_{ht}\rangle + a(\mathbf{e}_h, \mathbf{v}_h)\Big) - (\mathbf{e}_{h-}^{n-1}, \mathbf{v}_+^{n-1}) = ((I - P_{n-1})\mathbf{u}(t^{n-1}), \mathbf{v}_+^{n-1})$$
$$+ \int_{t^{n-1}}^{t^n} \Big((\mathbf{e}_p, \mathbf{v}_{ht}) - a(\mathbf{e}_p, \mathbf{v}_h) + b(\mathbf{v}_h, r_h)\Big),$$

and integrate parts in time to get

$$\int_{t^{n-1}}^{t^n} \Big(\langle \mathbf{e}_{ht}, \mathbf{v}_h\rangle + a(\mathbf{e}_h, \mathbf{v}_h)\Big) + (\mathbf{e}_{h+}^{n-1} - \mathbf{e}_{h-}^{n-1}, \mathbf{v}_+^{n-1}) = ((I - P_{n-1})\mathbf{u}(t^{n-1}), \mathbf{v}_+^{n-1})$$
$$+ \int_{t^{n-1}}^{t^n} \Big((\mathbf{e}_p, \mathbf{v}_{ht}) - a(\mathbf{e}_p, \mathbf{v}_h) + b(\mathbf{v}_h, r_h)\Big).$$

Letting $\mathbf{v}_h = \tilde{\mathbf{e}} \in \mathcal{P}_k[t^{n-1}, t^n; Z_h^n]$ be the discrete approximation of $\chi_{[t^{n-1},t)}\mathbf{e}_h$ characterized by equation (3.1) gives

$$\int_{t^{n-1}}^{t} (\mathbf{e}_{ht}, \tilde{\mathbf{e}}) + (\mathbf{e}_{h+}^{n-1} - \mathbf{e}_{h-}^{n-1}, \mathbf{e}_{h+}^{n-1}) = ((I - P_{n-1})\mathbf{u}(t^{n-1}), \mathbf{e}_{h+}^{n-1}) - \int_{t^{n-1}}^{t^n} \Big(a(\mathbf{e}_p, \tilde{\mathbf{e}}) + a(\mathbf{e}_h, \tilde{\mathbf{e}})\Big).$$

As in the previous proof, $\tilde{\mathbf{e}}(t) \in Z_h^n$ so that $b(\tilde{\mathbf{e}}, r_h) = 0$, and $\tilde{\mathbf{e}}_t \in \mathcal{P}_{k-1}[t^{n-1}, t^n, Z_h^n]$ so $\int_{t^{n-1}}^{t^n} (\mathbf{e}_p, \tilde{\mathbf{e}}_t) = \int_{t^{n-1}}^{t^n} ((I - \mathbb{P}_n^{loc})\mathbf{u}, \tilde{\mathbf{e}}_t) = 0$. Since $\mathbf{e}_{ht} \in \mathcal{P}_{k-1}[t^{n-1}, t^n, Z_h^n]$ the defining property of $\tilde{\mathbf{e}}$ allows the first term to be integrated exactly,

$$(1/2)\|\mathbf{e}_h(t)\|_{L^2(\Omega)}^2 + (1/2)\|[\mathbf{e}_h^{n-1}]\|_{L^2(\Omega)}^2 - (1/2)\|\mathbf{e}_{h-}^{n-1}\|_{L^2(\Omega)}$$
$$= ((I - P_{n-1})\mathbf{u}(t^{n-1}), \mathbf{e}_{h+}^{n-1}) - \int_{t^{n-1}}^{t^n} \Big(a(\mathbf{e}_p, \tilde{\mathbf{e}}) + a(\mathbf{e}_h, \tilde{\mathbf{e}})\Big).$$

Using Lemma 3.1 to bound $\tilde{\mathbf{e}}$ in terms of $\mathbf{e}_h$ we obtain

$$(1/2)\|\mathbf{e}_h(t)\|_{L^2(\Omega)}^2 + (1/2)\|[\mathbf{e}_h^{n-1}]\|_{L^2(\Omega)} - (1/2)\|\mathbf{e}_{h-}^{n-1}\|_{L^2(\Omega)}^2$$
$$\le ((I - P_{n-1})\mathbf{u}(t^{n-1}), \mathbf{e}_{h+}^{n-1}) + \nu C_k \int_{t^{n-1}}^{t^n} \Big(\|\mathbf{e}_p\|_{H^1(\Omega)}^2 + \|\mathbf{e}_h\|_{H^1(\Omega)}^2\Big).$$

As in the proof of Theorem 4.6 the first term on the right can be bounded by

$$\|(I - P_{n-1})\mathbf{u}(t^{n-1})\|_{L^2(\Omega)}^2 + (1/4)\|[\mathbf{e}_h^{n-1}]\|_{L^2(\Omega)}^2,$$

or

$$C_k/(\Delta t^n \nu)\|P_n(I - P_{n-1})\mathbf{u}(t^{n-1})\|_{H^{-1}(\Omega)}^2 + (\nu/4)\int_{t^{n-1}}^{t^n} \|\mathbf{e}_h\|_{H^1(\Omega)}^2,$$

with C_k depending only on k. The theorem then follows upon using Theorem 4.6 to bound $\|\mathbf{e}_{h-}^{n-1}\|_{L^2(\Omega)}$ and $\|\mathbf{e}_h\|_{L^2[0,T;H^1(\Omega)]}^2$. □

Combining the above with the triangle inequality provides various estimates for the projection error $\mathbf{u} - \mathbb{P}_h\mathbf{u}$ in terms of the local projection errors $\mathbf{u} - \mathbb{P}_h^{loc}\mathbf{u}$. The following corollary bounds the energy and $L^2(\Omega)$ norms which appear in the estimates for the Navier-Stokes equation below.

COROLLARY 4.8. *Under the hypotheses of the theorem there exists a constant $C = C(k, \Omega)$ such that*

$$\|\mathbf{u} - \mathbb{P}_h\mathbf{u}\|^2_{L^2[0,T;H^1(\Omega)]} \le C\|\mathbf{u} - \mathbb{P}_h^{loc}\mathbf{u}\|^2_{L^2[0,T;H^1(\Omega)]}$$
$$+(C/\nu)\sum_{i=0}^{n-1}\min\left(1/(\Delta t^{i+1}\nu)\left\|P_{i+1}(I-P_i)\mathbf{u}(t^i)\right\|^2_{H^{-1}(\Omega)}, \left\|(I-P_i)\mathbf{u}(t^i)\right\|^2_{L^2(\Omega)}\right),$$

and

$$\|\mathbf{u} - \mathbb{P}_h\mathbf{u}\|_{L^\infty[0,T;L^2(\Omega)]} \le \|\mathbf{u} - \mathbb{P}_h^{loc}\mathbf{u}\|_{L^\infty[0,T;L^2(\Omega)]} + C\sqrt{\nu}\|\mathbf{u} - \mathbb{P}_h^{loc}\mathbf{u}\|_{L^2[0,T;H^1(\Omega)]}$$
$$+C\left(\sum_{i=0}^{n-1}\min\left(1/(\Delta t^{i+1}\nu)\left\|P_{i+1}(I-P_i)\mathbf{u}(t^i)\right\|^2_{H^{-1}(\Omega)}, \left\|(I-P_i)\mathbf{u}(t^i)\right\|^2_{L^2(\Omega)}\right)\right)^{1/2}.$$

4.3. Rates of Convergence. Corollary 4.8 shows that the projection error $(I - \mathbb{P}_h)\mathbf{u}$ can be bounded by local projection errors of the form $(I - \mathbb{P}_h^{loc})\mathbf{u}$ and jumps in the projections at the discrete times $\{t^n\}$. In this section we verify that the projections $\mathbb{P}_h\mathbf{u}$ onto classical finite element subspaces will exhibit optimal rates of convergence when the solution $\mathbf{u}$ is sufficiently smooth.

THEOREM 4.9. *Assume the subspaces (U_h^n, Q_h^n) satisfy the inf-sup condition and approximation and inverse properties in Assumption 1 with constants independent of n and h, and assume that the time steps are quasi-uniform in the sense that there exists $0 \le \theta \le 1$ such that $\min_{1\le n\le N}\Delta t^n \le \theta \max_{1\le n\le N}\Delta t^n$. If $\mathbf{u} \in C[0,T;H^{\ell+1}(\Omega)\cap V(\Omega)]\cap H^{k+1}[0,T;H_0^1(\Omega)]$ then there exists a constant $C > 0$ independent of h, τ, and ν such that*

$$\|\mathbf{u} - \mathbb{P}_h\mathbf{u}\|_{L^2[0,T;H^1(\Omega)]} \le C\Big(\|\mathbf{u}\|_{L^2[0,T;H^{\ell+1}(\Omega)]}h^\ell + \|\mathbf{u}^{(k+1)}\|_{L^2[0,T;H^1(\Omega)]}\tau^{k+1}$$
$$+\|\mathbf{u}\|_{C[0,T;H^{\ell+1}(\Omega)]}\min\left(h^2/(\tau\nu), h/\sqrt{\tau\nu}\right)h^\ell\Big).$$

and

$$\|\mathbf{u} - \mathbb{P}_h\mathbf{u}\|_{L^\infty[0,T;L^2(\Omega)]} \le C\Big(\left(\sqrt{\nu}\|\mathbf{u}\|_{L^2[0,T;H^{\ell+1}(\Omega)]} + h\|\mathbf{u}\|_{L^\infty[0,T;H^{\ell+1}(\Omega)]}\right)h^\ell$$
$$+\left(\sqrt{\nu}\|\mathbf{u}^{(k+1)}\|_{L^2[0,T;H^1(\Omega)]} + \|\mathbf{u}^{(k+1)}\|_{L^2[0,T;L^2(\Omega)]}\right)\tau^{k+1}$$
$$+\|\mathbf{u}\|_{C[0,T;H^{\ell+1}(\Omega)]}\min\left(h^{3/2}/(\tau\sqrt{\nu}), \sqrt{h/\tau}\right)h^{\ell+1/2}\Big).$$

where $\tau = \max_{1\le n\le N}\Delta t^n$, and $\mathbf{u}^{(k+1)}$ denotes the $(k+1)^{st}$ time derivative of $\mathbf{u}$.

Proof. Theorem 4.3 and the approximation properties of the projections P_n given in Corollary 4.5 show

$$\|\mathbf{u} - \mathbb{P}_n^{loc}\mathbf{u}\|_{L^2[t^{n-1},t^n;H^1(\Omega)]} \le C\|\mathbf{u} - P_n\mathbf{u}\|_{L^2[t^{n-1},t^n;H^1(\Omega)]} + C(\Delta t^n)^{k+1}\|\mathbf{u}^{(k+1)}\|_{L^2[t^{n-1},t^n;H^1(\Omega)]}$$
$$\le C\|\mathbf{u}\|_{L^2[t^{n-1},t^n;H^{\ell+1}(\Omega]}h^\ell + C\tau^{k+1}\|\mathbf{u}^{(k+1)}\|_{L^2[t^{n-1},t^n;H^1(\Omega)]},$$

which shows

$$\|\mathbf{u}-\mathbb{P}_h^{loc}\mathbf{u}\|_{L^2[0,T;H^1(\Omega)]} \le C\Big(\|\mathbf{u}\|_{L^2[0,T;H^{\ell+1}(\Omega)]}h^\ell + \|\mathbf{u}^{(k+1)}\|_{L^2[0,T;H^1(\Omega)]}\tau^{k+1}\Big).$$

The estimates in Corollary 4.5 are used to estimate the jump terms.

$$\begin{aligned}\sum_{i=0}^{n-1}\min\left(1/(\Delta t^{i+1}\nu)\left\|P_{i+1}(I-P_i)\mathbf{u}(t^i)\right\|^2_{H^{-1}(\Omega)}, \left\|(I-P_i)\mathbf{u}(t^i)\right\|^2_{L^2(\Omega)}\right)\\ \le C\sum_{i=0}^{n-1}\min\left((1/\Delta t^{i+1}\nu)h^{2(\ell+2)}, h^{2(\ell+1)}\right)\|\mathbf{u}(t^i)\|^2_{H^{\ell+1}(\Omega)}\\ \le C(T,\theta)\|\mathbf{u}\|^2_{C[0,T;H^{\ell+1}(\Omega)]}\min\left(h^4/(\tau^2\nu), h^2/\tau\right)h^{2\ell}.\end{aligned}$$

Combining these two estimates with the bound in Corollary 4.8 establishes the bound in $L^2[0,T;H^1(\Omega)]$.

The first estimate of Theorem 4.3 shows

$$\begin{aligned}\|\mathbf{u}-\mathbb{P}_n^{loc}\mathbf{u}\|_{L^\infty[t^{n-1},t^n;L^2(\Omega)]} &\le \|\mathbf{u}-P_n\mathbf{u}\|_{L^\infty[t^{n-1},t^n;L^2(\Omega)]} + C_k(\Delta t^i)^{k+1}\|\mathbf{u}^{(k+1)}\|_{L^\infty[t^{n-1},t^n;L^2(\Omega)]}\\ &\le C\|\mathbf{u}\|_{L^\infty[t^{n-1},t^n;H^{\ell+1}(\Omega]}h^{\ell+1} + C\tau^{k+1}\|\mathbf{u}^{(k+1)}\|_{L^\infty[t^{n-1},t^n;L^2(\Omega)]}.\end{aligned}$$

Combining this with the above establishes the second statement of the theorem. □

4.4. Stability in $L^\infty[0,T;H^1(\Omega)]$. For the nonlinear problem, we will need a bound on $\|\mathbb{P}_h\mathbf{u}\|_{L^\infty[0,T;H^1(\Omega)]}$ when $\mathbf{u}_t-\nu\Delta\mathbf{u}\in L^2[0,T;L^2(\Omega)]$. That is, discontinuous Galerkin approximations of Stokes problem are bounded in $L^\infty[0,T;H^1(\Omega)]$ when $\mathbf{f}\in L^2[0,T;W(\Omega)]$ and $\mathbf{u}_0\in V(\Omega)$. The following proof uses the exponential interpolant of Definition 3.3.

THEOREM 4.10. *Let $\mathbf{u}_h$ be a solution of the discontinuous Galerkin scheme (4.3) with data $\mathbf{f}\equiv\mathbf{u}_t-\nu\Delta\mathbf{u}\in L^2[0,T;W(\Omega)]$ and $\mathbf{u}_0\in V(\Omega)$. Then,*

$$\|\mathbf{u}_h\|_{L^\infty[0,T;H^1(\Omega)]} \le C\left(\|\mathbf{u}_0\|_{H^1(\Omega)} + (1/\sqrt{\nu})\|\mathbf{f}\|_{L^2[0,T;L^2(\Omega)]}\right)$$

where C depends only on k and the domain Ω.

Proof. We construct a discrete approximation of the $\Delta\mathbf{u}_h$. At each time $t\in[t^{n-1},t^n)$ let $\mathbf{a}_p\in Z_h^n$ be the discrete approximation of $\Delta\mathbf{u}_h$ satisfying

$$(\mathbf{a}_p,\mathbf{w}_h) = (1/\nu)a(\mathbf{u}_h,\mathbf{w}_h), \qquad \forall\,\mathbf{w}_h\in Z_h^n.$$

Then $\mathbf{a}_p\in\mathcal{P}_k[t^{n-1},t^n;Z_h^n]$, and setting $\mathbf{w}_h(.)=\mathbf{u}_{ht}(.)\in Z_h^n$ and $\mathbf{w}_h(.)=\mathbf{a}_p(.)\in Z_h^n$ respectively, we obtain

$$(1/2)\frac{d}{dt}\|\nabla\mathbf{u}_h\|^2_{L^2(\Omega)} = (1/\nu)a(\mathbf{u}_h,\mathbf{u}_{ht}) = (\mathbf{a}_p,\mathbf{u}_{ht}) \quad\text{and}\quad a(\mathbf{u}_h,\mathbf{a}_p) = \nu\|\mathbf{a}_p\|^2_{L^2(\Omega)}. \tag{4.7}$$

Setting $\mathbf{v}_h=\mathbf{a}_p$ into equation (4.3), and using the last two equalities, gives

$$\begin{aligned}(1/2)\|\nabla\mathbf{u}^n_{h-}\|^2_{L^2(\Omega)} + (1/2)\|[\nabla\mathbf{u}^{n-1}_h]\|^2_{H^1(\Omega)} + \nu\int_{t^{n-1}}^{t^n}\|\mathbf{a}_p\|^2_{L^2(\Omega)}\\ = (1/2)\|\nabla\mathbf{u}^{n-1}_{h-}\|^2_{L^2(\Omega)} + \int_{t^{n-1}}^{t^n}(\mathbf{f},\mathbf{a}_p).\end{aligned}$$

Using Cauchy-Schwarz and Young inequalities for the last term we obtain the bound,

$$\|\nabla \mathbf{u}_{h-}^n\|_{L^2(\Omega)}^2 + \|[\nabla \mathbf{u}_h^{n-1}]\|_{L^2(\Omega)}^2 + (\nu/2)\int_{t^{n-1}}^{t^n} \|\mathbf{a}_p\|_{L^2(\Omega)}^2 \tag{4.8}$$
$$\leq \|\nabla \mathbf{u}_{h-}^{n-1}\|_{L^2(\Omega)}^2 + (1/\nu)\int_{t^{n-1}}^{t^n} \|\mathbf{f}\|_{L^2(\Omega)}^2.$$

This inequality implies the desired bound at partition points, and for the low order schemes, $k = 0$ and $k = 1$, also bounds $\|\mathbf{u}_h\|_{L^\infty[0,T;H^1(\Omega)]}$. For higher order schemes, we obtain an additional estimate using the discrete approximation of $\mathbf{a}_p e^{-\lambda(t-t^{n-1})}$. Let $\bar{\mathbf{a}}_p \in \mathcal{P}_k[t^{n-1}, t^n; Z_h^n]$, be the discrete approximation of $\mathbf{a}_p e^{-\lambda(t-t^{n-1})}$ constructed in Definition 3.3. Then

$$\int_{t^{n-1}}^{t^n} (\mathbf{u}_{ht}, \bar{\mathbf{a}}_p) = \int_{t^{n-1}}^{t^n} (\mathbf{u}_{ht}, \mathbf{a}_p) e^{-\lambda(t-t^{n-1})} = (1/2)\int_{t^{n-1}}^{t^n} e^{-\lambda(t-t^{n-1})} \frac{d}{dt}\|\nabla \mathbf{u}_h\|_{L^2(\Omega)}^2,$$

where at the last equality used (4.7). Integrating equation (4.3) by parts, setting $\mathbf{v}_h = \bar{\mathbf{a}}_p$, using the definition of $\mathbf{a}_p$, and Lemma 3.6, gives

$$(1/2)\|\nabla \mathbf{u}_{h-}^n\|_{L^2(\Omega)}^2 e^{-\lambda(t^n-t^{n-1})} + (1/2)\|[\nabla \mathbf{u}_h^{n-1}]\|_{L^2(\Omega)}^2 + \int_{t^{n-1}}^{t^n} (\lambda/2) e^{-\lambda(t-t^{n-1})} \|\nabla \mathbf{u}_h\|_{L^2(\Omega)}^2$$
$$\leq (1/2)\|\nabla \mathbf{u}_{h-}^{n-1}\|_{L^2(\Omega)}^2 + \int_{t^{n-1}}^{t^n} \big|(\mathbf{f}, \bar{\mathbf{a}}_p)\big| + \big|a(\mathbf{u}_h, \bar{\mathbf{a}}_p)\big|$$
$$\leq (1/2)\|\nabla \mathbf{u}_{h-}^{n-1}\|_{L^2(\Omega)}^2 + \int_{t^{n-1}}^{t^n} (1/4\nu)\|\mathbf{f}\|_{L^2(\Omega)}^2 + \nu\|\mathbf{a}_p\|_{L^2(\Omega)}^2 + \nu(\mathbf{a}_p, \bar{\mathbf{a}}_p)$$
$$\leq (1/2)\|\nabla \mathbf{u}_{h-}^{n-1}\|_{L^2(\Omega)}^2 + \int_{t^{n-1}}^{t^n} (1/4\nu)\|\mathbf{f}\|_{L^2(\Omega)}^2 + (1 + C_k)\nu\|\mathbf{a}_p\|_{L^2(\Omega)}^2.$$

Using equation (4.8) to bound the terms on the right, it follows that

$$\int_{t^{n-1}}^{t^n} \lambda e^{-\lambda(t-t^{n-1})} \|\nabla \mathbf{u}_h\|_{L^2(\Omega)}^2 \leq C\Big(\|\mathbf{u}_h^0\|_{H^1(\Omega)}^2 + (1/\nu)\int_0^{t^n} \|\mathbf{f}\|_{L^2(\Omega)}^2\Big),$$

where C depends on domain and k (but not on λ). Selecting $\lambda = 1/(t^n - t^{n-1})$ shows

$$\int_{t^{n-1}}^{t^n} e^{-1}/(t^n - t^{n-1}) \|\nabla \mathbf{u}_h\|_{L^2(\Omega)}^2 \leq C\Big(\|\mathbf{u}_h^0\|_{H^1(\Omega)}^2 + (1/\nu)\int_0^{t^n} \|f\|_{L^2(\Omega)}^2\Big).$$

Since $\nabla \mathbf{u}_h \in \mathcal{P}_k[t^{n-1}, t^n; L^2(\Omega)]$, Lemma 3.5 (with weight $w(t) \equiv 1$ and $p = \infty$) states

$$\|\nabla \mathbf{u}_h\|_{L^\infty[t^{n-1},t^n;L^2(\Omega)]}^2 \leq (C_k/(t^n - t^{n-1}))\|\nabla \mathbf{u}_h\|_{L^2[t^{n-1},t^n;L^2(\Omega)]}^2,$$

and the theorem follows. □

4.5. Estimates for the Stokes Problem. In this section we sketch how the arguments used in this section also provide error estimates for DG approximations of the Stokes problem. These results are of independent interest and are not used below.

The orthogonality condition for (4.3) takes the form.

$$\begin{cases} (\mathbf{e}_-^n, \mathbf{v}_-^n) + \displaystyle\int_{t^{n-1}}^{t^n} \Big(- \langle \mathbf{e}, \mathbf{v}_{ht} \rangle + a(\mathbf{e}, \mathbf{v}_h) + b(\mathbf{v}_h, p - p_h) \Big) - (\mathbf{e}_-^{n-1}, \mathbf{v}_+^{n-1}) = 0, \\ \displaystyle\int_{t^{n-1}}^{t^n} b(\mathbf{e}, q_h) = 0. \end{cases}$$

for all $(\mathbf{v}_h, q_h) \in \mathcal{P}_k[t^{n-1}, t^n; U_h^n] \times \mathcal{P}_k[t^{n-1}, t^n; Q_h^n]$. Here, $\mathbf{e} = \mathbf{u} - \mathbf{u}_h$, $(\mathbf{u}_h, p_h)$ is the solution of (4.3), and $(\mathbf{u}, p) \in L^2[0,T; V(\Omega)] \cap H^1[0,T; H^{-1}(\Omega)] \times L^2[0,T; L_0^2(\Omega)]$ is the solution of the parabolic Stokes' problem.

Estimates for $\mathbf{e}$ were obtained by selecting $\mathbf{v}_h \in \mathcal{P}_k[t^{n-1}, t^n; Z_h^n]$ which eliminates the discrete pressure; $\int_{t^{n-1}}^{t^n} b(\mathbf{v}_h, p - p_h) = \int_{t^{n-1}}^{t^n} b(\mathbf{v}_h, p - q_h)$, for all $q_h \in L^2[t^{n-1}, t^n; Q_h^n]$. In this situation the proof of Theorem 4.7 shows that the error $\mathbf{e}_h = \mathbb{P}_h^{loc}\mathbf{u} - \mathbf{u}_h$ satisfies

$$\|\mathbf{e}_h(t)\|_{L^2(\Omega)}^2 + \nu \int_0^{t^n} \|\mathbf{e}_h\|_{H^1(\Omega)}^2 + \sum_{i=0}^{n-1} \|[\mathbf{e}_h^i]\|_{L^2(\Omega)} \le C\Big(\|\mathbf{e}_h^0\|_{L^2(\Omega)}^2 + \int_0^{t^n} \nu \|(I - \mathbb{P}_h^{loc})\mathbf{u}\|_{H^1(\Omega)}^2$$
$$+ (1/\nu) \int_0^{t^n} \|p - q_h\|_{L^2(\Omega)}^2 + \sum_{i=0}^{n-1} \min\left(1/(\Delta t^{i+1}\nu) \left\|P_{i+1}(I - P_i)\mathbf{u}(t^i)\right\|_{H^{-1}(\Omega)}^2, \left\|(I - P_i)\mathbf{u}(t^i)\right\|_{L^2(\Omega)}^2 \right) \Big),$$

where C is a constant depending only on k and the domain Ω, and q_h is arbitrary.

Rates of convergence then follow as in Theorem 4.9

$$\|\mathbf{u} - \mathbb{P}_h\mathbf{u}\|_{L^2[0,T;H^1(\Omega)]} \le C\Big(\|\mathbf{u}\|_{L^2[0,T;H^{\ell+1}(\Omega)]} h^\ell + \|\mathbf{u}^{(k+1)}\|_{L^2[0,T;H^1(\Omega)]} \tau^{k+1} + (1/\sqrt{\nu}) \|\mathbf{u}(0)\|_{H^\ell(\Omega)} h^\ell$$
$$+ (1/\nu)\|p\|_{L^2[0,T;H^\ell(\Omega)]} h^\ell + \|\mathbf{u}\|_{C[0,T;H^{\ell+1}(\Omega)]} \min\left(h^2/(\tau\nu), h/\sqrt{\tau\nu}\right) h^\ell \Big).$$

and

$$\|\mathbf{u} - \mathbb{P}_h\mathbf{u}\|_{L^\infty[0,T;L^2(\Omega)]} \le C\Big(\left(\sqrt{\nu}\|\mathbf{u}\|_{L^2[0,T;H^{\ell+1}(\Omega)]} + h\|\mathbf{u}\|_{L^\infty[0,T;H^{\ell+1}(\Omega)]} \right) h^\ell$$
$$+ \left(\sqrt{\nu}\|\mathbf{u}^{(k+1)}\|_{L^2[0,T;H^1(\Omega)]} + \|\mathbf{u}^{(k+1)}\|_{L^\infty[0,T;L^2(\Omega)]} \right) \tau^{k+1} + \|\mathbf{u}(0)\|_{H^\ell(\Omega)} h^\ell$$
$$+ \|\mathbf{u}\|_{C[0,T;H^{\ell+1}(\Omega)]} \min\left(h^{3/2}/(\tau\sqrt{\nu}), \sqrt{h/\tau} \right) h^{\ell+1/2} + (1/\sqrt{\nu})\|p\|_{L^2[0,T;H^\ell(\Omega)]} h^\ell \Big).$$

where $\tau = \max_{1 \le n \le N} \Delta t^n$.

5. Analysis of the DG Scheme for the Navier-Stokes Equations.

5.1. Stability. The natural energy estimate for the discontinuous Galerkin scheme (2.4) is obtained by setting $\mathbf{v}_h = \mathbf{u}_h$ and using the skew symmetry of the trilinear form, $c(\mathbf{u}_h, \mathbf{u}_h, \mathbf{u}_h) = 0$, to get

$$(1/2)\|\mathbf{u}_-^n\|_{L^2(\Omega)}^2 + (1/2)\|[\mathbf{u}^{n-1}]\|_{L^2(\Omega)}^2 + \nu \int_{t^{n-1}}^{t^n} \|\mathbf{u}_h\|_{H^1(\Omega)}^2 \le (1/2)\|\mathbf{u}_-^{n-1}\|_{L^2(\Omega)}^2 + \int_{t^{n-1}}^{t^n} \langle \mathbf{f}, \mathbf{u}_h \rangle. \quad (5.1)$$

In general it is necessary to bound the last term on the right by

$$\langle \mathbf{f}, \mathbf{u}_h \rangle \le \|\mathbf{f}\|_{H^{-1}(\Omega)} \|\mathbf{u}_h\|_{H^1(\Omega)} \le (1/2\nu)\|\mathbf{f}\|_{H^{-1}(\Omega)}^2 + (\nu/2)\|\mathbf{u}_h\|_{H^1(\Omega)}^2,$$

which gives rise to a factor of $1/\nu$ on the right hand side.

The implicit Euler scheme ($k = 0$) has $\mathbf{u}_h$ is piecewise constant in time, so $\mathbf{u}_h = \mathbf{u}_-^n$ on (t^{n-1}, t^n) and the Cauchy Schwarz inequality can be used to bound the right hand side of equation (5.1). In this situation

$$(1 - \Delta t^n)\|\mathbf{u}_-^n\|^2_{L^2(\Omega)} + \|[\mathbf{u}^{n-1}]\|^2_{L^2(\Omega)} + 2\nu \int_{t^{n-1}}^{t^n} \|\mathbf{u}_h\|^2_{H^1(\Omega)} \leq \|\mathbf{u}_-^{n-1}\|^2_{L^2(\Omega)} + \int_{t^{n-1}}^{t^n} \|\mathbf{f}\|^2_{L^2(\Omega)},$$

which gives bounds independent of ν. Similarly, if $\mathbf{u}_h$ is piecewise linear an explicit computation shows

$$\int_{t^{n-1}}^{t^n} \|\mathbf{u}_h\|^2_{L^2(\Omega)} \leq \hat{C}\Delta t^n \left(\|\mathbf{u}_-^n\|^2_{L^2(\Omega)} + \|\mathbf{u}_-^{n-1}\|^2_{L^2(\Omega)} + \|[\mathbf{u}^{n-1}]\|^2_{L^2(\Omega)} \right),$$

where $\hat{C} = (6 + \sqrt{6})/4$. It follows that

$$(1-\hat{C}\Delta t^n)\|\mathbf{u}_-^n\|^2_{L^2(\Omega)} + c\|[\mathbf{u}^{n-1}]\|^2_{L^2(\Omega)} + 2\nu \int_{t^{n-1}}^{t^n} \|\mathbf{u}_h\|^2_{H^1(\Omega)} \leq (1+\hat{C}\Delta t^n)\|\mathbf{u}_-^{n-1}\|^2_{L^2(\Omega)} + \int_{t^{n-1}}^{t^n} \|\mathbf{f}\|^2_{L^2(\Omega)},$$

and if $\hat{C}\Delta t^n < 1$ the discrete Gronwall inequality is applicable. These observations are summarized in the following lemma.

LEMMA 5.1. *Let $(\mathbf{u}_h, p_h)$ be an approximate solution of the Navier-Stokes equations computed using the discontinuous Galerkin scheme (2.4) with $\mathbf{f} \in L^2[0, T; H^{-1}(\Omega)]$ and $\mathbf{u}_0 \in L^2(\Omega)$. Then,*

$$\|\mathbf{u}_-^n\|^2_{L^2(\Omega)} + \nu \int_0^{t^n} \|\mathbf{u}_h\|^2_{H^1(\Omega)} \leq \|\mathbf{u}_-^0\|^2_{L^2(\Omega)} + (C/\nu) \int_0^{t^n} \|\mathbf{f}\|^2_{H^{-1}(\Omega)}$$

where, C depends only on the domain Ω. Moreover, when $\mathbf{f} \in L^2[0, T; L^2(\Omega)]$, the low order DG schemes, $k = 0$ and $k = 1$ also satisfy

$$\|\mathbf{u}_-^n\|^2_{L^2(\Omega)} + \nu \int_0^{t^n} \|\mathbf{u}_h\|^2_{H^1(\Omega)} \leq Ce^{\hat{C}t^n} \left(\|\mathbf{u}_-^0\|^2_{L^2(\Omega)} + \int_0^{t^n} \|\mathbf{f}\|^2_{L^2(\Omega)} \right),$$

provided the maximal time step $\tau = \max_{1\leq n\leq N} \Delta t^n$ is smaller than some absolute constant $\hat{C}$.

5.2. Error estimates for the Navier-Stokes. Since solutions $(\mathbf{u}, p)$ of the Navier Stokes equations satisfy the Discontinuous Galerkin formulation (2.4) the error $\mathbf{e} = \mathbf{u} - \mathbf{u}_h$ satisfies the orthogonality relation,

$$\begin{cases} (\mathbf{e}_-^n, \mathbf{v}_-^n) + \displaystyle\int_{t^{n-1}}^{t^n} \Big(-(\mathbf{e}, \mathbf{v}_{ht}) + a(\mathbf{e}, \mathbf{v}_h) + b(\mathbf{v}_h, p - p_h) \Big) - (\mathbf{e}_-^{n-1}, \mathbf{v}_+^{n-1}) \\ \quad = -\displaystyle\int_{t^{n-1}}^{t^n} \Big(c(\mathbf{u}, \mathbf{u}, \mathbf{v}_h) - c(\mathbf{u}_h, \mathbf{u}_h, \mathbf{v}_h) \Big) \quad \forall\, \mathbf{v}_h \in \mathcal{P}_k[t^{n-1}, t^n; U_h^n], \\ \displaystyle\int_{t^{n-1}}^{t^n} b(\mathbf{e}, q_h) = 0, \qquad \forall\, q_h \in \mathcal{P}_k[t^{n-1}, t^n; Q_h^n]. \end{cases} \tag{5.2}$$

Our main theorem, stated next, is valid for both two and three dimensional domains and schemes of arbitrary order in space and time.

THEOREM 5.2. *Let $d = 2$ or 3 and let $(\mathbf{u}, p)$ be a solution of the Navier Stokes equations with $\mathbf{u} \in L^\infty[0, T; H^1(\Omega)]$. Let $(\mathbf{u}_h, p_h)$ be an approximate solution of $(\mathbf{u}, p)$ computed using the*

discontinuous Galerkin scheme (2.4), and write the error in the velocity as $\mathbf{e} = \mathbf{e}_p + \mathbf{e}_h$*, with* $\mathbf{e}_p = \mathbf{u} - \mathbb{P}_h\mathbf{u}$ *and* $\mathbf{e}_h = \mathbb{P}_h\mathbf{u} - \mathbf{u}_h$*, where* $\mathbb{P}_h\mathbf{u}$ *is the projection defined in Definition 4.2.*

Then there exists a constant $C = C(k, T, \Omega)$ *such that*

$$\|\mathbf{e}_{h-}^n\|_{L^2(\Omega)}^2 + \sum_{i=0}^{n-1} \|[\mathbf{e}_h^i]\|_{L^2(\Omega)}^2 + \nu \int_0^{t^n} \|\mathbf{e}_h\|_{H^1(\Omega)}^2 \leq (C/\nu) \exp(CT) \Big\{ \|\mathbf{e}_{h-}^0\|^2$$
$$+ \int_0^{t^n} \Big(\big(\|\mathbf{u}\|_{L^\infty[0,T;H^1(\Omega)]}^2 + \|\mathbb{P}_h\mathbf{u}\|_{L^\infty[0,T;H^1(\Omega)]}^2 \big) \|\mathbf{e}_p\|_{H^1(\Omega)}^2 + \|p - q_h\|_{L^2(\Omega)}^2 \Big) \Big\},$$

for any $q_h \in \{L^2[0,T;L^2(\Omega)] \mid q_h|_{(t^{n-1},t^n)} \in L^2[t^{n-1}, t^n; Q_h^n]\}$ *provided the maximum time step size* $\tau = \max_{1\leq n\leq N}(t^n - t^{n-1})$ *is sufficiently small. In particular, for the low order schemes,* $k = 0$ *and* $k = 1$*, we require* $C\|\mathbb{P}_h\mathbf{u}\|_{L^\infty[0,T;H^1(\Omega)]}^4 \tau/\nu^3 < 1$*, and additionally*

$$C\left(1/\nu + \|\mathbb{P}_h\mathbf{u}\|_{L^\infty[0,T;H^1(\Omega)]}\sqrt{\tau}\right) \|\mathbb{P}_h\mathbf{u}\|_{L^\infty[0,T;H^1(\Omega)]}^4 \sqrt{\tau}/\nu^4 < 1$$

for the higher order schemes, $k \geq 2$*.*

Recall that if $\mathbf{u}_t - \Delta\mathbf{u} \in L^2[0,T;L^2(\Omega)]$ then $\mathbf{u} \in L^\infty[0,T;H^1(\Omega)]$ and Theorem 4.10 shows that this is sufficient to guarantee that $\mathbb{P}_h\mathbf{u} \in L^\infty[0,T;H^1(\Omega)]$ too. Also, note that the DG approximation scheme is stable for any time step size $\tau > 0$, and that the restriction on τ and $L^\infty[0,T;H^1(\Omega)]$ bounds assumed in the theorem are only necessary in order to quantify the error.

Proof. **1) Orthogonality:** Introducing the splitting $\mathbf{e} = \mathbf{e}_p + \mathbf{e}_h$ into the orthogonality relation (5.2) gives

$$\begin{aligned}
&\int_{t^{n-1}}^{t^n} \Big((\mathbf{e}_{ht}, \mathbf{v}_h) + a(\mathbf{e}_h, \mathbf{v}_h) \Big) + (\mathbf{e}_{h+}^{n-1} - \mathbf{e}_{h-}^{n-1}, \mathbf{v}_{h+}^{n-1}) \\
&\quad = -\int_{t^{n-1}}^{t^n} \Big(c(\mathbf{u}, \mathbf{u}, \mathbf{v}_h) - c(\mathbf{u}_h, \mathbf{u}_h, \mathbf{v}_h) - b(\mathbf{v}_h, p - p_h) \Big) \\
&\quad - \int_{t^{n-1}}^{t^n} \Big((\mathbf{e}_{pt}, \mathbf{v}_h) + a(\mathbf{e}_p, \mathbf{v}_h) \Big) - (\mathbf{e}_{p+}^{n-1} - \mathbf{e}_{p-}^{n-1}, \mathbf{v}_{h+}^{n-1}).
\end{aligned}$$

for all $\mathbf{v}_h \in \mathcal{P}_k[t^{n-1}, t^n; Q_h^n]$. By construction $\mathbf{e}_p$ satisfies the orthogonality condition (4.4), which, after integration by parts in time, states

$$\int_{t^{n-1}}^{t^n} \Big((\mathbf{e}_{pt}, \mathbf{v}_h) + a(\mathbf{e}_p, \mathbf{v}_h) + b(\mathbf{v}_h, r_h) \Big) + (\mathbf{e}_{p+}^{n-1} - \mathbf{e}_{p-}^{n-1}, \mathbf{v}_{h+}^{n-1}) = 0.$$

Combining the last two equations, we obtain

$$\begin{aligned}
&\int_{t^{n-1}}^{t^n} \Big((\mathbf{e}_{ht}, \mathbf{v}_h) + a(\mathbf{e}_h, \mathbf{v}_h) \Big) + (\mathbf{e}_{h+}^{n-1} - \mathbf{e}_{h-}^{n-1}, \mathbf{v}_{h+}^{n-1}) \\
&\quad = \int_{t^{n-1}}^{t^n} \Big(c(\mathbf{u}, \mathbf{u}, \mathbf{v}_h) - c(\mathbf{u}_h, \mathbf{u}_h, \mathbf{v}_h) - b(\mathbf{v}_h, p - p_h - r_h) \Big).
\end{aligned} \tag{5.3}$$

We will always select $\mathbf{v}_h \in \mathcal{P}_k[t^{n-1}, t^n; Z_h^n]$ so that the pressure term becomes

$$\int_{t^{n-1}}^{t^n} b(\mathbf{v}_h, p - p_h - r_h) = \int_{t^{n-1}}^{t^n} b(\mathbf{v}_h, p) = \int_{t^{n-1}}^{t^n} b(\mathbf{v}_h, p - q_h).$$

In this equality we may choose $q_h \in L^2[t^{n-1}, t^n; Q_h^n]$, i.e., q_h it does not need to have polynomial time dependence. Letting $\mathbf{u}_p = \mathbb{P}_h \mathbf{u}$, algebraic manipulation using trilinearity of $c(.,.,.)$ shows

$$c(\mathbf{u}, \mathbf{u}, \mathbf{v}_h) - c(\mathbf{u}_h, \mathbf{u}_h, \mathbf{v}_h) = c(\mathbf{e}_p, \mathbf{u}, \mathbf{v}_h) + c(\mathbf{u}_p, \mathbf{e}_p, \mathbf{v}_h) + c(\mathbf{e}_h, \mathbf{u}_p, \mathbf{v}_h) + c(\mathbf{u}_h, \mathbf{e}_h, \mathbf{v}_h). \tag{5.4}$$

It remains to bound each term on the right when $\mathbf{v}_h = \mathbf{e}_h$.

2) Bound the Nonlinear Terms: The skew symmetry properties and bounds of Lemma 2.2 show that $c(\mathbf{u}_h, \mathbf{e}_h, \mathbf{e}_h) = 0$ and

$$\begin{aligned}
\int_{t^{n-1}}^{t^n} \Big| c(\mathbf{e}_p, \mathbf{u}, \mathbf{e}_h) \Big| &\le C \int_{t^{n-1}}^{t^n} \|\mathbf{e}_p\|_{H^1(\Omega)} \|\mathbf{u}\|_{H^1(\Omega)} \|\mathbf{e}_h\|_{H^1(\Omega)} \\
&\le (C/\epsilon) \|\mathbf{u}\|^2_{L^\infty[0,T;H^1(\Omega)]} \int_{t^{n-1}}^{t^n} \|\mathbf{e}_p\|^2_{H^1(\Omega)} + \epsilon \int_{t^{n-1}}^{t^n} \|\mathbf{e}_h\|^2_{H^1(\Omega)} \\
\int_{t^{n-1}}^{t^n} \Big| c(\mathbf{u}_p, \mathbf{e}_p, \mathbf{e}_h) \Big| &\le C \int_{t^{n-1}}^{t^n} \|\mathbf{u}_p\|_{H^1(\Omega)} \|\mathbf{e}_p\|_{H^1(\Omega)} \|\mathbf{e}_h\|_{H^1(\Omega)} \\
&\le (C/\epsilon) \|\mathbf{u}_p\|^2_{L^\infty[0,T;H^1(\Omega)]} \int_{t^{n-1}}^{t^n} \|\mathbf{e}_p\|^2_{H^1(\Omega)} + \epsilon \int_{t^{n-1}}^{t^n} \|\mathbf{e}_h\|^2_{H^1(\Omega)} \\
\int_{t^{n-1}}^{t^n} \Big| c(\mathbf{e}_h, \mathbf{u}_p, \mathbf{e}_h) \Big| &\le C \int_{t^{n-1}}^{t^n} \|\mathbf{e}_h\|^{1/2}_{L^2(\Omega)} \|\mathbf{e}_h\|^{1/2}_{H^1(\Omega)} \|\mathbf{u}_p\|_{H^1(\Omega)} \|\mathbf{e}_h\|_{H^1(\Omega)} \\
&\le (C/\epsilon^3) \|\mathbf{u}_p\|^4_{L^\infty[0,T;H^1(\Omega)]} \int_{t^{n-1}}^{t^n} \|\mathbf{e}_h\|^2_{L^2(\Omega)} + \epsilon \int_{t^{n-1}}^{t^n} \|\mathbf{e}_h\|^2_{H^1(\Omega)}.
\end{aligned}$$

where C denotes a constant that depends only on Ω. Setting $\mathbf{v}_h = \mathbf{e}_h$ in equation (5.3) and collecting the above bounds with $\epsilon = O(\nu)$ we obtain

$$\begin{aligned}
\|\mathbf{e}^n_{h-}\|^2_{L^2(\Omega)} + \|[\mathbf{e}^{n-1}_h]\|^2_{L^2(\Omega)} + \int_{t^{n-1}}^{t^n} (\nu/2) \|\mathbf{e}_h\|^2_{H^1(\Omega)} &\le \|\mathbf{e}^{n-1}_{h-}\|^2_{L^2(\Omega)} \\
+ (C_1^4/\nu^3) \int_{t^{n-1}}^{t^n} \|\mathbf{e}_h\|^2_{L^2(\Omega)} &+ (C/\nu) \int_{t^{n-1}}^{t^n} \big(C_2^2 \|\mathbf{e}_p\|^2_{H^1(\Omega)} + \|p - q_h\|^2_{L^2(\Omega)} \big),
\end{aligned} \tag{5.5}$$

where $C = C(\Omega)$ is independent of ν and C_1 and C_2 take the form

$$\begin{aligned}
C_1 &= C \|\mathbf{u}_p\|_{L^\infty[0,T;H^1(\Omega)]}, \\
C_2 &= C(\|\mathbf{u}\|_{L^\infty[0,T;H^1(\Omega)]} + \|\mathbf{u}_p\|_{L^\infty[0,T;H^1(\Omega)]})
\end{aligned} \tag{5.6}$$

For the low order schemes, $k = 0$ and $k = 1$, $\|\mathbf{e}_h\|_{L^2(\Omega)}$ is bounded by $\mathbf{e}^n_-$, $\mathbf{e}^{n-1}_-$ and $[\mathbf{e}^{n-1}]$ as in the discussion preceding Lemma 5.1. An application of Gronwall's inequality then completes the proof. For the higher order schemes it is necessary to bound $\int_{t^{n-1}}^{t^n} \|\mathbf{e}_h\|^2_{L^2(\Omega)}$.

3) Bounding $\|\mathbf{e}_h\|_{L^2(\Omega)}$: Fix $\mathbf{z}_h \in Z_h^n$ and $t \in (t^{n-1}, t^n)$ and select $\mathbf{v}_h(s) = \phi(s)\mathbf{z}_h$ where $\mathbf{v}_h \in \mathcal{Z}_h^n$ and $\phi \in \mathcal{P}_k(t^{n-1}, t^n)$ satisfies

$$\phi(t^{n-1}) = 1, \qquad \int_{t^{n-1}}^{t^n} \phi\psi = \int_{t^{n-1}}^{t} \psi, \qquad \psi \in \mathcal{P}_{k-1}(t^{n-1}, t^n).$$

(That is, $\mathbf{v}_h$ is the discrete approximation of $\chi_{[t^{n-1},t)}\mathbf{z}_h$ constructed in Section 3.1). Recall that Lemma 3.2 shows that $\|\phi\|_{L^\infty(t^{n-1},t^n)} \le C_k$ where C_k is a constant independent of t. Since

$\mathbf{e}_{ht} \in \mathcal{P}_{k-1}[t^{n-1}, t^n; Z_h^n]$ it follows that

$$\int_{t^{n-1}}^{t^n} (\mathbf{e}_{ht}, \mathbf{v}_h) = \int_{t^{n-1}}^{t} (\mathbf{e}_{ht}, \mathbf{z}_h) = (\mathbf{e}_h(t) - \mathbf{e}_{h+}^{n-1}, \mathbf{z}_h).$$

With this choice of $\mathbf{v}_h$ equation (5.3) becomes

$$(\mathbf{e}_h(t) - \mathbf{e}_{h-}^{n-1}, \mathbf{z}_h) = \int_{t^{n-1}}^{t^n} \Big(- a(\mathbf{e}_h, \mathbf{z}_h) + c(\mathbf{u}_h, \mathbf{u}_h, \mathbf{z}_h) - c(\mathbf{u}, \mathbf{u}, \mathbf{z}_h) - b(\mathbf{z}_h, p - q_h)\Big)\phi.$$

Splitting the trilinear terms as in equation (5.4) we obtain

$$\begin{aligned}
(\mathbf{e}_h(t) - \mathbf{e}_{h-}^{n-1}, \mathbf{z}_h) &\le C \int_{t^{n-1}}^{t^n} \Big(\nu \|\mathbf{e}_h\|_{H^1(\Omega)} + \|\mathbf{e}_p\|_{H^1(\Omega)} \|\mathbf{u}\|_{H^1(\Omega)} + \|\mathbf{u}_p\|_{H^1(\Omega)} \|\mathbf{e}_p\|_{H^1(\Omega)} \\
&\quad + \|\mathbf{e}_h\|_{H^1(\Omega)} \|\mathbf{u}_p\|_{H^1(\Omega)} + \|\mathbf{u}_h\|_{H^1(\Omega)} \|\mathbf{e}_h\|_{H^1(\Omega)} + \|p - q_h\|_{L^2(\Omega)} \Big) \|\mathbf{z}_h\|_{H^1(\Omega)} \\
&\le C \Big(\big(\nu + \|\mathbf{u}_p\|_{L^\infty[t^{n-1}, t^n; H^1(\Omega)]}\big) \tau^{1/2} \|\mathbf{e}_h\|_{L^2[t^{n-1}, t^n; H^1(\Omega)]} \\
&\qquad + \big(\|\mathbf{u}\|_{L^\infty[t^{n-1}, t^n; H^1(\Omega)]} + \|\mathbf{u}_p\|_{L^\infty[t^{n-1}, t^n; H^1(\Omega)]} \big) \tau^{1/2} \|\mathbf{e}_p\|_{L^2[t^{n-1}, t^n; H^1(\Omega)]} \\
&\qquad + \|\mathbf{u}_h\|_{L^2[0,T; H^1(\Omega)]} \|\mathbf{e}_h\|_{L^2[t^{n-1}, t^n; H^1(\Omega)]} + \tau^{1/2} \|p - q_h\|_{L^2[t^{n-1}, t^n; L^2(\Omega)]} \Big) \|\mathbf{z}_h\|_{H^1(\Omega)} \\
&\le C \Big((\nu + C_1) \sqrt{\tau} \|\mathbf{e}_h\|_{L^2[t^{n-1}, t^n; H^1(\Omega)]} + C_2 \sqrt{\tau} \|\mathbf{e}_p\|_{L^2[t^{n-1}, t^n; H^1(\Omega)]} \\
&\qquad + (1/\nu) \|\mathbf{e}_h\|_{L^2[t^{n-1}, t^n; H^1(\Omega)]} + \tau^{1/2} \|p - q_h\|_{L^2[t^{n-1}, t^n; L^2(\Omega)]} \Big) \|\mathbf{z}_h\|_{H^1(\Omega)}.
\end{aligned}$$

where the constant $C = (\Omega)$ is independent of ν and C_1, C_2 depend upon $\mathbf{u}$ as in equation (5.6). Now note that $\nu + C_1 \sim C_1$, select $\mathbf{z}_h = \mathbf{e}_h(t)$ and integrate with respect to t to get,

$$\begin{aligned}
\int_{t^{n-1}}^{t^n} (\mathbf{e}_h(t) - \mathbf{e}_{h-}^{n-1}, \mathbf{e}_h(t)) &\le C \Big(C_1 \sqrt{\tau} \|\mathbf{e}_h\|_{L^2[t^{n-1}, t^n; H^1(\Omega)]} + C_2 \sqrt{\tau} \|\mathbf{e}_p\|_{L^2[t^{n-1}, t^n; H^1(\Omega)]} \\
&\qquad + (1/\nu) \|\mathbf{e}_h\|_{L^2[t^{n-1}, t^n; H^1(\Omega)]} + \tau^{1/2} \|p - q_h\|_{L^2[t^{n-1}, t^n; L^2(\Omega)]} \Big) \int_{t^{n-1}}^{t^n} \|\mathbf{e}_h\|_{H^1(\Omega)}
\end{aligned}$$

Therefore, with $\Delta t^n = t^n - t^{n-1}$ we have

$$\begin{aligned}
\int_{t^{n-1}}^{t^n} \|\mathbf{e}_h\|_{L^2(\Omega)}^2 &\le \Delta t^n \|\mathbf{e}_h^{n-1}\|_{L^2(\Omega)}^2 \\
&\quad + C \sqrt{\tau} \int_{t^{n-1}}^{t^n} \Big((1/\nu + C_1 \sqrt{\tau}) \|\mathbf{e}_h\|_{H^1(\Omega)}^2 + (C_2 \sqrt{\tau}) \|\mathbf{e}_p\|_{H^1(\Omega)}^2 \Big) + \sqrt{\tau} \|p - q_h\|_{L^2(\Omega)}^2 \Big).
\end{aligned} \tag{5.7}$$

4) Combining the Estimates: Using inequality (5.7) to bound the integral of $\|\mathbf{e}_h\|_{L^2(\Omega)}^2$ on the right hand side of equation (5.5) we obtain

$$\begin{aligned}
&\|\mathbf{e}_{h-}^n\|_{L^2(\Omega)}^2 + \|[\mathbf{e}_h^{n-1}\|_{L^2(\Omega)}^2 + \int_{t^{n-1}}^{t^n} \left(\nu/2 - (1/\nu + C_1 \sqrt{\tau}) C_1^4 \sqrt{\tau}/\nu^3 \right) \|\mathbf{e}_h\|_{H^1(\Omega)}^2 \\
&\le (1 + C_1^4 \Delta t^n/\nu^3) \|\mathbf{e}_{h-}^{n-1}\|_{L^2(\Omega)}^2 + C \int_{t^{n-1}}^{t^n} (C_2/\nu + C_1^4 \tau/\nu^3) C_2 \|\mathbf{e}_p\|_{H^1(\Omega)}^2 + (1/\nu + C_1^4 \tau/\nu^3) \|p - q_h\|_{L^2(\Omega)}^2.
\end{aligned}$$

Under the assumptions on the maximal time step τ, application of the discrete Gronwall Lemma completes the proof. □

The technical development in Step 3 of our proof was necessary since the discrete energy estimate does not naturally bound the $L^2(\Omega)$ norm of the solution at times between the partition points. In Appendix A it is shown that in two dimensions that it is possible to bound the discrete solution $\mathbf{u}_h$ in $L^\infty[0,T;L^2(\Omega)]$.

Theorem 4.9 can now be used to establish rates of convergence for smooth solutions.

THEOREM 5.3. *Let $\Omega \subset \mathbb{R}^d$ with $d = 2$ or 3 and for $h > 0$ let $\{t_h^n\}_{n=0}^{N_h}$ be a partition of $[0,T]$ and let $\{(U_h^n, Q_h^n)\}_{n=0}^{N_h}$ be finite element subspaces of $H_0^1(\Omega)^d \times L_0^2(\Omega)$. Assume that the time partition is quasi uniform in the sense that there exists $0 \le \theta \le 1$ such that $\min_{1\le n\le N} \Delta t_h^n \le \theta \max_{1\le n\le N} \Delta t_h^n$, and that the finite element subspaces satisfy the approximation, inf-sup, and inverse hypotheses in Assumption 1.*

Let $\{(\mathbf{u}_h, p_h)\}_{h>0}$ be approximate solutions of the Navier Stokes equations computed using the discontinuous Galerkin scheme (2.4) and let the solution $(\mathbf{u}, p)$ of the Navier Stokes equations satisfy

$$\mathbf{u} \in C[0,T;H^{\ell+1}(\Omega)] \cap H^{k+1}[0,T;L^2(\Omega)], \qquad \text{and} \qquad p \in L^2[0,T;H^\ell(\Omega)],$$

where $\ell \ge 1$ and $k \ge 0$ are the polynomial degrees of the space and time dependence of $(\mathbf{u}_h, p_h)$.

If the initial data $\mathbf{u}_-^0$ satisfies $\|\mathbf{u}_-^0 - P_0\mathbf{u}(0)\|_{L^2(\Omega)} \le Ch^\ell \|\mathbf{u}(0)\|_{H^\ell(\Omega)}$, there exists constant $C = C(k,T,\Omega) > 0$ independent of $0 < \nu \le 1$ such that the error $\mathbf{e} = \mathbf{u} - \mathbf{u}_h$ satisfies

$$\begin{aligned}\|\mathbf{e}_-^n\|_{L^2(\Omega)} + \sqrt{\nu}\|\mathbf{e}\|_{L^2[0,T;H^1(\Omega)]} &\le C\big(\exp(CT/\nu^3)\big)^{1/2}\Big\{1/\sqrt{\nu}\|\mathbf{u}_-^0\|_{H^\ell(\Omega)}h^\ell \\ &+ (C_2/\sqrt{\nu})\Big(\|\mathbf{u}\|_{L^2[0,T;H^{\ell+1}(\Omega)]}h^\ell + \|\mathbf{u}^{(k+1)}\|_{L^2[0,T;H^1(\Omega)]}\tau^{k+1} \\ &+ \|\mathbf{u}\|_{C[0,T;H^{\ell+1}(\Omega)]}\min\{\frac{h^2}{\tau\nu}, \frac{h}{\sqrt{\tau\nu}}\}h^\ell\Big) + (1/\sqrt{\nu})\|p\|_{L^2[0,T;H^\ell(\Omega)]}h^\ell\Big\},\end{aligned}$$

provided the maximum time step size $\tau = \max_{1\le n\le N_h}(t_h^n - t_h^{n-1})$ is sufficiently small. In particular, for the low order schemes, $k = 0$ and $k = 1$, we require $C\|\mathbf{u}\|^4_{L^\infty[0,T;H^1(\Omega)]}\tau/\nu^3 < 1$, and additionally

$$C\left(1/\nu + \|\mathbf{u}\|_{L^\infty[0,T;H^1(\Omega)]}\sqrt{\tau}\right)\|\mathbf{u}\|^4_{L^\infty[0,T;H^1(\Omega)]}\sqrt{\tau}/\nu^4 < 1$$

for the higher order schemes, $k \ge 2$. Here $C_2 = C(\|\mathbf{u}\|_{L^\infty[0,T;H^1(\Omega)]} + \|\mathbf{u}_p\|_{L^\infty[0,T;H^1(\Omega)]})$, and $\mathbf{u}^{(k+1)}$ denotes the $(k+1)^{st}$ time derivative of $\mathbf{u}$.

5.3. Comments on Regularity. While the DG scheme for the Navier Stokes equations will converge under minimal regularity guaranteed by the energy estimate, in order to get rates of convergence we needed to assume additionally that $\mathbf{u} \in L^\infty[0,T;H^1(\Omega)]$. The regularity theorems in [12, 29] show that if $\mathbf{f} \in L^2[0,T;W(\Omega)]$ and $\mathbf{u}_0 \in V(\Omega)$ then the solution $(\mathbf{u}, p)$ of equations (2.1) satisfies

$$(\mathbf{u}, p) \in L^2[0,T;H^2(\Omega) \cap V(\Omega)] \cap H^1[0,T;W(\Omega)] \times L^2[0,T;H^1(\Omega) \cap L_0^2(\Omega)].$$

More precisely, when $d = 3$ and $\Omega \subset \mathbb{R}^3$ is open bounded and of class C^2, there exists a strong solution of the Navier Stokes equations on $[0,T]$ when

$$0 \le T < \frac{3}{8\hat{C}(1 + \|\mathbf{u}_0\|^2_{H^1(\Omega)})},$$

where $\hat{C} = C\max(1/\nu^4, (1/\nu)\|\mathbf{f}\|_{L^\infty[0,T;L^2(\Omega)]})$. That is, $\mathbf{u} \in L^\infty[0,T;V(\Omega)] \cap L^2[0,T;H^2(\Omega)]$ and [30, Relation 3.28]

$$\|u(t)\|^2_{H^1(\Omega)} \leq 2(1 + \|\mathbf{u}_0\|^2_{H^1(\Omega)}), \qquad t \in [0,T].$$

When $d = 2$ the "smallness assumption" on data, $\mathbf{f} \in L^2[0,T;W(\Omega)]$ and $\mathbf{u}_0 \in V$, is not required. In this situation there exists $C > 0$ depending only upon Ω, $\|\mathbf{f}\|_{L^2[0,T;H^1(\Omega)]}$, and $\|\mathbf{u}_0\|_{H^1(\Omega)}$, such that [30, Relation (3.14)]

$$\sup_{t\in[0,T]} \|u(t)\|^2_{H^1(\Omega)} \leq (C/\nu)\exp(C/\nu^3).$$

Comprehensive surveys of various regularity results for the Navier-Stokes equations can be found in [12, 29, 30].

An alternative weak formulation for the Navier-Stokes equations (1.1) seeks a velocity field $\mathbf{u} \in L^2[0,T;V(\Omega)] \cap H^1[0,T;V(\Omega)^*]$ such that

$$\begin{cases} \langle \mathbf{u}_t, \mathbf{v}\rangle + a(\mathbf{u},\mathbf{v}) + c(\mathbf{u},\mathbf{u},\mathbf{v}) &= \langle \mathbf{f}, \mathbf{v}\rangle \qquad \forall\, \mathbf{v} \in V(\Omega), \\ \mathbf{u}(0,x) &= \mathbf{u}_0, \end{cases}$$

with data $\mathbf{f} \in L^2[0,T;V(\Omega)^*]$ and $\mathbf{u}_0 \in W(\Omega)$. Then the associated pressure p satisfies (1.1) in a distributional sense [29, 25]. Similar considerations also hold for the linear problem.

While the energy estimate bounds $\mathbf{u}$ when $\mathbf{f} \in L^2[0,T;H^{-1}(\Omega)]$ and $\mathbf{u}_0 \in L^2(\Omega)$, regularity of p and $\mathbf{u}_t$ in $L^2[0,T;H^{-1}(\Omega)]$ has to be assumed. This is the case even for the (linear) Stokes equations. The construction of the projection $\mathbb{P}_h$ requires

$$\mathbf{u} \in L^2[0,T;H^1_0(\Omega)] \cap H^1[0,T;H^{-1}(\Omega)],$$

and this holds when $\mathbf{f} \in L^2[0,T;W(\Omega)]$ and $\mathbf{u}_0 \in V(\Omega)$; see also [20] for semi-discrete (in space) approximations.

REFERENCES

[1] D. N. Arnold, F. Brezzi, B. Cockburn and D. L. Marini, Unified analysis of discontinuous Galerkin methods for elliptic problems, *SIAM J. Numer. Anal.* **39**, 2001-02, pp 1749-1779.

[2] G. Baker, V. Dougalis and O. Karakashian, On a higher order accurate fully discrete Galerkin approximation to the Navier-Stokes equations, *Math. Comp.* **39** 1982, pp 339-375.

[3] R. E. Bank and R.F. Santos, Analysis of some moving space-time finite element methods, *SIAM J. Numer. Anal.* **30**, 1993, pp 1-18

[4] M. Bause, On optimal convergence rates for higher-order Navier-Stokes approximations I: Error estimates for spatial discretization, *IMA J. Numer. Anal.* **25** 2005, pp 812-841.

[5] F. Brezzi, On the existence, uniqueness and approximation of saddle point problems arising from Lagrangian multipliers, *RAIRO Anal. Numer.* **8** 1974, pp 129-151.

[6] F. Brezzi and M. Fortin Mixed and Hybrid Finite Element Methods, Springer-Verlag, 1991.

[7] K. Chrysafinos and N. J. Walkington, Error estimates for the discontinuous Galerkin methods for parabolic equations, *SIAM J. Numer. Anal.*, **44** No1, 2006, pp 349-366.

[8] K. Chrysafinos and N. J. Walkington, Error estimates for the discontinuous Galerkin methods for implicit parabolic equations, *SIAM J. Numer. Anal.*, **43** No6, 2006, pp 2478-2499.

[9] K. Chrysafinos and Noel J. Walkington, Lagrangian and moving mesh finite element methods for the advection-diffusion equation, *ESAIM M²AN*, **42** No1, 2008, pp 27-56.

[10] P. G. Ciarlet, The finite element method for elliptic problems, SIAM Classics In Applied Math, 2002

[11] B. Cockburn, G. E. Karnadiakis, and C. W. Shu, The development of discontinuous Galerkin methods, in Discontinuous Galerkin methods (Newport, RI, 1999), Springer, Berlin, 2000, pp 3-50.
[12] P. Constantin and C. Foias, Navier-Stokes equations, Chicago Lectures in Mathematics, 1988.
[13] M. Delfour, W. Hager and F. Trochu, Discontinuous Galerkin Methods for Ordinary Differential Equations, *Math. Comp*, **36**, 1981, pp 455-473.
[14] T. F. Dupont and Yingjie Liu, Symmetric error estimates for moving mesh Galerkin methods for advection-diffusion equations, *SIAM J. Numer. Anal.* **40**, 2002, pp 914-927.
[15] K. Eriksson, C. Johnson and V. Thomée, Time discretization of parabolic problems by the discontinuous Galerkin method, *RAIRO Modél. Math. Anal. Numér.*, **29**, 1985, pp 611-643.
[16] V. Girault and P-A. Raviart, Finite element methods for Navier-Stokes equation, Spinger-Verlag, Berlin, 1986.
[17] J. G. Heywood and R. Rannacher, Finite element approximation of the nonstationary Navier-Stokes problem. I. Regularity of solutions and second-order error estimates for spatial discretization , *SIAM. J. Numer. Anal.* **19** 1982, pp 275-311.
[18] J. G. Heywood and R. Rannacher, Finite element approximation of the nonstationary Navier-Stokes problem, III. Smoothing property and higher order estimates for spatial discretization, *SIAM. J. Numer. Anal.* **25** 1988, pp 489-512.
[19] J. G. Heywood and R. Rannacher, Finite element approximation of the nonstationary Navier-Stokes problem. IV. Error analysis for second order discretization, *SIAM J. Numer. Anal.* **27** 1990, pp. 355-384.
[20] L. S. Hou, Error estimates for semidiscrete finite element approximations of the Stokes equations under minimal regularity assumptions, *J. Sci. Comput.* **16**, 2001, pp 287-317.
[21] P. Jamet, Galerkin-type approximations which are discontinuous in time for parabolic equations in a variable domain, *SIAM J. Numer. Anal.* **15**, 1978, pp 912-928.
[22] C. Johnson, Numerical Solution of Partial Differential Equations by the Finite Element Method, Cambridge, 1987.
[23] T. Katsaounis, Fully discrete nonconforming finite element schemes for the nonstationary Navier-Stokes equations, *East-West J. Numer. Math.*, **6**, 1998, pp 273-298.
[24] P. Lasaint and P.-A. Raviart, On a finite element method for solving the neutron transport equation, in Mathematical aspects of finite elements in partial differential equations, C. de Boor, ed., Academic Press, New York, 1974, pp. 89-123.
[25] P. L. Lions, *Mathematical Topics in Fluid Mechanics, Volume 1: Incompressible Models*, Oxford Press, Oxford, U.K., 1996.
[26] Yingjie Liu, R.E. Bank, T.F Dupont, S. Garcia and R. P. Santos, Symmetric error estimates for moving mesh mixed methods for advection-diffusion equations, *SIAM J. Numer. Anal.* **40**, 2003, pp 2270-2291.
[27] O. Pironeau, On the transport diffusion algorithm and its application to the Navier-Stokes equations, *Numer. Math.*, **38**, 1982, pp 309-332.
[28] E. Süli, Convergence and nonlinear stability of the Lagrange Galerkin method for the Navier-Stokes equations, *Numer. Math.*, **53**, 1998, pp. 457-483.
[29] R. Temam, Navier-Stokes Equations, North Holland, 1979.
[30] R. Temam, Navier-Stokes equations and nonlinear functional analysis, Second Edition, CBNS-NSF Regional Conference Series in Applied Mathematics, SIAM, Philadelphia, 1995.
[31] V. Thomée, Galerkin finite element methods for parabolic problems, Spinger-Verlag, Berlin, 1997
[32] N. J. Walkington, Convergence of the discontinuous Galerkin method for discontinuous solutions, *SIAM J. Numer. Anal.*, **42**, 2005, pp 1801-1817.

Appendix A. Bounding $\mathbf{u}_h$ in Two Dimensions.

The sharper estimates available for the trilinear forms in two dimensions enable us to bound the approximate solutions in $L^\infty[0,T;L^2(\Omega)]$.

Theorem A.1. *Let $d=2$ and $(\mathbf{u}_h, p_h)$ be an approximate solution of the Navier-Stokes equations computed using the discontinuous Galerkin scheme (2.4) with $\mathbf{f} \in L^2[0,T;H^{-1}(\Omega)]$ and $\mathbf{u}_0 \in W(\Omega)$. Then, there exists $C>0$ depending only on the domain and the constant k such that:*

$$\|\mathbf{u}_h\|_{L^\infty[0,T;L^2(\Omega)]} \leq C\Big(\|\mathbf{u}^0\|^2_{L^2(\Omega)} + (1/\nu^2)\|\mathbf{f}\|^2_{L^2[0,T;H^{-1}(\Omega)]}\Big).$$

Proof. Let $\bar{\mathbf{u}}_h$ denote the exponential interpolant of $\mathbf{u}_h$ constructed in Definition 3.3. Since $\mathcal{Z}_h^n = \mathcal{P}_k[t^{n-1}, t^n; Z_h^n]$ it follows that $\bar{\mathbf{u}}_h \in Z_h^n$ when $\mathbf{u}_h \in \mathcal{Z}_h^n$. Setting $\mathbf{v}_h = \bar{\mathbf{u}}_h$ into (2.4) the pressure term vanishes and we obtain

$$\int_{t^{n-1}}^{t^n} \Big((\mathbf{u}_{ht}, \bar{\mathbf{u}}_h) + a(\mathbf{u}_h, \bar{\mathbf{u}}_h) + c(\mathbf{u}_h, \mathbf{u}_h, \bar{\mathbf{u}}_h) \Big) + (\mathbf{u}_+^{n-1} - \mathbf{u}_-^{n-1}, \mathbf{u}_+^{n-1}) = \int_{t^{n-1}}^{t^n} \langle \mathbf{f}, \bar{\mathbf{u}}_h \rangle.$$

Since $\mathbf{u}_{ht} \in \mathcal{P}_{k-1}[t^{n-1}, t^n; U_h^n]$ the construction of $\bar{\mathbf{u}}_h$ gives.

$$\begin{aligned}
&\int_{t^{n-1}}^{t^n} (\mathbf{u}_{ht}, \bar{\mathbf{u}}_h) = \int_{t^{n-1}}^{t^n} (\mathbf{u}_{ht}, \mathbf{u}_h) e^{-\lambda(t-t^{n-1})} \\
&= (1/2)\|\mathbf{u}_-^n\|_{L^2(\Omega)}^2 e^{-\lambda(t^n - t^{n-1})} - (1/2)\|\mathbf{u}_+^{n-1}\|^2 + (\lambda/2) \int_{t^{n-1}}^{t^n} \|\mathbf{u}_h(t)\|_{L^2(\Omega)}^2 e^{-\lambda(t-t^{n-1})}.
\end{aligned}$$

Upon recalling that $\|\nabla \bar{\mathbf{u}}_h\|_{L^2[t^{n-1},t^n;L^2(\Omega)]} \le C\|\nabla \mathbf{u}_h\|_{L^2[t^{n-1},t^n;L^2(\Omega)]}$ it follows that

$$\begin{aligned}
&(1/2)\|\mathbf{u}_-^n\|_{L^2(\Omega)}^2 e^{-\lambda(t^n - t^{n-1})} + (1/2)\|[\mathbf{u}_h^{n-1}]\|_{L^2(\Omega)}^2 + (\lambda/2) \int_{t^{n-1}}^{t^n} \|\mathbf{u}_h\|_{L^2(\Omega)}^2 e^{-\lambda(t-t^{n-1})} \\
&= (1/2)\|\mathbf{u}_-^{n-1}\|_{L^2(\Omega)}^2 + \int_{t^{n-1}}^{t^n} \Big(\langle \mathbf{f}, \bar{\mathbf{u}}_h \rangle - a(\mathbf{u}_h, \bar{\mathbf{u}}_h) - c(\mathbf{u}_h, \mathbf{u}_h, \bar{\mathbf{u}}_h) \Big) \qquad \text{(A.1)} \\
&\le (1/2)\|\mathbf{u}_-^{n-1}\|_{L^2(\Omega)}^2 + \int_{t^{n-1}}^{t^n} \Big((1/2\nu)\|\mathbf{f}\|_{H^{-1}(\Omega)}^2 + C\nu\|\nabla \mathbf{u}_h\|_{L^2(\Omega)}^2 + |c(\mathbf{u}_h, \mathbf{u}_h, \bar{\mathbf{u}}_h)| \Big)
\end{aligned}$$

To bound the trilinear term first note that $c(\mathbf{u}_h, \mathbf{u}_h, \bar{\mathbf{u}}_h) = c(\mathbf{u}_h, \mathbf{u}_h, \bar{\mathbf{u}}_h - \mathbf{u}_h)$. Then, inequality (2.2) and Lemma 3.6 imply

$$\begin{aligned}
&\int_{t^{n-1}}^{t^n} \big| c(\mathbf{u}_h, \mathbf{u}_h, \bar{\mathbf{u}}_h - \mathbf{u}_h) \big| \\
&\le C \int_{t^{n-1}}^{t^n} \|\mathbf{u}_h\|_{L^2(\Omega)}^{1/2} \|\mathbf{u}_h\|_{H^1(\Omega)}^{1/2} \Big(\|\mathbf{u}_h\| \|\bar{\mathbf{u}}_h - \mathbf{u}_h\|_{L^2(\Omega)}^{1/2} \|\bar{\mathbf{u}}_h - \mathbf{u}_h\|_{H^1(\Omega)}^{1/2} \\
&\qquad + \|\bar{\mathbf{u}}_h - \mathbf{u}_h\|_{H^1(\Omega)} \|\mathbf{u}_h\|_{L^2(\Omega)}^{1/2} \|\mathbf{u}_h\|_{H^1(\Omega)}^{1/2} \Big) \\
&\le C\|\mathbf{u}_h\|_{L^\infty[t^{n-1},t^n;L^2(\Omega)]}^{1/2} \|\bar{\mathbf{u}}_h - \mathbf{u}_h\|_{L^\infty[t^{n-1},t^n;L^2(\Omega)]}^{1/2} \int_{t^{n-1}}^{t^n} \|\mathbf{u}_h\|_{H^1(\Omega)}^{3/2} \|\bar{\mathbf{u}}_h - \mathbf{u}_h\|_{H^1(\Omega)}^{1/2} \\
&\qquad + C\|\mathbf{u}_h\|_{L^\infty[t^{n-1},t^n;L^2(\Omega)]} \int_{t^{n-1}}^{t^n} \|\bar{\mathbf{u}}_h - \mathbf{u}_h\|_{H^1(\Omega)} \|\mathbf{u}_h\|_{H^1(\Omega)} \\
&\le C\lambda(t^n - t^{n-1}) \|\mathbf{u}_h\|_{L^\infty[t^{n-1},t^n;L^2(\Omega)]} \int_{t^{n-1}}^{t^n} \|\mathbf{u}_h\|_{H^1(\Omega)}^2.
\end{aligned}$$

Substituting the above inequality into (A.1), and setting $\lambda = 1/(t^n - t^{n-1})$ it follows that

$$\begin{aligned}
\|\mathbf{u}_-^n\|_{L^2(\Omega)}^2 e^{-\lambda(t^n - t^{n-1})} + \|[\mathbf{u}_h^{n-1}]\|_{L^2(\Omega)}^2 + \int_{t^{n-1}}^{t^n} (e^{-1}/(t^n - t^{n-1})) \|\mathbf{u}_h\|_{L^2(\Omega)}^2 &\le \|\mathbf{u}_-^{n-1}\|_{L^2(\Omega)}^2 \\
+ C \int_{t^{n-1}}^{t^n} \Big((1/\nu)\|\mathbf{f}\|_{H^{-1}(\Omega)}^2 + \nu\|\nabla \mathbf{u}_h\|_{L^2(\Omega)}^2 + \|\mathbf{u}_h\|_{L^\infty[t^{n-1},t^n;L^2(\Omega)]} \|\mathbf{u}_h\|_{H^1(\Omega)}^2 \Big).
\end{aligned}$$

Using the Lemma 3.5 to bound $\|\mathbf{u}_h\|^2_{L^\infty[t^{n-1},t^n;L^2(\Omega)]}$ by $(C_k/(t^n-t^{n-1}))\|\mathbf{u}_h\|^2_{L^2[t^{n-1},t^n;L^2(\Omega)]}$ gives

$$\begin{aligned}
&\|\mathbf{u}^n_-\|^2_{L^2(\Omega)}e^{-\lambda(t^n-t^{n-1})} + \|[\mathbf{u}^{n-1}_h]\|^2_{L^2(\Omega)} + (1/C_k)\|\mathbf{u}_h\|^2_{L^\infty[t^{n-1},t^n;L^2(\Omega)]}\\
&\le \|\mathbf{u}^{n-1}_-\|^2_{L^2(\Omega)} + C\int_{t^{n-1}}^{t^n}\left((1/\nu)\|\mathbf{f}\|^2_{H^{-1}(\Omega)} + \nu\|\nabla\mathbf{u}_h\|^2_{L^2(\Omega)}\right) + C(k)\Big(\int_{t^{n-1}}^{t^n}\|\mathbf{u}_h\|^2_{H^1(\Omega)}\Big)^2.
\end{aligned}$$

Using Lemma 5.1 to bound the last term on the right completes the proof. □

Appendix B. The Stokes Projection.

Estimates for the parabolic Stokes projection $\mathbb{P}_h$ used the properties of the stationary Stokes projection in an essential fashion. For completeness we sketch the proofs these properties stated in Lemma 4.4 and Corollary 4.5.

Proof. (of Lemma 4.4) Define $\mathbf{f} \in H^{-1}(\Omega)$ by $\mathbf{f}(\mathbf{v}) = a(\mathbf{u},\mathbf{v})$; that is, $\mathbf{f} = -\nu\Delta\mathbf{u}$. If $\mathbf{u} \in V(\Omega)$ then $(\mathbf{u}, 0)$ is the solution of the weak statement

$$a(\mathbf{u},\mathbf{v}) + b(\mathbf{v},p) = \mathbf{f}(\mathbf{v}),$$

$$b(\mathbf{u},q) = 0,$$

for all $(\mathbf{v},q) \in H^1_0(\Omega)\times L^2_0(\Omega)$. If $(\mathbf{u}_h,p_h) \in U_h \times Q_h$ is the Galerkin approximation, then $\mathbf{u}_h = \Pi_h\mathbf{u}$, and classical finite element approximation theory [5, 16, 29] states

$$\begin{aligned}
\|\mathbf{u}-\mathbf{u}_h\|_{H^1(\Omega)} + (1/\nu)\|0-p_h\|_{L^2(\Omega)} &\le C\left(\inf_{\mathbf{v}_h\in U_h}\|\mathbf{u}-\mathbf{v}_h\|_{H^1(\Omega)} + (1/\nu)\inf_{q_h\in Q_h}\|0-q_h\|_{L^2(\Omega)}\right)\\
&\le C\inf_{\mathbf{v}_h\in U_h}\|\mathbf{u}-\mathbf{v}_h\|_{H^1(\Omega)}.
\end{aligned}$$

Duality is used to bound the $L^2(\Omega)$ norm of the projection error. Let $(\mathbf{v},r) \in H^1_0(\Omega)\times L^2_0(\Omega)$ satisfy

$$a(\mathbf{w},\mathbf{v}) + b(\mathbf{w},r) = (\mathbf{u}-\mathbf{u}_h,\mathbf{w}),$$

$$b(\mathbf{v},s) = 0,$$

for all $(\mathbf{w},s) \in H^1_0(\Omega)\times L^2_0(\Omega)$. Then

$$\|\mathbf{u}-\mathbf{u}_h\|^2_{L^2(\Omega)} = a(\mathbf{u}-\mathbf{u}_h,\mathbf{v}) + b(\mathbf{u}-\mathbf{u}_h,r).$$

Since

$$a(\mathbf{u}-\mathbf{u}_h,\mathbf{v}_h) = b(\mathbf{v}_h,p_h) = -b(\mathbf{v}-\mathbf{v}_h,p_h) \qquad \mathbf{v}_h \in U_h,$$

and

$$b(\mathbf{u}-\mathbf{u}_h,r) = -b(\mathbf{u}_h,r) = -b(\mathbf{u}_h,r-r_h) = b(\mathbf{u}-\mathbf{u}_h,r-r_h) \qquad r_h \in Q_h,$$

it follows that

$$\begin{aligned}
\|\mathbf{u}-\mathbf{u}_h\|^2_{L^2(\Omega)} &= a(\mathbf{u}-\mathbf{u}_h,\mathbf{v}-\mathbf{v}_h) - b(\mathbf{v}-\mathbf{v}_h,p_h) + b(\mathbf{u}-\mathbf{u}_h,r-r_h)\\
&\le C(\|\mathbf{u}-\mathbf{u}_h\|_{H^1(\Omega)} + \|p_h\|_{L^2(\Omega)}/\nu)\left(\nu\|\mathbf{v}-\mathbf{v}_h\|_{H^1(\Omega)} + \|r-r_h\|_{L^2(\Omega)}\right)\\
&\le C\inf_{w_h\in U_h}\|\mathbf{u}-\mathbf{w}_h\|_{H^1(\Omega)}\left(\nu\|\mathbf{v}-\mathbf{v}_h\|_{H^1(\Omega)} + \|r-r_h\|_{L^2(\Omega)}\right).
\end{aligned}$$

If Ω is sufficiently smooth to guarantee $H^2(\Omega) \times H^1(\Omega)$ regularity for the Stokes problem, the last term is dominated by $C\|\mathbf{u} - \mathbf{u}_h\|_{L^2(\Omega)} h$ and the proof follows. ∎

Estimates for the jump terms were based upon the statement of Corollary 4.5 proved next.

Proof. (of Corollary 4.5) If the mesh is quasi-uniform inverse estimates establish optimal interpolation of P_n on $H^1(\Omega)$.

$$\begin{aligned}
\|\mathbf{u} - P_n\mathbf{u}\|_{H^1(\Omega)} &\le \|\mathbf{u} - \Pi_n\mathbf{u}\|_{H^1(\Omega)} + \|\Pi_n\mathbf{u} - P_n\mathbf{u}\|_{H^1(\Omega)} \\
&\le \|\mathbf{u} - \Pi_n\mathbf{u}\|_{H^1(\Omega)} + (C/h)\|\Pi_n\mathbf{u} - P_n\mathbf{u}\|_{L^2(\Omega)} \\
&\le \|\mathbf{u} - \Pi_n\mathbf{u}\|_{H^1(\Omega)} + (2C/h)\|\Pi_n\mathbf{u} - \mathbf{u}\|_{L^2(\Omega)} \\
&\le C \inf_{\mathbf{v}_h \in U_h^n} \|\mathbf{u} - \mathbf{v}_h\|_{H^1(\Omega)}.
\end{aligned}$$

This shows that $\|P_n\mathbf{u}\|_{H^1(\Omega)} \le C\|\mathbf{u}\|_{H^1(\Omega)}$ which we now use to estimate $\|P_n(I - P_m)\mathbf{u}\|_{H^{-1}(\Omega)}$.

$$\begin{aligned}
\|P_n(I - P_m)\mathbf{u}\|_{H^{-1}(\Omega)} &= \sup_{\mathbf{v} \in H^1(\Omega)} \frac{(\mathbf{u} - P_m\mathbf{u}, P_n\mathbf{v})}{\|\mathbf{v}\|_{H^1(\Omega)}} \\
&= \sup_{\mathbf{v} \in H^1(\Omega)} \frac{(\mathbf{u} - P_m\mathbf{u}, P_n\mathbf{v} - \mathbf{w}_h)}{\|\mathbf{v}\|_{H^1(\Omega)}}, \qquad \mathbf{w}_h \in Z_h^m \\
&\le C \sup_{\mathbf{v}_h \in Z_h^n} \frac{(\mathbf{u} - P_m\mathbf{u}, \mathbf{v}_h - \mathbf{w}_h)}{\|\mathbf{v}_h\|_{H^1(\Omega)}} \\
&\le C\|\mathbf{u} - P_m\mathbf{u}\|_{L^2(\Omega)} \inf_{\mathbf{w}_h \in Z_h^m} \sup_{\mathbf{v}_h \in Z_h^n} \frac{\|\mathbf{v}_h - \mathbf{w}_h\|_{L^2(\Omega)}}{\|\mathbf{v}_h\|_{H^1(\Omega)}} \\
&\le Ch \inf_{\mathbf{u}_h \in U_h^m} \|\mathbf{u} - \mathbf{u}_h\|_{L^2(\Omega)} \inf_{\mathbf{w}_h \in Z_h^m} \sup_{\mathbf{v}_h \in Z_h^n} \frac{\|\mathbf{v}_h - \mathbf{w}_h\|_{L^2(\Omega)}}{\|\mathbf{v}_h\|_{H^1(\Omega)}}
\end{aligned}$$

Under the hypotheses of the lemma the last term is bounded by a constant of the form Ch. To establish this, fix $\mathbf{v}_h \in Z_h^n$ and let $\mathbf{v}$ be the solution of the Stokes problem with right hand side $\mathbf{f}(\mathbf{z}) = a(\mathbf{v}_h, \mathbf{z})$,

$$a(\mathbf{v}, \mathbf{z}) + b(\mathbf{z}, p) = a(\mathbf{v}_h, \mathbf{z}), \qquad b(\mathbf{v}, q) = 0, \qquad (\mathbf{z}, q) \in H_0^1(\Omega) \times L_0^2(\Omega).$$

Then $\|\mathbf{v}\|_{H^1(\Omega)} \le \|\mathbf{v}_h\|_{H^1(\Omega)}$ and $\mathbf{v}_h = \Pi_n\mathbf{v}$. It follows that $\|\mathbf{v} - \mathbf{v}_h\|_{L^2(\Omega)} \le Ch\|\mathbf{v}_h\|_{H^1(\Omega)}$. Then $\mathbf{w}_h = \Pi_m\mathbf{v}$ satisfies $\|\mathbf{v} - \mathbf{w}_h\|_{L^2(\Omega)} \le Ch\|\mathbf{w}_h\|_{H^1(\Omega)}$, and the triangle inequality, $\|\mathbf{v}_h - \mathbf{w}_h\|_{L^2(\Omega)} \le \|\mathbf{v} - \mathbf{v}_h\|_{L^2(\Omega)} + \|\mathbf{v} - \mathbf{w}_h\|_{L^2(\Omega)}$ can then be used to show

$$\|P_n(I - P_m)\mathbf{u}\|_{H^{-1}(\Omega)} \le Ch^2 \inf_{\mathbf{u}_h \in U_h^m} \|\mathbf{u} - \mathbf{u}_h\|_{L^2(\Omega)} \qquad \mathbf{u} \in V(\Omega).$$

∎