EDGE-COLORING CLIQUES WITH THREE COLORS ON ALL 4-CLIQUES

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Abstract. A coloring of the edges of K_n is constructed such that every copy of K_4 has at least three colors on its edges. As $n \to \infty$, the number of colors used is $e^{O(\sqrt{\log n}\,)}$. This improves upon the previous probabilistic bound of $O(\sqrt{n}\,)$ due to Erdős and Gyárfás.

1. The Problem

The classical Ramsey problem asks for the minimum n such that every k-coloring of the edges of K_n yields a monochromatic K_p. For each n below this threshold, there is a k-coloring such that every p-clique receives at least 2 colors. Since the thresholds are unknown, we may study the problem by fixing n and asking for the minimum k such that $E(K_n)$ can be k-colored with each p-clique receiving at least 2 colors. This generalizes naturally as follows.

Definition. For integers n, p, q, a (p, q)-coloring of K_n is a coloring of the edges of K_n in which the edges of every p-clique together receive at least q colors. Let $f(n, p, q)$ denote the minimum number of colors in a (p, q)-coloring of K_n.

The function $f(n, p, q)$ was first studied by Elekes, Erdős and Füredi (as described in Section 9 of [1]). Erdős and Gyárfás [2] later improved the results, using the Local Lemma to prove an upper bound of $O(n^{c_{p,q}})$, where $c_{p,q} = \frac{p-2}{\binom{p}{2}-q+1}$. In addition they determined for each p the smallest q such that $f(n, p, q)$ is linear in n and the smallest q such that $f(n, p, q)$ is quadratic in n. Many small cases remain unresolved, most notably the determination of $f(n, 4, 3)$. Indeed, the Local Lemma shows only that $f(n, 4, 3) = O(\sqrt{n}\,)$, but it remains open even whether $f(n, 4, 3)/\log n \to \infty$.

In this note we show that the optimal (4,3)-coloring of K_n uses many fewer colors than the random (4,3)-coloring. We do this by explicitly constructing a (4,3) coloring of K_n. Our main theorem is the following:

Theorem. *$f(n, 4, 3) < e^{\sqrt{c \log n}\,(1+o(1))}$, where $c = 4 \log 2$.*

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2. The Coloring

In this section we describe the coloring of $E(K_n)$.

We write $[n]$ for $\{1, 2, \ldots, n\}$. The symmetric difference of sets A and B is $A \triangle B = (A - B) \cup (B - A)$. For integers $t < m$, let $\binom{[m]}{t}$ denote the family of all t-subsets of $[m]$.

Let G be the complete graph on $\binom{m}{t}$ vertices. Let $V(G) = \binom{[m]}{t}$, and for each t-set T of $[m]$, rank the $2^t - 1$ proper subsets of T according to some linear order. Color the edge AB with the two dimensional vector

$$c(AB) = (c_0(AB), c_1(AB))$$

where

$$c_0(AB) = \min\{i : i \in A \triangle B\}.$$

Set

$$S = \begin{cases} A & \text{if } c_0(AB) \in A \\ B & \text{if } c_0(AB) \in B. \end{cases}$$

Let $c_1(AB)$ be the rank of $A \cap B$ in the linear order associated with the proper subsets of S.

In this construction, the number of colors used is at most $(2^t - 1)(m - 1)$.

Remark: This construction is valid even if we let the vertex set consist of all subsets of $[m]$ of size at most t, but the gain in the number of vertices is asymptotically negligible.

3. The Proof

We now check that our coloring is a (4,3) coloring of K_n. First observe that there are no monochromatic triangles. Indeed, if ABC is one such triangle, and $c_0(AB) = i \in A$, then, since $c(AB) = c(BC)$ certainly implies that $c_0(AB) = c_0(BC)$, we have $i \in C$. But now $i \notin A \triangle C$, so $c(AC) \neq c(AB)$.

Since monochromatic triangles are forbidden, the only types of 2-colored K_4's that can occur are those in Figure 1.

Fig. 1: The 2-colored K_4's

Type 1: Here one color class is the path $ABCD$, while the other is the path $BDAC$. Suppose $c_0(AB) = i$.

Case 1: $i \in A$. Then $i \in C$ and $i \notin B, D$. Moreover,

$$A \cap [i-1] = B \cap [i-1] = C \cap [i-1] = D \cap [i-1]$$

because i is the smallest element in $A \triangle B$ and $c(AB) = c(BC) = c(CD)$. This implies that $c_0(AC) > i = c_0(AD)$. Thus $c(AC) \neq c(AD)$.

Case 2: $i \in B$. Then $i \in D$ and $i \notin A, C$. Reversing the labels on the path $ABCD$ now puts us back in Case 1.

Type 2: Here one color class is the 4-cycle $ABCD$, while the other contains the edges AC and BD. By symmetry we may assume that $c_0(AB) \in A - B$; and hence also $c_0(AB) \in C - D$. Thus $c_0(AD) = c_0(AB) \in (A \cap C) - (B \cup D)$, which implies that

1) $c_1(AB)$ is the rank of $A \cap B$ in A, and
2) $c_1(AD)$ is the rank of $A \cap D$ in A.

Since the rank of a subset in a set identifies the subset, we have $A \cap B = A \cap D$. Interchanging the roles of A and C, we obtain $C \cap B = C \cap D$.

Because $c(AC) = c(BD)$, we may assume that $c_0(AC) = c_0(BD) = i$. Thus either $i \in (A \cap B) - (C \cup D)$, or $i \in (A \cap D) - (C \cup B)$, or $i \in (C \cap B) - (A \cup D)$, or $i \in (C \cap D) - (A \cup B)$. Each of these four cases contradicts either $A \cap B = A \cap D$ or $C \cap B = C \cap D$.

Proof of Theorem. Set $t = \left\lceil \sqrt{\log n} / \sqrt{\log 2} \right\rceil$ and choose m such that $\binom{m}{t} < n \leq \binom{m+1}{t}$. Since f is a nondecreasing function of n and $(m/t)^t < \binom{m}{t}$ for $t < m$, we have

$$
\begin{aligned}
f(n,4,3) &\leq f\left(\binom{m+1}{t}, 4, 3\right) \\
&\leq (2^t - 1)m \\
&< 2^t t\, n^{1/t} \\
&= (1+o(1))\; e^{2\sqrt{\log 2 \log n} + \frac{\log\log n - \log\log 2}{2}} \\
&= e^{\sqrt{4 \log 2 \log n}\ (1+o(1))}. \quad \square
\end{aligned}
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References

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