# When is an almost monochromatic $K_4$ guaranteed?

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**Abstract**

Suppose that $n > (\log k)^{ck}$, where $c$ is a fixed positive constant. We prove that no matter how the edges of $K_n$ are colored with $k$ colors, there is a copy of $K_4$ whose edges receive at most two colors. This improves the previous best bound of $k^{c'k}$, where $c'$ is a fixed positive constant, which follows from results on classical Ramsey numbers.

## 1 Introduction

Let $p, q$ be positive integers with $2 \leq q \leq \binom{p}{2}$. A $(p,q)$-coloring of $K_n$ is an edge-coloring such that every copy of $K_p$ receives at least $q$ distinct colors on its edges. Let $f(n,p,q)$ denote the minimum number of colors in a $(p,q)$-coloring of $K_n$. This parameter, introduced in [1] and subsequently investigated by Erdős and Gyárfás [2] is a generalization of the classical Ramsey numbers. Indeed, if $R_k(p)$ denotes the minimum $n$ so that every $k$-edge-coloring of $K_n$ results in a monochromatic $K_p$, then determining all $R_k(p)$ is equivalent to determining all $f(n,p,2)$. Many special cases of $f(n,p,q)$ lead to nontrivial problems (see, e.g. [3, 5, 7, 8]). One particular interesting case is $f(n,4,3)$. In [1] it was observed that an easy application of the probabilistic method yields $f(n,4,3) = o(n)$. This was subsequently improved in [2] to $f(n,4,3) = O(\sqrt{n})$ via the Local Lemma. The second author [4] then improved the upper bound further to $e^{O(\sqrt{\log n})} = n^{o(1)}$, and this is the current best known upper bound. The lower bound follows from the well-known fact $R_k(4) < k^{O(k)}$, which implies that there is a constant $c$ such that

$$f(n,4,3) \geq f(n,4,2) > \frac{c \log n}{\log \log n}.$$

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Here we give the first improvement of this lower bound.

**Theorem 1** *Let $a \geq 1$ be fixed. There is a constant $c$ depending on $a$ such that for all $n \geq 2a$,*

$$f(n, 2a, a+1) > \frac{c \log n}{\log \log \log n}.$$

Let $R_k(p, q)$ be the minimum $n$ so that every $k$-edge-coloring of $K_n$ yields a copy of $K_p$ with at most $q-1$ colors. Then $R_k(p, q) \leq n$ implies that $f(n, p, q) > k$. Theorem 1 therefore follows from

$$R_k(2a, a+1) \leq c'(\log k)^{c'k} \tag{1}$$

where $c'$ is a positive constant depending only on $a$.

## 2 The setup of the proof

Let $a \geq 1$ be a positive integer throughout the rest of the paper.

Clearly, $f(n, 2, 2) = 0$ for $n \geq 2$. The idea of our proof is to run induction on something related to $a$, but not on $a$ itself, since in this case the scale would be too rough. To facilitate the induction, we introduce some definitions.

**Definition 2** *A $k$-edge-coloring $\chi$ of $K_n$ is a* $(\gamma_1, \ldots, \gamma_k)$-coloring *if, for each $i \in [k]$, color $i$ does not appear in any subgraph $K_{2\gamma_i+2}$ whose edges are colored with at most $\gamma_i + 1$ colors. In particular, if $\gamma_i = 0$, then color $i$ does not appear in any subgraph $K_2$ whose edges are colored with 1 color, that is, does not appear at all.*

Note that a $k$-edge-coloring of $K_N$ is a $(2a, a+1)$-coloring iff it is an $(a-1, \ldots, a-1)$-coloring. Consequently, Equation (1) states that if $K_N$ admits an $(a-1, \ldots, a-1)$-coloring with $k$ colors, then $N \leq c'(\log k)^{c'k}$, where $c'$ depends only on $a$.

**Definition 3** *For an edge-coloring $\chi$ of $K_n$ and a color $i$, the* weakness *$\gamma_i(\chi)$ of $i$ is the minimum $p$ such that color $i$ does not appear in a $K_{2p+2}$ with at most $p+1$ colors. In particular, $\gamma_i(\chi) = 0$ iff color $i$ is not present in $\chi$ at all. Then $\gamma(\chi) = \sum_{i=1}^{k} \gamma_i(\chi)$ is called the* weakness *of $\chi$.*

Note that by definition, each edge-coloring $\chi$ of $K_n$ is a $(\gamma_1(\chi), \ldots, \gamma_k(\chi))$-coloring. Also by definition, the weakness of any $(a-1, \ldots, a-1)$-coloring with $k$ colors is at most $(a-1)k$. Then (1) will follow from the following fact.

**Theorem 4** *There is a positive constant $c_1$ such that if $\chi$ is an edge-coloring of $K_N$, then*

$$N \leq c_1(\log \gamma(\chi))^{c_1 \gamma(\chi)}.$$

In everything that follows, let $\gamma_0$ be sufficiently large so that for $\gamma \geq \gamma_0$, we have $\log\log\gamma > 1$,

$$\left(\frac{\log\gamma}{1000\log\log\gamma}\right)^{15} > \frac{\log\gamma}{4500\log\log\gamma}, \qquad \text{and} \qquad 10^4\left(\frac{\log\gamma}{1000\log\log\gamma}\right)^5 \log\log\gamma > \log 2\gamma.$$

Let

$$\epsilon = \epsilon_\gamma = \frac{1000\log\log\gamma}{\log\gamma} < \frac{1}{100}, \qquad t = t_\gamma = \lceil \epsilon^{-10} \rceil, \qquad s = s_\gamma = \left\lceil \frac{(t-1)^{1/4}}{\sqrt{20}} \right\rceil > \frac{40}{\epsilon}. \tag{2}$$

Let $c = R_{\gamma_0}(2\gamma_0)$ and define $g(\gamma) = c(\log\gamma)^{1000\gamma} = c\gamma^{\epsilon\gamma}$.

We will prove Theorem 4 by showing the following:

Suppose that $\chi$ is a $(\gamma_1, \ldots, \gamma_k)$-coloring of $K_N$ and $\gamma = \sum_i \gamma_i$. Then $N < g(\gamma)$. $\qquad (*)$

We will prove $(*)$ by induction on $\gamma$ and $k$. If $0 \leq \gamma \leq \gamma_0$, then certainly $N < c \leq g(\gamma)$, so we may assume that $\gamma > \gamma_0$. If some $\gamma_i = 0$, then color $i$ cannot appear at all, so we apply induction on $k$ since the bound does not depend on $k$. Thus, we may assume that each $\gamma_i$ is positive; in particular, $k \leq \gamma$. We will also assume that $N \geq g(\gamma) = c(\log\gamma)^{1000\gamma} = c\gamma^{\epsilon\gamma}$ and proceed to get a contradiction.

For a vertex $x$ in a colored $K_n$ and a color $i$, let $d_i(x)$ denote the number of edges of color $i$ incident to $x$.

**Claim 5** *For $\gamma > \gamma_0$ and $\epsilon, t, s$ defined as above, we have $2t^{2s} < \gamma^{0.1s\epsilon - 2}$.*

**Proof.** Since $2 < t^s$ and $s > 400/\epsilon$, the result follows from $t^{3s} < \gamma^{s\epsilon/20}$, which is equivalent to $60\log t < \epsilon\log\gamma$. Since $t < \epsilon^{-11}$, we have

$$\frac{60\log t}{\epsilon} < \frac{660\log\epsilon^{-1}}{\epsilon} < \frac{660\log\gamma}{1000\log\log\gamma}\log\left[\frac{\log\gamma}{1000\log\log\gamma}\right] < \frac{\log\gamma}{\log\log\gamma}\log\log\gamma = \log\gamma. \qquad \square$$

In the next section we prove the technical statement that every dense bipartite graph $F(V_1, V_2; E)$ contains a 'large' subset $M$ of $V_1$ in which every $t$-element subset has 'many' common neighbors in $V_2$. In Section 4 we prove the main result.

## 3 A Probabilistic Lemma

One of our main tools is the following lemma, essentially Lemma 1 in [6]. The proof uses ideas of Sudakov [9]. By $N(A)$ we denote the set of common neighbors of all vertices in $A$.

**Lemma 6** *Let positive integers $m, n, h, d$ and reals $\alpha, \beta$ be such that*

$$m^{d/h} < \beta. \tag{3}$$

*Let $F = (V_1, V_2; E)$ be a bipartite graph with $|V_1| = m$, $|V_2| = n$ such that*

$$\deg_F(v) \geq n/\alpha \quad \text{for each } v \in V_1.$$

*Then there is a subset $V_1''$ of $V_1$ with $|V_1''| > m/\alpha^h - 1$ such that every $d$-tuple $D$ of vertices in $V_1''$ has at least $n/\beta$ common neighbors.*

**Proof.** Let $x_1, ..., x_h$ be a sequence of $h$ not necessarily distinct vertices of $V_2$, which we choose uniformly and independently at random and denote $S = \{x_1, ..., x_h\}$. Denote by $V_1'$ the set $N(S)$ of common neighbors of vertices in $S$. Note that the size of $V_1'$ is a random variable and that $S \subseteq N(v)$ for every $v \in V_1'$. Then, using (3), we can estimate the expected size of $V_1'$ as follows

$$\mathbf{E}\big(|V_1'|\big) = \sum_{v \in V_1} \mathbf{Pr}\big(v \in V_1'\big) = \sum_{v \in V_1} \left(\frac{|N(v)|}{n}\right)^h \geq m\,\alpha^{-h}. \tag{4}$$

On the other hand, by definition, the probability that a given set of vertices $W \subset V_1$ is contained in $V_1'$ equals $\big(|N(W)|/n\big)^h$. Denote by $Z$ the number of subsets $W$ of $V_1'$ of size $d$ with $|N(W)| < n/\beta$. Then by (3) the expected value of $Z$ is at most

$$\mathbf{E}\big(Z\big) = \sum_{W \subseteq V_1 \,:\, |W|=d,\, |N(W)|<n/\beta} \mathbf{Pr}\big(W \subset V_1'\big) \leq \binom{m}{d}\left(\frac{1}{\beta}\right)^h \leq m^d \left(\frac{1}{\beta}\right)^h < 1. \tag{5}$$

Hence, the expectation of $|V_1'| - Z$ is greater than $m\,\alpha^{-h} - 1$ and thus, there is a choice $S_0$ of $S$ such that the corresponding value of $|V_1'(S_0)| - Z(S_0)$ is greater than $m\,\alpha^{-h} - 1$. For every $d$-tuple $D$ of vertices of $V_1'(S_0)$, delete a vertex $v_D \in D$ from $V_1'(S_0)$. The resulting set $V_1''$ satisfies the lemma. □

# 4 Proof of the Theorem

Call a $t$-set of vertices *rainbow* if its edges are colored with at least $10t^{3/2}$ colors.

**Claim 7** *Suppose that $n \geq \gamma > \gamma_0$, the edges of $K_n$ are colored (with any number of colors) and $d_i(x) \leq 2n\gamma^{-\epsilon/10}$ for each $x \in V(K_n)$ and each color $i$. Then the number of $t$-sets that are not rainbow is at most $\binom{n}{t}/\gamma$.*

**Proof.** First, let us estimate $\nu(i, t, n)$ — the number of $t$-sets in $K_n$ in which there is a vertex incident with at least $s$ edges of color $i$ in this $t$-set. We can first choose the vertex, then choose $s$

incident edges of color $i$ and include the other ends of these edges, and then add $n-s-1$ other vertices. This gives

$$\nu(i,t,n) \le \sum_{x\in V(K_n)} \binom{d_i(x)}{s}\binom{n-1-s}{t-1-s} \le n\binom{\frac{2n}{\gamma^{\epsilon/10}}}{s}\binom{n-1-s}{t-1-s} \le \binom{n}{t}\gamma^{-s\epsilon/10}\, t^{2s}.$$

Similarly, let $\psi(i,t,n)$ be the number of $t$-sets in $K_n$ in which there is a matching of color $i$ of size at least $s$. Let $e_i$ be the number of edges of color $i$. Since

$$e_i \le \frac{n}{2}\max_{x\in V(K_n)} d_i(x) \le n^2\gamma^{-\epsilon/10},$$

we have

$$\psi(i,t,n) \le \binom{e_i}{s}\binom{n-2s}{t-2s} \le \binom{\frac{n^2}{\gamma^{\epsilon/10}}}{s}\binom{n-2s}{t-2s} \le \binom{n}{t}t^{2s}\gamma^{-s\epsilon/10}.$$

Now Claim 5 implies that

$$\nu(i,t,n)+\psi(i,t,n) \le 2\binom{n}{t}t^{2s}\gamma^{-s\epsilon/10} < \frac{1}{\gamma^2}\binom{n}{t}.$$

Suppose that a $t$-set $T$ contains more than $s^2$ edges of color $i$ and let $G_i$ be the graph of these edges. Either $G_i$ has a vertex incident with at least $s$ edges, or Vizing's Theorem implies that $G_i$ has a proper edge-coloring with at most $s$ colors. In the latter case, $G_i$ has a matching of size at least $s^2/s = s$. We have already shown that the number of $t$-sets that contain a monochromatic matching of size $s$ or a vertex with $s$ edges of the same color is at most $\binom{n}{t}/\gamma^2$. Consequently, the number of $t$-sets that contain more than $s^2$ edges of some color is at most

$$k\binom{n}{t}/\gamma^2 \le \binom{n}{t}/\gamma.$$

Each $t$-set not included above has at most $s^2$ edges in each color and therefore at least $\binom{t}{2}/s^2$ colors. By the choice of $s$, this is at least $10t^{3/2}$. Hence the number of rainbow $t$-sets is at least $(1-1/\gamma)\binom{n}{t}$. $\square$

**Claim 8** *Let $u \in V(K_N)$ and $S = S(u) = \{j \in [k] \ :\ d_j(u) \le N/\gamma^{1+\epsilon/2}\}$. Then for every $i \in [k]-S$ and $j \in [k]$, the number of vertices $x \in N_i(u)$ for which*

$$|N_j(x)\cap N_i(u)| \ge 2d_i(u)/\gamma^{\epsilon/10} \tag{6}$$

*is at most $\gamma^{\epsilon\gamma-3}$.*

**Proof.** Suppose the contrary. Then there are colors $i \in [k]-S(u)$ and $j \in [k]$ such that $N_i(u)$ contains a set $M$ of $\lceil\gamma^{\epsilon\gamma-3}\rceil$ vertices $x$ such that (6) holds. Consider the bipartite graph $F(V_1,V_2;E)$

with partite sets $V_1 = M$ and $V_2 = N_i(u) - M$ whose edges are all edges of color $j$ in our $K_N$ connecting $V_1$ with $V_2$. By (6) and since $|M| = \lceil \gamma^{\epsilon\gamma-3} \rceil < \lceil N/\gamma^3 \rceil < d_i(u)/\gamma^{\epsilon/10}$, we have for every $v \in V_1$,

$$\deg_F(v) > \frac{2d_i(u)}{\gamma^{\epsilon/10}} - |M| > \frac{d_i(u)}{\gamma^{\epsilon/10}} > \frac{|V_2|}{\gamma^{\epsilon/10}}.$$

Observe that graph $F$ satisfies the conditions of Lemma 6 with

$$m = |M|, \quad n = |V_2|, \quad h = \gamma/\sqrt{t}, \quad d = t, \quad \alpha = \gamma^{\epsilon/10}, \quad \beta = 2m^{t/h}.$$

Hence, there is a subset $M'$ of $V_1$ with

$$|M'| > m/\alpha^h - 1 \geq \gamma^{\epsilon\gamma-3}\alpha^{-h} - 1 > \gamma^{\epsilon\gamma-3}\gamma^{-(\gamma/\sqrt{t})\epsilon/10} - 1 > \gamma^{0.9\epsilon\gamma} \tag{7}$$

such that every $d$-tuple $D$ of vertices in $M'$ has at least $n/\beta$ common neighbors.

We will construct a sequence $M_0 \subset M_1 \subset \cdots$ of subsets of $M'$ as follows. Let $M_0 = M'$. Suppose that $M_0, M_1, \ldots, M_l$ are constructed. If there is a vertex $x_{l+1} \in M_l$ and a color $j_{l+1}$ such that $|N_{j_{l+1}}(x_{l+1}) \cap M_l| \geq |M_l|\gamma^{-\epsilon/10}$, then we let $M_{l+1} = N_{j_{l+1}}(x_{l+1}) \cap M_l$, otherwise we stop. Suppose that we stop at Step $q$. Each color $i$ appears at most $2\gamma_i + 1$ times in $\{j_1, \ldots, j_q\}$ since otherwise we have a monochromatic $K_{2\gamma_i+2}$ which is forbidden. Consequently, $q \leq \sum_i (2\gamma_i + 1) = 2\gamma + k \leq 3\gamma$. From this and (7),

$$|M_q| > |M_0|(\gamma^{-\epsilon/10})^{3\gamma} = |M_0|\gamma^{-3\gamma\epsilon/10} > \gamma^{0.9\epsilon\gamma}\gamma^{-3\gamma\epsilon/10} = \gamma^{0.6\gamma\epsilon} > \gamma.$$

Hence, by Claim 7, $M_q$ contains a rainbow $t$-tuple $D$ (in fact it contains many). Let $N_F(D) = U$. By Lemma 6, $|U| \geq n/\beta$. Now suppose $\ell$ is a color that appears in $D$. Then the weakness of $\ell$ within $U$ is strictly smaller than $\gamma_\ell$, since if $\ell$ appears in a $K_{2p}$ within $U$ that receives at most $p$ colors, then this copy together with an edge of color $\ell$ from $D$ yields a $K_{2(p+1)}$ with at most $p+1$ colors (the only new color is possibly $j$). Therefore, the weakness of $\chi$ when restricted to $U$ is at most $\gamma' = \gamma - 10t^{3/2}$. Hence by the induction hypothesis, $|U| < g(\gamma') = c(\log \gamma')^{1000\gamma'}$. Since $|U| \geq n/\beta$,

$$n \leq \beta c(\log \gamma')^{1000\gamma'}.$$

On the other hand, since $|M| < d_i(u)/2$,

$$n = |V_2| = d_i(u) - |M| > \frac{d_i(u)}{2} > \frac{N}{2\gamma^{1+\epsilon/2}}.$$

This gives

$$N < 2\gamma^{1+\epsilon/2}(2m^{t/h})c(\log \gamma')^{1000\gamma'} = 4\gamma^{1+\epsilon/2}m^{t\sqrt{t}/\gamma}c(\log \gamma')^{1000\gamma'} < \gamma^{2+\epsilon t^{3/2}}c(\log \gamma')^{1000\gamma'},$$

where the last inequality holds because $m = |M| < \gamma^{\epsilon\gamma}$. As $N \geq g(\gamma) = c(\log \gamma)^{1000\gamma}$, we get

$$(\log \gamma)^{1000\gamma} < \gamma^{2+\epsilon t^{3/2}}(\log \gamma')^{1000\gamma'} < \gamma^{2+\epsilon t^{3/2}}(\log \gamma)^{1000\gamma'}.$$

Taking logs, this reduces to

$$1000\gamma \log\log\gamma < (2+\epsilon t^{3/2})\log\gamma + 1000\gamma' \log\log\gamma.$$

Consequently,

$$(1000\log\log\gamma)10t^{3/2} < (2+\epsilon t^{3/2})\log\gamma = 2\log\gamma + 1000t^{3/2}\log\log\gamma.$$

Simplifying, we obtain $9000t^{3/2}\log\log\gamma < 2\log\gamma$. Finally, this yields

$$\left(\frac{\log\gamma}{1000\log\log\gamma}\right)^{15} = \epsilon^{-15} \le t^{3/2} < \frac{\log\gamma}{4500\log\log\gamma},$$

which contradicts our choice of $\gamma$. □

**Claim 9** *For every $u \in V(K_N)$, the number of rainbow $t$-sets on $V(K_N)-\{u\}$ all of whose vertices are connected with $u$ by edges of the same color is at least $0.3\sum_{i=1}^{k}\binom{d_i(u)}{t}$.*

**Proof.** Fix some $u \in V(K_N)$. Let $S = \{i \in [k] \,:\, d_i(u) \le N/\gamma^{1+\epsilon/2}\}$. Then

$$\sum_{i\in S}\binom{d_i(u)}{t} \le k\binom{\left\lfloor \frac{N}{\gamma^{1+\epsilon/2}}\right\rfloor}{t} \le k\binom{\lfloor \frac{N}{k}\rfloor}{t}\gamma^{-\epsilon t/2} \le 2\gamma^{-\epsilon t/2}\sum_{i=1}^{k}\binom{d_i(u)}{t}.$$

We put the factor 2 since $d_1(u)+\ldots+d_k(u) = N-1$ and not $N$. Since $t = \lceil \epsilon^{-10}\rceil$, we have $t\epsilon > 20$ and hence

$$\sum_{i\in S}\binom{d_i(u)}{t} \le \gamma^{-10}\sum_{i=1}^{k}\binom{d_i(u)}{t}. \tag{8}$$

Now, let $i \notin S$. Let $M$ be the set of vertices $x \in N_i(u)$ such that for some color $j$ (6) holds. Let $\overline{M} = N_i(u) - M$. By Claim 8,

$$|M| < \gamma^{\epsilon\gamma - 2} < \frac{N}{\gamma^2} < \frac{|N_i(u)|}{t}.$$

Hence for the subgraph $F$ of our $K_N$ on $\overline{M}$, the conditions of Claim 7 are satisfied since $|\overline{M}| > (1-1/t)d_i(u) > 0.9d_i(u) > \gamma$. Thus by Claim 7, at least $(1-1/\gamma)\binom{|\overline{M}|}{t}$ $t$-sets in $\overline{M}$ are rainbow. Now

$$\frac{\gamma-1}{\gamma}\binom{|\overline{M}|}{t} \ge \frac{\gamma-1}{\gamma}\binom{d_i(u)(1-1/t)}{t}.$$

For large $\gamma$, the last expression is at least

$$0.9\left(\frac{t-1}{t}\right)^{t}\binom{d_i(u)}{t} \ge \frac{1}{3}\binom{d_i(u)}{t}.$$

Combining this with (8), we finish the proof. □

By Claim 9, the total number of $(t+1)$-sets $\{u_0, u_1, \ldots, u_t\}$ of vertices of $V(K_N)$ such that the $t$-set $\{u_1, \ldots, u_t\}$ is rainbow and all edges from $u_0$ to $u_1, \ldots, u_t$ are of the same color is at least

$$0.3 \sum_{u \in V(K_N)} \sum_{i=1}^{k} \binom{d_i(u)}{t} \geq 0.3N \cdot k \binom{(N-1)/k}{t} \geq N \cdot (2k)^{1-t} \binom{N}{t}.$$

It follows that some rainbow $t$-set $\{u_1, \ldots, u_t\}$ is contained in at least $N \cdot (2k)^{1-t}$ such $(t+1)$-sets. Let $U$ be the set of all vertices $u_0$ in these $(t+1)$-sets containing our chosen $\{u_1, \ldots, u_t\}$. Then, for some $1 \leq i \leq k$ the size of the subset $U_i$ of $U$ that is connected with each of $u_1, \ldots, u_t$ by an edge of color $i$ is at least $2N \cdot (2k)^{-t}$. Since $\{u_1, \ldots, u_t\}$ is rainbow, it contains edges of at least $10t^{3/2}$ colors. For every color $\ell$ that appears within $\{u_1, \ldots, u_t\}$, the weakness of $\ell$ when restricted to $U_i$ is at most $\gamma_\ell - 1$. Hence by the induction hypothesis, $|U_i| \leq g(\gamma') = c(\log \gamma')^{1000\gamma'}$, where $\gamma' = \gamma - 10t^{3/2}$. Since $|U_i| \geq 2N/(2k)^t$ and $N \geq g(\gamma)$, we obtain

$$c(\log \gamma)^{1000\gamma} \leq N < (2k)^t c(\log \gamma')^{1000\gamma'} < (2k)^t c(\log \gamma)^{1000\gamma'}.$$

Dividing by $c$ and taking logs,

$$1000\gamma \log \log \gamma < t \log 2\gamma + 1000\gamma' \log \log \gamma.$$

Consequently,

$$(1000 \log \log \gamma) 10t^{3/2} < t \log 2\gamma.$$

Plugging in the values of $t$ and $\epsilon$, we obtain

$$10^4 \left( \frac{\log \gamma}{1000 \log \log \gamma} \right)^5 \log \log \gamma = 10^4 \epsilon^{-5} \log \log \gamma < 10^4 \sqrt{t} \log \log \gamma < \log 2\gamma.$$

This contradicts our choice of $\gamma$ and completes the proof. □

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