Risk Minimization Control for Beating the Market Strategies

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Abstract. We examine the situation when the investor wants to outperform a certain benchmark by actively trading in this asset, typically a stock index. We consider an investor who wants to minimize the expected shortfall in the case he fails to achieve this goal. Using recently developed techniques of Föllmer and Leukert, we can relate this optimal investment problem to option hedging. This allows us to obtain analytical characterizations of the optimal strategy in special cases.

1 Introduction

This paper studies a dynamic portfolio selection problem, assuming continuous time model with a risky asset and a money market. We study the case when the investor wants to outperform a certain benchmark (we call it index) by trading in this asset. The trading is continuous and the performance of the investor is measured by the shortfall risk. He wants to minimize this risk while trying to outperform the index. There are several techniques which could be applied in solving the problem, in particular, constraint optimization, convex duality methods or methods of stochastic optimal control. An alternative approach is to use the technique of Neyman Pearson lemma as suggested by Föllmer and Leukert (1999, 2000).

There are several papers which solve the problem of finding optimal strategy which maximizes the probability of beating the index. Heath (1993) solved the problem when the underlying process is a Brownian motion with a drift. Karatzas (1997) extended this result to a random drift and Browne (1999) pointed out the link between certain digital options and strategies which maximized the probability of beating the index. Föllmer and Leukert (1999, 2000) generalized these methods to the case of quantile hedging, and further introduced efficient hedging with the criterion of minimizing shortfall risk. Using the method of Neyman Pearson lemma, they provided existence results in the general semimartingale model. The optimal strategies can be characterized by the Radon-Nikodym derivative in the complete market case. They also provided explicit solutions for the Black-Scholes model with insufficient initial capital. Basak, Shapiro and Teplá (2002) in their work provide an independent approach, which also features an investor targeting outperformance of a benchmark with some permitted shortfall.

In this paper we extend the use of the techniques developed by Föllmer and Leukert (2000) to study strategies which try to beat the market. The no arbitrage argument precludes any existence of a trading strategy which would outperform the market with probability 1. The best we can hope for is to find an optimal strategy which would beat the benchmark in the sense that it minimizes the expected shortfall from reaching this goal. This result can have an impact on the choice of trading strategies of many hedge and mutual funds which use a stock index as a benchmark of their performance. The paper is structured in the following way. After model independent problem formulation, we solve for optimal strategies in both Brownian and Poisson model.

2 Problem Formulation

Suppose that the investor can trade in the index S_t and in the money market with constant interest rate r. His wealth X_t evolves according to the self-financing strategy

$$dX_t = q_t dS_t + r(X_t - q_t S_t)dt, \tag{1}$$

where q_t is number of shares held at time t. We do not impose any constraints on q_t except that the wealth X_t stays nonnegative for all t, i.e.,

$$X_t \geq 0, \quad \forall t \in [0, T]. \tag{2}$$

Let us consider the situation when the investor wants to outperform the index itself by selecting the trading strategy q_t. Thus he wants to compare the resulting wealth X_T with respect to the index level S_T at time T. To measure the risk involved in such a strategy, we introduce the loss function f. For computational simplicity we can assume that f is of the form

$$f(x) = \frac{x^p}{p}. \tag{3}$$

We will study the case $p > 1$ for risk averse investors. The objective of the trader is to beat the market by a factor of α at a fixed time horizon T, while minimizing the expected shortfall

$$\mathbb{E}f[((1+\alpha)S_T - X_T)^+], \tag{4}$$

when he fails to meet this target. This problem is non-trivial when the initial capital X_0 is less than the super-hedging price $(1+\alpha)S_0$. Otherwise, the target can always be met.

In summary, we can describe the problem as: for any fixed constant $0 < X_0 < (1+\alpha)S_0$, we need to solve

$$\min_{X \in \mathcal{X}(X_0)} \mathbb{E}f[((1+\alpha)S_T - X_T)^+] \tag{Main Problem}$$

where $\mathcal{X}(X_0) = \{\, X_t \geq 0 \,:\, X_t = X_0 + \int_0^t q_u dS_u + \int_0^t r(X_u - q_u S_u)du\,,\, 0 \leq t \leq T \,\}$,

and $\alpha > 0$ is a constant, $f(x) = \dfrac{x^p}{p},\ p > 1$.

One can relate this situation to option hedging. The value $(1+\alpha)S_T$ can be viewed as an option payoff. The objective of the trader is to deliver this payoff, but he is short of the initial capital to do so with probability 1. Föllmer and Leukert were able to characterize the optimal solution for the option hedging problem in general. Suppose that H_T is a payoff at time T (a nonnegative $\mathcal{F}_T$ - measurable random variable). The objective of the trader is to replicate this payoff by creating a self-financing trading account X_t having dynamics (1). In a complete market, there would exist a hedging strategy q_t providing perfect option replication $X_T = H_T$, given that the trader starts with initial capital equal to discounted expected payoff under the risk-neutral measure $X_0 = e^{-rT}\widetilde{\mathbb{E}}H_T$. However, if the initial capital X_0 is smaller than $e^{-rT}\widetilde{\mathbb{E}}H_T$, there is positive probability that $X_T < H_T$, i.e., the trader would fail to deliver the option payoff in all possible scenarios. However, one can try to minimize the resulting expected loss by minimizing

$$\mathbb{E}f[(H_T - X_T)^+] \tag{5}$$

for some loss function f.

Since it is impossible to perfectly replicate the contingent claim being short of the initial capital, one can selectively lower the payoff of this claim in such a way that it is possible to hedge it perfectly. For this procedure, called modification of the payoff, let us introduce the following set

$$\mathcal{R} = \{\phi : \Omega \to [0,1] \mid \phi \text{ is } \mathcal{F}_T - measurable\}. \tag{6}$$

The modified payoff is ϕH_T. The following theorem identifies the optimal modification of the payoff which minimizes the shortfall risk for convex loss function $l(x)$ in complete market case, i.e. there is a unique risk-neutral measure $\tilde{\mathbb{P}}$.

Theorem 2.1 (Föllmer and Leukert, 2000) *The solution $\hat{\phi}$ to the above optimization problem is given by*

$$\hat{\phi} = 1 - \frac{I(cZ)}{H_T} \wedge 1, \tag{7}$$

where $Z = \frac{d\widetilde{\mathbb{P}}}{d\mathbb{P}}$, $I(x) = \left[\frac{df(x)}{dx}\right]^{-1}$, and the constant c is determined by the initial condition

$$e^{-rT}\widetilde{\mathbb{E}}\left[\hat{\phi}H_T\right] = X_0. \tag{8}$$

We can now apply this result to our case using the payoff $H_T = (1+\alpha)S_T$ and a loss function of the form (3).

Remark 2.2 (Controlling the downside risk) The investor might wish to guarantee that he would end up with at least a fraction $0 \le \underline{\alpha} < 1$ of the benchmark $\underline{\alpha}S_T$ at time T. It is easy to see by no arbitrage argument that in this case his wealth must satisfy $X_t \ge \underline{\alpha}S_t$ at all times $0 \le t \le T$. If we introduce the residual wealth $\tilde{X}_t = X_t - \underline{\alpha}S_t$, we can transform the problem of controlling the downside risk to the original problem by the following simple observation

$$\min_{X_t \ge \underline{\alpha}S_t} \mathbb{E}f[((1+\alpha)S_T - X_T)^+] = \min_{\tilde{X}_t \ge 0} \mathbb{E}f[((1+\alpha-\underline{\alpha})S_T - \tilde{X}_T)^+], \tag{9}$$

as long as $\alpha > 0$. Therefore the investor can use his residual wealth to try to deliver $(1+\alpha-\underline{\alpha})S_T$ at time T. His remaining wealth $\underline{\alpha}S_t$ should be fully invested in the benchmark, leading to wealth of $\underline{\alpha}S_T$ at time T. Without loss of generality, we will consider only the case in the (Main Problem).

3 Minimizing Shortfall Risk in a Brownian Model

Suppose that the index evolves according to the stochastic differential equation

$$dS_t = S_t(\mu dt + \sigma dW_t) \tag{10}$$

under the real physical measure, where μ is the drift and σ is the volatility of the underlying asset. The stock dynamics under the risk neutral measure is given by

$$dS_t = S_t(rdt + \sigma d\widetilde{W}_t). \tag{11}$$

We assume that $\mu > r$. The risk-neutral measure and the original market measure are related by the Radon-Nikodym derivative

$$Z = \frac{d\widetilde{\mathbb{P}}}{d\mathbb{P}} = e^{-\frac{\mu-r}{\sigma}W_T - \frac{1}{2}\left(\frac{\mu-r}{\sigma}\right)^2 T} = \text{const } {S_T}^{-\frac{\mu-r}{\sigma^2}}. \tag{12}$$

The process $\widetilde{W}_t$ given by $\widetilde{W}_t = W_t + \frac{\mu-r}{\sigma}t$ is a Brownian motion under $\widetilde{\mathbb{P}}$.

In this section we provide solutions to the problem of risk minimization control of the beating the market strategies.

Proposition 3.1 *For $0 < X_0 < (1+\alpha)S_0$, the optimal control to (Main Problem) is to replicate an option with the payoff*

$$\hat{X}_T = (1+\alpha)\left[S_T - L^{aq+1}{S_T}^{-aq}\right] I_{[S_T > L]}, \tag{13}$$

where $a = \frac{\mu-r}{\sigma^2}$ and $q = \frac{1}{p-1}$. The constant L is determined by the initial condition

$$X_0 = (1+\alpha)\left[S_0\Phi\left(\frac{\ln(\frac{S_0}{L})+(r+\frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}\right) - S_0^{-aq}L^{aq+1}e^{\gamma T}\Phi\left(\frac{\ln(\frac{S_0}{L})+(r+\frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}} + (aq+1)\sigma\sqrt{T}\right)\right], \tag{14}$$

where $\Phi(\cdot)$ is the standard normal cumulative distribution function and $\gamma = \sigma^2(aq+1)^2 - (aq+1)(r+\frac{1}{2}\sigma^2)$. The optimal wealth process is

$$\hat{X}(t,S_t) = (1+\alpha)\left[S_t\Phi\left(\frac{\ln(\frac{S_t}{L})+(r+\frac{1}{2}\sigma^2)\tau}{\sigma\sqrt{\tau}}\right) - S_t^{-aq}L^{aq+1}e^{\gamma\tau}\Phi\left(\frac{\ln(\frac{S_t}{L})+(r+\frac{1}{2}\sigma^2)\tau}{\sigma\sqrt{\tau}} + (aq+1)\sigma\sqrt{\tau}\right)\right], \tag{15}$$

and the optimal strategy is to invest $\hat{q}_t$ shares in the index S_t where

$$q_t = \hat{X}_s(t,S_t). \tag{16}$$

Proof. We can apply Theorem 2.1 to identify the optimal modified payoff for the claim $(1+\alpha)S_T$. The modification is given by

$$\hat{\phi} = 1 - \frac{I(cZ)}{(1+\alpha)S_T} \wedge 1,$$

where c is determined by

$$\widetilde{\mathbb{E}}\left[\hat{\phi}(1+\alpha)S_T\right] = S_0 e^{rT},$$

with

$$Z = \frac{d\widetilde{\mathbb{P}}}{d\mathbb{P}} \text{ and } I(x) = \left[\frac{df(x)}{dx}\right]^{-1}.$$

In our case

$$f(x) = \frac{x^p}{p}\ ,\ \frac{df(x)}{dx} = x^{p-1}\ ,\ I(x) = x^q,$$

where $q = \frac{1}{p-1} > 0$. Hence the optimal solution is the strategy that replicates a contract with the payoff:

$$\hat{\phi}(1+\alpha)S_T = (1+\alpha)S_T - \left[(cZ)^q \wedge (1+\alpha)S_T\right].$$

Now,

$$Z = \frac{d\widetilde{\mathbb{P}}}{d\mathbb{P}} = \text{const } {S_T}^{-a},$$

where $a = \frac{\mu - r}{\sigma^2} > 0$ and

$$(cZ)^q = \psi S_T^{-aq}$$

for some constant $\psi > 0$. Since $g_1(x) = \psi x^{-aq}$ is a convex and decreasing function, there is at most one point of intersection with function $g_2(x) = (1+\alpha)x$. Let's call L the point of intersection. Then $(1+\alpha)L = \psi L^{-aq}$, from which we get $\psi = (1+\alpha)L^{aq+1}$. Therefore the payoff of this claim can be written as:

$$\hat{\phi}(1+\alpha)S_T = \begin{cases} 0, & \text{if } S_T \leq L \\ (1+\alpha)\left[S_T - L^{aq+1}{S_T}^{-aq}\right], & \text{if } S_T > L. \end{cases}$$

The constant L is given by:

$$e^{-rT}\widetilde{\mathbb{E}}\Bigg[(1+\alpha)\left[S_T - L^{aq+1}{S_T}^{-aq}\right]1_{\{S_T > L\}}\Bigg] = S_0.$$

The final wealth of the optimal strategy is equal to the modified payoff $\hat{X}_T = \hat{\phi}(1+\alpha)S_T$, and the wealth process can be computed using standard risk-neutral pricing method

$$\begin{aligned} \hat{X}(t, S_t) &= e^{-r\tau}\widetilde{\mathbb{E}}\Bigg[(1+\alpha)\left[S_T - L^{aq+1}{S_T}^{-aq}\right]1_{\{S_T > L\}}\Big|\mathcal{F}_t\Bigg] \\ &= (1+\alpha)\Bigg[S_t\Phi\left(\frac{\ln(\frac{S_t}{L}) + (r+\frac{1}{2}\sigma^2)\tau}{\sigma\sqrt{\tau}}\right) - S_t^{-aq}L^{aq+1}e^{\gamma\tau}\Phi\left(\frac{\ln(\frac{S_t}{L}) + (r+\frac{1}{2}\sigma^2)\tau}{\sigma\sqrt{\tau}} + (aq+1)\sigma\sqrt{\tau}\right)\Bigg], \end{aligned}$$

where $\tau = T - t$ and $\gamma = \sigma^2(aq+1)^2 - (aq+1)(r + \frac{1}{2}\sigma^2)$. The optimal strategy is the usual delta-hedge. $\diamond$

Remark 3.2 The optimal investor would bankrupt the fund in the extreme case when $S_T < L$. The more averse investor, the lower is the level L. He would under-perform the market for $S_T < L(1+\frac{1}{\alpha})^{\frac{1}{aq+1}}$, but he would outperform the market otherwise. This trader would be increasingly successful as the market goes up, but he would lose some money when the market declines ($S_T < L(1+\frac{1}{\alpha})^{\frac{1}{aq+1}}$), and it is even possible that he would bankrupt the fund when the market goes down significantly ($S_T < L$). Notice that the optimal solution never exceeds the level $(1+\alpha)S_T$, so that $\mathbb{P}(X_T > (1+\alpha)S_T) = 0$. See Figure 1 for illustration. As mentioned earlier in remark 2.2, it is possible to reformulate the problem in such a way that the trader would never bankrupt the fund, having always at least $\underline{\alpha}$ fraction of the market.

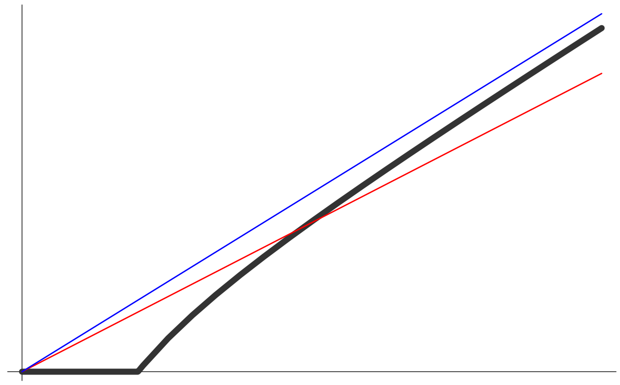

Figure 1: Payoffs of S_T (bottom straight line), $(1+\alpha)S_T$ (top straight line) and a typical optimal payoff profile (thick black line) as a function of the stock price S_T. The optimal payoff profile is 0 for $S_T < L$, it is below S_T up to the point $L(1+\frac{1}{\alpha})^{\frac{1}{\alpha q+1}}$, otherwise it exceeds the index S_T.

4 Minimizing Shortfall Risk in a Poisson Model

Suppose that the stock evolves according to the stochastic differential equation

$$dS_t = S_{t-}[\mu dt - (1-\delta)(dN_t - \lambda dt)] \tag{17}$$

under the real physical measure $\mathbb{P}$, where μ is the drift, N_t is the Poisson process with constant intensity λ, and $0 < \delta < 1$ is the faction of the price after a jump. The solution to (17) is given by

$$S_t = S_0 e^{\mu t + \lambda(1-\delta)t}\delta^{N_t} = S_0 e^{N_t \ln\delta + \mu t + \lambda(1-\delta)t}. \tag{18}$$

The stock dynamics under the risk neutral measure is given by

$$dS_t = S_{t-}[rdt - (1-\delta)(dN_t - \lambda^* dt)], \tag{19}$$

where $\lambda^* = \frac{\mu - r}{1-\delta} + \lambda \triangleq \lambda(1+\nu)$ is the intensity of the Poisson process under $\widetilde{\mathbb{P}}$. We assume that $\mu > r$. Consequently, $\nu > 0$ and $\lambda^* > \lambda$. The risk neutral measure and the original market measure are related by the Radon-Nikodym derivative process

$$Z_t = \mathbb{E}\left[\frac{d\widetilde{\mathbb{P}}}{d\mathbb{P}}\,\middle|\,\mathcal{F}_t\right] = e^{N_t \ln\frac{\lambda^*}{\lambda} - (\lambda^* - \lambda)t} = e^{N_t \ln(1+\nu) - \lambda\nu t}. \tag{20}$$

The solution to the (Main Problem) in the Poisson model is given as follows.

Proposition 4.1 *For $0 < x < (1+\alpha)S_0$, there exists an integer $N \geq 0$, and a constant $0 < c < (1+\alpha)S_0$ such that the initial condition is satisfied:*

$$x = (1+\alpha)S_0 \sum_{k=0}^{N} x_k^{\delta}(T) - ce^{\beta T} \sum_{k=0}^{N} x_k^{(1+\nu)^q}(T), \tag{21}$$

where $\nu = \frac{\mu - r}{\lambda(1-\delta)}$, $q = \frac{1}{p-1}$, $\beta = -r + q\lambda + ((1+\nu)^q - 1 - q)\lambda^*$, $x_k^{\eta}(t) = e^{-\eta\lambda^* t}\frac{(\eta\lambda^* t)^k}{k!}$. Let $\tau = T - t$, the wealth process of the optimal strategy is

$$\hat{X}_t = \begin{cases} 0, & \text{if } N_t > N; \\ (1+\alpha)S_t \sum_{k=0}^{N-N_t} x_k^{\delta}(\tau) - cK(t)S_t^{-b}e^{\beta\tau} \sum_{k=0}^{N-N_t} x_k^{(1+\nu)^q}(\tau), & \text{if } N_t \leq N, \end{cases} \tag{22}$$

where $b = -\frac{q\ln(1+\nu)}{\ln\delta} > 0$, and $K(t) = S_0^b e^{q(\mu b+\lambda(1-\delta)b-\lambda\nu)t} > 0$. In particular, at final time T, the optimal wealth is equal to the modified payoff

$$\hat{X}_T = \hat{\phi}(1+\alpha)S_T = \begin{cases} 0, & \text{if } N_T > N; \\ (1+\alpha)S_T - cK(T)S_T^{-b}, & \text{if } N_T \le N. \end{cases} \tag{23}$$

The optimal strategy is to invest in $\hat{q}_t$ shares of stocks where

$$\hat{q}_t = \frac{\hat{X}(t,\delta S_{t-}) - \hat{X}(t,S_{t-})}{\delta S_{t-} - S_{t-}} = \frac{\hat{X}(t,N_{t-}+1) - \hat{X}(t,N_{t-})}{(\delta-1)S_0 e^{N_{t-}\ln\delta+\mu t+\lambda(1-\delta)t}}. \tag{24}$$

Remark 4.2 *Since there is one-to-one correspondence between the values of N_t and S_t, the optimal wealth process can be written as $\hat{X}_t = \hat{X}(t,N_t) = \hat{X}(t,S_t)$. Note $N_t \le N$ is equivalent to $S_t \le S_0 e^{N\ln\delta+\mu t+\lambda(1-\delta)t}$.*

Proof. As computed in proposition 3.1, the optimal final wealth should equal to the modified payoff

$$\begin{aligned} \hat{X}_T &= \hat{\phi}(1+\alpha)S_T \\ &= (1+\alpha)S_T - I(\tilde{c}Z_T) \wedge (1+\alpha)S_T \\ &= (1+\alpha)S_T - cZ_T^q \wedge (1+\alpha)S_T \\ &= \begin{cases} 0, & cZ_T^q \ge (1+\alpha)S_T \\ (1+\alpha)S_T - cZ_T^q, & cZ_T^q < (1+\alpha)S_T \end{cases} \end{aligned}$$

Recall

$$S_T = S_0 e^{N_T\ln\delta+\mu T+\lambda(1-\delta)T}, \quad Z_T = e^{N_T\ln(1+\nu)-\lambda\nu T}. \tag{25}$$

Obviously, Z_T^q is an increasing function of N_T and S_T is a decreasing function of N_T. If $c \ge (1+\alpha)S_0$ (or equivalently, $cZ_0^q \ge (1+\alpha)S_0$), then $cZ_T^q \ge (1+\alpha)S_T$ always holds and $\hat{X}_T \equiv 0$. Since $x > 0$, this not the case we are studying. We also have $x < (1+\alpha)S_0$. Therefore, $\hat{X}_T \le (1+\alpha)S_T$ and $c > 0$. For any $0 < c < (1+\alpha)S_0$, suppose $cZ_T^q < (1+\alpha)S_T$ for $N_T \le N$ and $cZ_T^q \ge (1+\alpha)S_T$ for $N_T > N$. The optimal wealth process is computed as the conditional expectation of the final wealth

$$\begin{aligned} \hat{X}_t &= e^{-r\tau}\mathbb{E}[\hat{X}_T|\mathcal{F}_t] \\ &= \begin{cases} 0, & \text{if } N_t > N; \\ e^{-r\tau}\mathbb{E}[((1+\alpha)S_T - cZ_T^q)1_{\{N_T\le N\}}|\mathcal{F}_t], & \text{if } N_t \le N. \end{cases} \\ &= \begin{cases} 0, & \text{if } N_t > N; \\ e^{-r\tau}(1+\alpha)S_t\mathbb{E}[e^{N_\tau\ln\delta+\mu\tau+\lambda(1-\delta)\tau}1_{\{N_\tau\le N-N_t\}}] & \\ -e^{-r\tau}cZ_t^q\mathbb{E}[e^{N_\tau q\ln(1+\nu)-q\lambda\nu\tau}1_{\{N_\tau\le N-N_t\}}], & \text{if } N_t \le N. \end{cases} \\ &= \begin{cases} 0, & \text{if } N_t > N; \\ (1+\alpha)S_t\sum_{k=0}^{N-N_t} x_k^\delta(\tau) - cZ_t^q e^{\beta\tau}\sum_{k=0}^{N-N_t} x_k^{(1+\nu)^q}(\tau), & \text{if } N_t \le N. \end{cases} \end{aligned}$$

Since $Z_t^q = K(t)S_t^{-b}$ where $b = -\frac{q\ln(1+\nu)}{\ln\delta} > 0$, and $K(t) = S_0^b e^{q(\mu b+\lambda(1-\delta)b-\lambda\nu)t} > 0$, we get formula (22). In particular, we can find an integer $N \ge 0$, and a constant $0 < c < (1+\alpha)S_0$ such that the initial condition is satisfied:

$$x = (1+\alpha)S_0 \sum_{k=0}^{N} x_k^{\delta}(T) - ce^{\beta T} \sum_{k=0}^{N} x_k^{(1+\nu)^q}(T).$$

The optimal strategy for the jump is easily computed using Ito's formula

$$\hat{q}_t = \frac{\hat{X}(t, \delta S_{t-}) - \hat{X}(t, S_{t-})}{\delta S_{t-} - S_{t-}}.$$

$\diamond$

Remark 4.3 The conclusion is similar to the Brownian case. The optimal investor would bankrupt the fund in the extreme case when $S_T < S_0 e^{N \ln \delta + \mu t + \lambda(1-\delta)t}$. He would under-perform the market for $S_T < (\frac{cK(T)}{\alpha})^{\frac{1}{b+1}}$, but he would outperform the market otherwise.

5 Conclusion

We have determined the optimal strategy of a trader who wants to outperform the market and we have related this problem to option hedging. Beating the market could be viewed as an act of hedging the option with insufficient initial capital for a perfect hedge. However, the trader can choose a set of scenarios when he outperforms the market, but at the cost of under-performance in some other set of scenarios. The paper identifies the optimal choice for the shortfall measure associated with power loss function. The risk averse investor's optimal choice is to outperform the market consistently when the stock is going up, but he might under perform the market when the market goes down. In the extreme case of market crash he can even bankrupt the fund. It is straightforward to modify this problem in such a way that the bankruptcy never occurs – the value of the fund being always greater than a certain fraction of the market.

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