

Two-regular subgraphs of hypergraphs

Dhruv Mubayi *

Jacques Verstraëte †

April 25, 2008

Abstract

We prove that the maximum number of edges in a k -uniform hypergraph on n vertices containing no 2-regular subhypergraph is $\binom{n-1}{k-1}$ if $k \geq 4$ is even and n is sufficiently large. Equality holds only if all edges contain a specific vertex v . For odd k we conjecture that this maximum is $\binom{n-1}{k-1} + \lfloor \frac{n-1}{k} \rfloor$, with equality only for the hypergraph described above plus a maximum matching omitting v .

1 Introduction

One of the most basic facts in combinatorics is that an acyclic graph on n vertices has at most $n - 1$ edges, with equality only for trees. A natural generalization to hypergraphs (see Berge [3] for more details) is obtained by defining a circuit to be a hypergraph consisting of distinct vertices v_1, v_2, \dots, v_k and distinct edges e_1, \dots, e_k such that $v_i \in e_i$ for $i = 1, 2, \dots, k$, $v_{i+1} \in e_i$ for $i = 1, 2, \dots, k - 1$, and $v_1 \in e_k$. Then a hypergraph H with no circuit satisfies

$$\sum_{e \in H} (|e| - 1) \leq |V(H)| - 1.$$

In this paper, we consider a generalization to hypergraphs in a different direction. Since a cycle is a 2-regular graph, we may ask for the maximum number of edges that a hypergraph on n vertices can have without a 2-regular subgraph – i.e. a subhypergraph in which every vertex has degree two. Throughout the paper, hypergraphs where all edges have size k are called k -uniform hypergraphs or, simply, k -graphs. A star is a hypergraph in which there is a vertex v such that all possible edges containing v are present and there are no other edges. Our main result shows that stars are the extremal hypergraphs not containing a 2-regular subgraph when k is even:

Theorem 1. *For every even integer $k > 2$, there exists an integer n_k such that for $n \geq n_k$, if H is an n -vertex k -graph with no 2-regular subgraph, then $|H| \leq \binom{n-1}{k-1}$. Equality holds if and only if H is a star.*

*Department of Mathematics, Statistics and Computer Science, University of Illinois, Chicago, Illinois 60607; research partially supported by NSF grants DMS 0400812, 0653946, and an Alfred P. Sloan Research Fellowship. E-mail: mubayi@math.uic.edu

†Department of Mathematics, University of California at San Diego, 9500 Gilman Drive, La Jolla, California 92093-0112, USA; research supported by an Alfred P. Sloan Research Fellowship. E-mail: jverstra@ucsd.edu

The non-uniform analog of this theorem, which is much simpler, is proved in Section 2. As one might expect, the proof of Theorem 1 needs completely new techniques than the graph case. The result is proved via the stability approach. Stability results were introduced in extremal graph theory by Erdős and Simonovits [14] in the 60's. The program of using stability to prove exact results has been recently used with great success in extremal set theory (see [5, 6, 7, 8, 9, 10, 11]). Perhaps the main difficulty in passing to an exact result when k is odd is that stars are not extremal when k is odd: it is possible to add to a star on n vertices a matching of size $\lfloor \frac{n-1}{k} \rfloor$, resulting in an n -vertex k -graph with no 2-regular subgraph with a few more edges than a star. We conjecture that this “star-plus-matching” construction is the unique extremal configuration when k is odd:

Conjecture 1. *For every odd integer $k \geq 3$, there exists an integer n_k such that for $n \geq n_k$, if H is an n -vertex k -graph with no 2-regular subgraph then $|H| \leq \binom{n-1}{k-1} + \lfloor \frac{n-1}{k} \rfloor$. Equality holds if and only if H is a star with center v together with a maximal matching omitting v .*

Conjecture 1 is a weaker version of a conjecture due to Füredi, that for $k > 3$, a k -graph containing no two pairs of disjoint sets with the same union has at most $\binom{n-1}{k-1} + \lfloor \frac{n-1}{k} \rfloor$ edges. For odd $k > 3$, this implies Conjecture 1; in fact two pairs of disjoint sets with the same union is the smallest possible 2-regular k -graph when k is odd. The question of determining the maximum number of edges $f_k(n)$ of a k -graph on n vertices containing no two pairs of disjoint edges with the same union was originally raised by Erdős (see [4]). This problem was studied by Frankl and Füredi [4], and the authors [12], who showed that $f_k(n) < 3\binom{n}{k-1}$, and this is the current best upper bound on $f_k(n)$.

This paper is organized as follows. In the next section, we prove the nonuniform analogue of Theorem 1, that a collection of subsets of an n -element set with no 2-regular subsystem has size at most 2^{n-1} with equality (for $n \geq 3$) only for a star. In Section 3, we present two lemmas used to prove Theorem 1. The proof of Theorem 1 is in Sections 4–6, and has three parts. First we shall show (see Section 4) that if H is an n -vertex k -graph with no 2-regular subgraph, then $|H| \lesssim \binom{n-1}{k-1}$. Using this result, we prove the stability result (see Section 5), which says that if $|H| \sim \binom{n-1}{k-1}$ then $\Delta(H) \sim \binom{n-1}{k-1}$. Finally, we use this stability theorem to prove Theorem 1 in Section 6. The final section mentions related open problems.

Terminology. We denote by $V(H)$ the set of vertices of a hypergraph H . The degree of a vertex v , written $d(v)$, is the number of edges containing that vertex. A matching is a hypergraph whose edges are pairwise disjoint – equivalently this is a hypergraph in which every vertex has degree one. A k -graph is a hypergraph where all sets have size k , and a hypergraph is r -regular if all its vertices have degree r . We write $\binom{X}{k}$ for the collection of all k -sets of X . A star is a hypergraph or a k -graph consisting of all possible edges containing a fixed vertex. For a hypergraph H , denote by $\Delta(H)$ its maximum degree. For $v \in V(H)$, let $H - \{v\} = \{e \in H : v \notin e\}$ and $H_v = \{e \setminus \{v\} : v \in e \in H\}$. If $f, g : \mathbb{N} \rightarrow \mathbb{R}$ are two functions then we write $f(n) \gtrsim g(n)$ to denote that $f(n) \geq g(n)h(n)$ for some function $h(n)$ such that $\liminf_{n \rightarrow \infty} h(n) = 1$. This is an equivalent but more convenient way to write $f(n) \geq (1 + o(1))g(n)$. In the case $f(n) = (1 + o(1))g(n)$ we write $f(n) \sim g(n)$. If there is a

constant $c > 0$ such that $f(n) \geq cg(n)$ for all n , then we write $f(n) \gg g(n)$. Throughout this paper, k is always fixed relative to n .

2 Non-uniform hypergraphs

In this section, we prove the nonuniform analogue of Theorem 1. We stipulate that edges of a hypergraph are non-empty sets. A star on n vertices is a hypergraph consisting of all 2^{n-1} sets containing a fixed vertex.

Theorem 2. *Let $n \geq 1$ and let H be a hypergraph on n vertices containing no 2-regular subgraph. Then $|H| \leq 2^{n-1}$. If $n \geq 3$ and equality holds, then H is a star.*

Proof. We remark that it is easy to obtain an upper bound 2^{n-1} : if H has no 2-regular subgraph, then H contains at most one complementary pair – a complementary pair consists of the edge e and the edge $V(H) \setminus \{e\}$. This shows $|H| \leq 2^{n-1} + 1$, but if H contains both edges of some complementary pair, then $V(H)$ cannot be an edge of H , showing $|H| \leq 2^{n-1}$. For the characterization of equality, we proceed by induction on n for $n \geq 3$.

It is straightforward to check the case $n = 3$; we omit the details. Now we proceed to the induction step. Let us assume that $n \geq 4$ and H has size $|H| = 2^{n-1}$ and no 2-regular subgraph. We will show that H is a star. Since a star is a maximal 2-regular subgraph, this proves Theorem 2. First we show that every $v \in V(H)$, apart from at most one vertex, has degree exactly 2^{n-2} . If there is a vertex $v \in V(H)$ with $d(v) < 2^{n-2}$, then $H - \{v\}$ has a 2-regular subgraph, by induction. So every vertex of H has degree at least 2^{n-2} . Pick a vertex $x \in V(H)$. If x is contained in every set in H , then H is a star with center x and we're done. We may therefore assume that there is a set $e \in H$ missing x . Assume that $|e| = k$ where $1 \leq k \leq n$. For each subset $f \subset V(H) \setminus (e \cup \{x\})$, the number of sets in H containing x whose intersection with $V(H) \setminus (e \cup \{x\})$ is f is at most 2^{k-1} , for otherwise two of these sets have complementary intersections in e and these together with e give a 2-regular subgraph, a contradiction. Hence the number of sets containing x is at most $2^{n-k-1}2^{k-1} = 2^{n-2}$. So x has degree exactly 2^{n-2} , in which case $|H - \{x\}| = 2^{n-2}$. By induction, $H - \{x\}$ is a star with center at some vertex w . But now it is easy to see that all sets containing x must also contain w , else we find a 2-regular subgraph (either x lies in 2 sets omitting w , or one set omitting w and one containing w since the degree of x is $2^{n-2} > 1$; in each case choose an appropriate set containing w to form a 2-regular subgraph). Therefore H is a star with center w . \square

3 Preliminary Lemmas

In this section, we present two lemmas which will be used in proving Theorem 1. The first lemma involves matchings. If M_1 and M_2 are distinct matchings and $V(M_1) = V(M_2)$, then $M_1 \triangle M_2$ is a hypergraph whose vertices all have degree two. This observation is the key point of the following lemma.

Lemma 1. *Let H be a k -graph on n vertices containing no 2-regular subgraph. Then $|H| \leq 6n\Delta^{(k-1)/k}$ or $|H| < 2k\Delta$.*

Proof. Let $|H| = dn/k$ and suppose $|H| \geq 2k\Delta$. Then it is enough to prove that $\Delta \geq (1/k)(d/6)^{k/(k-1)}$ to prove the lemma. Suppose, for a contradiction, that this is not true. We count matchings in H of size $m = \lfloor |H|/k\Delta \rfloor$ to show that H contains a 2-regular subgraph. Note that $m \geq 2$ since $|H| \geq 2k\Delta$. For a lower bound on the number of matchings of size m , we may greedily pick disjoint edges f_1, f_2, \dots, f_m where at each step we exclude all edges that intersect previously chosen edges. Since at each step we exclude at most $k\Delta$ new edges, the number of matchings of size m in H is at least

$$\frac{1}{m!} \prod_{i=0}^{m-1} (|H| - k\Delta i) = \frac{1}{m!} |H|^m \prod_{i=0}^{m-1} \left(1 - \frac{k\Delta i}{|H|}\right) \geq \frac{1}{m!} |H|^m \prod_{i=0}^{m-1} \left(1 - \frac{i}{m}\right) \geq (k\Delta)^m.$$

To complete the proof, we show that there exist distinct matchings M_1, M_2 of H such that $\bigcup_{f \in M_1} f = \bigcup_{f \in M_2} f$. This suffices, since the edges in $M_1 \Delta M_2$ form a 2-regular subgraph, thus contradicting the fact that H has no 2-regular subgraph. Using the upper bound on Δ :

$$\binom{n}{mk} < \left(\frac{3n}{mk}\right)^{mk} \leq \left(\frac{6k\Delta}{d}\right)^{km} < (k\Delta)^m.$$

Here we used $m \geq dn/2k^2\Delta$ and then the assumed upper bound on Δ . Since $\binom{n}{mk}$ is the number of sets of mk vertices of H , and there are more than $(k\Delta)^m$ matchings of size m in H , we find the two required distinct matchings M_1, M_2 . \square

Our second lemma involves circuits in hypergraphs. A circuit is a hypergraph consisting of distinct vertices v_1, v_2, \dots, v_k and distinct edges e_1, \dots, e_k such that $v_i \in e_i$ for $i = 1, 2, \dots, k$, $v_{i+1} \in e_i$ for $i = 1, 2, \dots, k-1$, and $v_1 \in e_k$. We require the following lemma on 2-regular subgraphs arising from circuits in hypergraphs of a certain bipartite structure:

Lemma 2. *Let G be a k -graph and $V(G) = A \cup B$, where $A \cap B = \emptyset$, all edges $e \in G$ have $|e \cap A| = k-1$, and every $(k-1)$ -set in A lies in at least two edges of G . If G has no 2-regular subgraph, then*

$$|G| < 2|B| \binom{|A| + k - 3}{k - 2}.$$

Proof. It is enough to show $|G| < 2|B| \binom{|A|-1}{k-2}$ when $|A| = a(k-1)$ for some integer a , since we may always add at most $k-2$ points to A so that $k-1$ divides $|A|$. Baranyai's Theorem [2] states that if s divides n , then the complete s -graph on n vertices can be partitioned into $\binom{n-1}{s-1}$ perfect matchings. Using this theorem with $s = k-1$, we write

$$\binom{A}{k-1} = M^1 \cup M^2 \cup \dots \cup M^{\binom{|A|-1}{k-2}}$$

where each M^i is a matching and the matchings are edge-disjoint. For each matching $M^i = \{e_1^i, \dots, e_a^i\}$, let G^i be the set of edges in E whose intersection with A is e_j^i for some j . Let

f_j^i be the set of vertices $v \in B$ such that $e_j^i \cup \{v\} \in G^i$. Consider the hypergraph H_j^i with edges f_j^i , $j = 1, \dots, a$. If H_j^i contains a circuit with vertices v_1, v_2, \dots, v_p , then G contains the 2-regular subgraph with edges

$$e_1^i \cup \{v_1\} \quad e_1^i \cup \{v_2\} \quad e_2^i \cup \{v_2\} \quad e_2^i \cup \{v_3\} \quad e_p^i \cup \{v_p\} \quad e_p^i \cup \{v_1\}$$

which contradicts that G has no 2-regular subgraph. Consequently, H_j^i has no circuit. It is well-known that a hypergraph H with no circuit satisfies

$$\sum_{e \in H} (|e| - 1) \leq (|V(H)| - 1). \quad (1)$$

By hypothesis, $|f_j^i| \geq 2$ for all i, j . Applying (1) to H_j^i , we therefore obtain

$$\sum_j |f_j^i| \leq \sum_j 2(|f_j^i| - 1) < 2|B|. \quad (2)$$

Adding (2) over different i, j , we obtain

$$|G| = \sum_i \sum_j |f_j^i| \leq 2|B| \binom{|A| - 1}{k - 2}. \quad \square$$

4 The Asymptotic Result

Theorem 3. *Let $k \geq 3$ and let H be an n -vertex k -graph with no 2-regular subgraph. Then $|H| - \binom{n-1}{k-1} \ll n^{k-1-1/11}$.*

Proof. We prove the following more precise statement: for all $n > k^{100}$,

$$|H| < (1 + cn^{-\gamma}) \binom{n-1}{k-1}$$

where $c = 2(k+1)!$ and $\gamma = \frac{1}{11}$. Define $\alpha = (k+1)/(3k-1)$ for $k > 3$ and $\alpha = 7/11$ for $k = 3$. Suppose, for a contradiction, that $|H|$ equals this upper bound (we may assume it is an integer for this theorem) for some H . Let T denote the set of vertices of H of degree at least $D = n^{k-1-\alpha}$, and set $t = |T|$. Then $tD \leq k|H|$ and, since $n > k^{100}$,

$$t < D^{-1}k(1 + cn^{-\gamma}) \binom{n-1}{k-1} < kn^\alpha. \quad (3)$$

Let $H_i = \{e \in H : |e \cap T| = i\}$ for $i \leq k$, and define $G = \{e \in H_1 : \nexists f \in H_1 : e \setminus T = f \setminus T\}$. In particular, it is clear that $|G| \leq \binom{n-1}{k-1}$.

Claim 1. $|H_i| < \begin{cases} 6n^{1+(k-1)(k-1-\alpha)/k} & \text{for } i = 0 \\ |G| + 2kn^{k-2+\alpha} & \text{for } i = 1 \end{cases}$

Proof. Since $\Delta(H_0) < D$, by definition of T , the first bound follows from Lemma 1. For the second bound, we apply Lemma 2 to $H_1 \setminus G$ with $A = V(H) \setminus T$ and $B = T$ to obtain $|H_1 \setminus G| < 2|T| \binom{n+k-3}{k-2} < 2tn^{k-2}$. The bound on $|H_1|$ now follows from (3).

Claim 2. $|H \setminus (H_0 \cup H_1)| < \begin{cases} k^2 n^{k-2+2\alpha} & \text{for } k > 3 \\ 6(n^{1+\alpha} + n^{3\alpha}) & \text{for } k = 3 \end{cases}$

Proof. For $k > 3$, by definition, every edge in $H \setminus (H_0 \cup H_1)$ consists of two vertices of T and $k - 2$ vertices of $V(H)$, so certainly $|H \setminus (H_0 \cup H_1)| \leq \binom{|T|}{2} n^{k-2}$. Now apply (3). For $k = 3$, observe that $|H_3| < \binom{|T|}{3}$. Furthermore, by Lemma 2, with $A = T$ and $B = V(H) \setminus T$, $|H_2| < 2|T|(n - |T|) + \binom{|T|}{2} < 2tn$. Here we note that there could be $\binom{|T|}{2}$ pairs in T contained in only one triple of H_2 . Those contained in two triples or more are the ones to which Lemma 2 applies, giving the bound $2|T|(n - |T|)$ for those triples. Using (3) gives the claim.

Now we complete the proof. By definition of α , the bounds in Claims 1 and 2 are all of order at most $n^{k-1-\gamma}$ (the case $i = 0$ in Claim 1 needs a somewhat tedious calculation). Specifically,

$$|H \setminus G| = |H_0| + |H_1 \setminus G| + |H \setminus (H_0 \cup H_1)| < (6 + k^2 + 2k)n^{k-1-\gamma} < 4k^2 n^{k-1-\gamma}. \quad (4)$$

Using the bound $|G| \leq \binom{n-1}{k-1}$ in (4), we obtain

$$|H| = |G| + |H \setminus G| < \binom{n-1}{k-1} + 4k^2 n^{k-1-\gamma} < (1 + cn^{-\gamma}) \binom{n-1}{k-1}.$$

This contradiction completes the proof. □

5 Stability

Theorem 4. *Let $k \geq 3$ and let H_n be an n -vertex k -graph with no 2-regular subgraph. If $|H_n| \sim \binom{n-1}{k-1}$, then $\Delta(H_n) \sim \binom{n-1}{k-1}$.*

Proof. For simplicity of notation, we let $H = H_n$ and omit the subscript n when dealing with hypergraphs constructed from H . As in the proof of Theorem 3, let T denote the set of vertices in H of degree at least $n^{k-1-\alpha}$, $H_1 = \{e \in H : |e \cap T| = 1\}$ and $G = \{e \in H_1 : \nexists f \in H_1 : e \setminus T = f \setminus T\}$. Define

$$G' = \{e \setminus T : e \in G\}.$$

For each $x \in T$, let $G_x = \{e \in G' : e \cup \{x\} \in G\}$. Let v be a vertex such that $|G_v| = \max_{x \in T} |G_x|$. Note that all sets in G have size k , and all sets in G' or any G_x have size $k - 1$. By (4), $|G'| = |G| \sim |H| \sim \binom{n-1}{k-1}$, so it suffices to prove $|G_v| \sim |G'|$ for some v to prove the theorem. Suppose, for a contradiction, that for some positive $\varepsilon < \frac{1}{2}$,

$$|G_v| \lesssim (1 - \varepsilon)|G'|. \quad (5)$$

For any hypergraph F , define $P(F) = \{\{e, f\} \subset F : |e \cap f| = 1\}$. Define $P_1(G') \subset P(G')$ to be the set of pairs $\{e, f\} \in P(G')$ such that $e, f \in G_x$ for some x , and $P_2(G') = P(G') \setminus P_1(G')$. The strategy is to use (5) to derive a contradiction by finding edges $e, e' \in G_x$ and $f, f' \in G_y$, for some $x \neq y$, such that $|e \cap f| = 1 = |e' \cap f'|$, $e \Delta f = e' \Delta f'$ and $e \cap f \neq e' \cap f'$ (sometimes the latter condition will be guaranteed by $e \cap e' = \emptyset = f \cap f'$). For in this case, the edges

$$e \cup \{x\} \quad e' \cup \{x\} \quad f \cup \{y\} \quad f' \cup \{y\} \tag{6}$$

form a 2-regular subgraph of H .

Claim 1. $|P_2(G')| \leq \frac{1}{2} \binom{t}{2} \binom{2k-4}{k-2} \binom{n-1}{2k-4}$.

Proof. Fix distinct vertices $x, y \in T$. We show that the number of $\{e, f\} \in P_2(G')$ such that $e \in G_x$ and $f \in G_y$ for $x \neq y$ is at most $\frac{1}{2} \binom{2k-4}{k-2} \binom{n-1}{2k-4}$. This completes the proof, since there are $\binom{t}{2}$ choices for x and y .

First observe that there are at most $\binom{n-1}{2k-4}$ choices for $e \cup f$ with $\{e, f\} \in P_2(G')$, since $|e| = |f| = k - 2$ and $e \cap f = \emptyset$. We now count the number of pairs $\{e', f'\} \in P_2(G')$ such that $e' \in G_x$ and $f' \in G_y$ and $e' \cup f' = e \cup f$. Let $\{e_i, f_i\} : i = 1, 2, \dots, I$ be an enumeration of these pairs with $e_i \cup f_i = e \cup f$ and $e_i \in G_x$ and $f_i \in G_y$. If $e_i \cap f_i \neq e_j \cap f_j$ for some $i \neq j$, then we obtain a 2-regular subgraph as in (6). So we may assume that $e_i \cap f_i = e_j \cap f_j$ for all $i \neq j$ and $i, j \leq I$. Since we also have $e_i \Delta f_i = e_j \Delta f_j$, it must be that

$$I \leq \frac{1}{2} \binom{2k-4}{k-2}$$

since the right hand side is the number of ways to partition $e \cup f$ into two sets of size $k - 2$, and all the pairs $\{e_i, f_i\}$ are distinct. So the number of pairs $\{e, f\} \in P_2(G')$ with $e \in G_x$ and $f \in G_y$ is bounded above by

$$\binom{n-1}{2k-4} \cdot I \leq \binom{n-1}{2k-4} \cdot \frac{1}{2} \binom{2k-4}{k-2}.$$

This completes the proof of the claim.

For the rest of the proof, let $\psi(\varepsilon) = ((1 - \varepsilon)2 + \varepsilon 2)^{1/2}$. For $i \in \{1, 2\}$, let $Q_i(G')$ denote the set of pairs $\{\{e, f\}, \{e', f'\}\}$ such that $\{e, f\}, \{e', f'\} \in P_i(G')$, $e \cap e' = \emptyset = f \cap f'$ and $e \Delta f = e' \Delta f'$. These are called type i quadrilaterals of G' . For $x \in T$, define $Q_1(G_x)$ to be the collection of pairs $\{\{e, f\}, \{e', f'\}\} \in Q_1(G')$ such that $\{e, f, e', f'\} \subset G_x$. These are type 1 quadrilaterals of G_x . Let K be the complete $(k - 1)$ -graph on $V(G')$. Recall that $P(K)$ is the number of pairs $\{e, f\} \subset K$ such that $|e \cap f| = 1$. So in the case that $k = 2$, when K is the complete graph, this is just the number of paths of length two.

Claim 2. $|P_1(G')| \lesssim \psi(\varepsilon) \cdot |P(K)|$.

Proof. Let $\{\{e, f\}, \{e', f'\}\} \in Q_1(G')$. If $e, f \in G_x$ and $e', f' \in G_y$ with $x \neq y$, then we obtain a 2-regular subgraph similar to that in (6). We conclude that if $e, f \in G_x$, then also $e', f' \in G_x$. It follows that

$$|Q_1(G')| = \sum_{x \in T} |Q_1(G_x)|. \quad (7)$$

For a pair $\{g, h\}$ of disjoint sets of size $k-2$ in $V(G')$, let $p(g, h)$ denote the number of pairs $\{e, f\} \in P_1(G')$ with $e \setminus f = g$ and $f \setminus e = h$. The number of such pairs $\{g, h\}$ is at most

$$\binom{\binom{n-1}{k-2}}{2} := N.$$

Note also that the sum of $p(g, h)$ over all $\{g, h\} \subset V(G')$ is exactly $|P_1(G')|$. By convexity of binomial coefficients,

$$|Q_1(G')| = \sum_{\{g, h\}} \binom{p(g, h)}{2} \gtrsim \binom{|P_1(G')|/N}{2} \cdot N \sim \frac{|P_1(G')|^2}{\binom{n-1}{k-2}^2}. \quad (8)$$

The first equality is the hypergraph analog of the fact that the number of quadrilaterals in a graph F is exactly $\sum_{u, v \in V(F)} \binom{p(u, v)}{2}$ where $p(u, v)$ is the number of paths of length two from u to v in F . On the other hand, we observe that $|Q_1(G_x)| \leq \frac{1}{2}(k-1)^2 \binom{|G_x|}{2}$, since if we fix two disjoint edges, say $e, e' \in G_x$, then the number of type 1 quadrilaterals of the form $\{\{e, f\}, \{e', f'\}\}$ is at most $(k-1)^2$. The same type 1 quadrilaterals are counted if we had fixed the two disjoint edges $f, f' \in G_x$ instead of e, e' , and this gives the observation. Therefore, by (7),

$$|Q_1(G')| \leq \frac{1}{2}(k-1)^2 \sum_{x \in T} \binom{|G_x|}{2}.$$

By convexity, this sum is a maximum when $|G_v| \sim (1-\varepsilon)|G'|$ and $|G_w| \sim \varepsilon|G'|$ for some $w \neq v$, and the rest of the $|G_x|$ s are zero. Therefore

$$|Q_1(G')| \lesssim \frac{1}{4}(k-1)^2 \psi(\varepsilon) 2|G'|^2. \quad (9)$$

Combining (8) and (9), and $|G'| \sim \binom{n-1}{k-1}$, we obtain

$$|P_1(G')| \lesssim \psi(\varepsilon) \cdot \frac{1}{2}(k-1)|G'| \binom{n-1}{k-2} \lesssim \psi(\varepsilon)|P(K)|.$$

This proves Claim 2.

We now complete the proof for $k > 3$. Since $|G'| \sim |K|$, it is straightforward to see that $|P(G')| \sim |P(K)|$, where we recall

$$|P(K)| = n \binom{\binom{n-1}{k-2}}{2}.$$

If d_x is the number of sets in $K \setminus G'$ which contain $x \in V(G')$, then

$$\begin{aligned} |P(K) \setminus P(G')| &\leq \sum_{x \in V(G')} \binom{d_x}{2} + \sum_{x \in V(G')} d_x \left(\binom{n-2}{k-2} - d_x \right) \\ &= (k-1)|K \setminus G'| \binom{n-2}{k-2} - \frac{1}{2} \sum_{x \in V(G')} d_x^2 - \frac{1}{2}(k-1)|K \setminus G'|. \end{aligned} \quad (10)$$

Here we used that

$$\sum_{x \in V(G')} d_x = (k-1)|K \setminus G'|.$$

Now since $|P(K)|$ is of order n^{2k-3} , and $|G'| \sim |K|$, we see that all terms in (10) are negligible relative to $|P(K)|$, except possibly the sum of d_x^2 . We wish to find

$$\max \sum_{x \in V(G')} d_x^2 \quad \text{if} \quad \sum_{x \in V(G')} d_x = |K \setminus G'|.$$

The maximum possible value of d_x is $\binom{n-2}{k-2}$. For a maximum of the sum of squares, we let

$$\frac{(k-1)|K \setminus G'|}{\binom{n-2}{k-2}}$$

of the d_x take the value $\binom{n-2}{k-2}$, and the rest are zero. Therefore

$$\max \sum_{x \in V(G')} d_x^2 \leq (k-1)|K \setminus G'| \binom{n-2}{k-2}$$

and this is negligible relative to $|P(K)|$ since $|K| \sim |G'|$ and $|P(K)|$ has order n^{2k-3} .

By (3), $t \leq kn^\alpha$ where $\alpha < \frac{1}{2}$ (this relies on $k > 3$). Therefore

$$\begin{aligned} |P(K)| \sim |P(G')| &= |P_1(G')| + |P_2(G')| \\ &\lesssim \psi(\varepsilon)|P(K)| + 2 \binom{t}{2} \binom{2k-4}{k-2} \binom{n}{2k-4} \\ &\sim \psi(\varepsilon)|P(K)|. \end{aligned} \quad (11)$$

However, $\psi(\varepsilon) < 1$, so the above inequality is a contradiction.

For $k = 3$, G' is a graph and $P(G')$ is the set of paths of length two in G' . The problem with the above arguments for $k = 3$ is that (3) only gives $t \leq 3n^{7/11}$, which is too large for (11) to hold and provide a contradiction. Therefore we go one step further, and count paths of length three in G' instead of paths of length two. Let $P_3(G')$ be the number of paths of length three in G' with edges from three different G_x s. By Claim 2,

$$|P_2(G')| = |P(G')| - |P_1(G')| \gtrsim (1 - \psi(\varepsilon))|P(K)| \gg n^3. \quad (12)$$

As in Claim 1, if $\{\{e, f\}, \{e', f'\}\}$ is a type 2 quadrilateral of G and $e, e' \in G_x$ and $f, f' \in G_y$, then we obtain a 2-regular subgraph of H . So each type 2 quadrilateral contains edges from

at least three different G_x s, and these edges form a path of length three in G' . Consequently, as in (8), the convexity of binomial coefficients and (12) give

$$|P_3(G')| \geq \frac{1}{4}|Q_2(G')| \geq \frac{1}{4} \binom{|P_2(G')|/N}{2} N \gg n^4$$

since $N = \binom{n-1}{2}$. Let (A, B) be a random partition of $V(G')$, defined by placing a vertex in A with probability $\frac{1}{2}$ and in B with probability $\frac{1}{2}$, independently for each vertex of $V(G')$. Let G^* denote the graph consisting of all edges between A and B . Then the expected value of $|P_3(G^*)|$ is exactly $\frac{1}{8}|P_3(G')|$, so there is a partition of G' for which

$$|P_3(G^*)| \geq \frac{1}{8}|P_3(G')| \gg n^4. \quad (13)$$

Let $e_1e_2e_3$ and $f_1f_2f_3$ be two paths in G^* with the same pair of endpoints. Suppose $e_i \in G_{j(i)}$ and $f_i \in G_{h(i)}$ where $\{j(1), j(2), j(3)\} = \{h(1), h(2), h(3)\}$. Since G^* is bipartite, amongst these edges there is a cycle C of length four or six containing exactly zero or two edges from each $G_{j(i)}$, $i = 1, 2, 3$. It is easily checked that the unique edges of H' which contain the edges of C form a 2-regular subgraph of H , which is a contradiction. We conclude that at most $\binom{t}{3}$ paths of length three in G^* with edges in different G_i s have the same pair of endpoints. It follows that

$$|P_3(G^*)| \leq \binom{t}{3} \binom{n}{2} \ll n^{4-\frac{1}{11}}$$

using (3). This contradicts (13), and completes the proof of Theorem 4. \square

The following corollary is an immediate consequence of Theorem 4.

Corollary 1. *Fix $k \geq 3$. For every $\varepsilon > 0$, there exists $n_{k,\varepsilon}$ such that if $n > n_{k,\varepsilon}$ and H is an n -vertex k -graph containing no 2-regular subgraph, with $|H| \geq \binom{n-1}{k-1}$, then there is a vertex $v \in V(H)$ such that $|H - \{v\}| < \varepsilon n^{k-1}$.*

6 The Exact Result

In this section we prove Theorem 1. Our main tools are the asymptotic and stability result. Let H be an n -vertex k -graph containing no 2-regular subgraph, where $k \geq 4$ is even, and suppose $|H| = \binom{n-1}{k-1}$. Let $\varepsilon = \frac{1}{100k^{4k}}$. By Corollary 1, for large enough n , there is a vertex $v \in V(H)$ such that

$$|H - \{v\}| \leq \varepsilon n^{k-1}. \quad (14)$$

Let $H^* = H - \{v\}$. To complete the proof, we show $|H^*| = 0$. Suppose, for a contradiction, that $|H^*| > 0$. For $|e| = k - 2$, let $d_v(e)$ be the degree of a set e in H_v . Let

$$s = n - k + 1 - \frac{2k|H^*|}{\binom{n-1}{k-2}}.$$

Claim 1. There are pairwise disjoint $(k-2)$ -sets $e_1, e_2, \dots, e_k \subset V(H) \setminus \{v\}$ such that $d_v(e_i) \geq s$ for $i \in \{1, 2, \dots, k\}$.

Proof. Let F be the family of $(k-2)$ -sets in $V(H_v)$ whose degree is at least s , and let F^c be the rest of the $(k-2)$ -sets in $V(H_v)$. Then

$$(k-1)|H_v| = \sum_e d_v(e) \leq |F|(n-k+1) + |F^c|s.$$

where the sum is over $e \subset V(H_v)$ of size $k-2$. As $|F| + |F^c| = \binom{n-1}{k-2}$, this implies

$$\frac{2k|H^*||F|}{\binom{n-1}{k-2}} \geq (k-1)|H_v| - s \binom{n-1}{k-2} = 2k|H^*| - (k-1)|H^*|$$

since $|H^*| = \binom{n-1}{k-1} - |H_v|$. Hence $|F| \geq (1 - \frac{k-1}{2k}) \binom{n-1}{k-2} > \frac{1}{2} \binom{n-1}{k-2}$. Let $\{e_1, e_2, \dots, e_l\}$ be a maximum matching in F . If $l < k$, then all other sets of F have an element within $e_1 \cup e_2 \cup \dots \cup e_l$, which implies (since we may take n large enough) that

$$|F| \leq (k-1)(k-2) \binom{n-1}{k-3} < k^2 \binom{n-1}{k-3} < \frac{1}{2} \binom{n-1}{k-2}.$$

This contradiction shows that $l \geq k$ and the claim is proved.

Let $W = \{w \in V(H_v) \mid \exists i : e_i \cup \{v, w\} \notin H\}$. By Claim 1, $|W| < k(n-s)$. By adding points arbitrarily to W , we may assume that $|W| = \lceil k(n-s) \rceil$. Define, for each $i \in \{0, 1, \dots, k\}$, $H_i = \{e \in H^* : |e \cap W| = i\}$ and let $G = H_0 \cup H_1 \cup \dots \cup H_{k-2}$. Note that the H_i partition H^* .

Claim 2. $|H_{k-1}| \leq \binom{|W|}{k-1}$.

Proof. Suppose there exists a $(k-1)$ -set $e \subset W$ and elements $y, z \notin W$ such that $e \cup \{y\}, e \cup \{z\} \in H_{k-1}$. Since $|e| = k-1$, by Claim 1 and the definition of W there exists i such that $e_i \cap e = \emptyset$ and $e_i \cup \{v, y\}, e_i \cup \{v, z\} \in H$. Together with $e \cup \{y\}$ and $e \cup \{z\}$, this yields a 2-regular subgraph in H . This contradiction implies that we may count sets in H_{k-1} by their intersection with W to obtain $|H_{k-1}| \leq \binom{|W|}{k-1}$.

Claim 3. $|H^*| \geq \binom{n-k-1}{k/2-1}$.

Proof. Since $|H^*| \geq 1$, there exists $e \in H^*$. Let e' be a $\frac{k}{2}$ -subset of e . Now for each choice of a $(\frac{k}{2}-1)$ -set $f \subset V(H_v) \setminus e$, one of the sets $f \cup e' \cup \{v\}$ or $f \cup (e \setminus e') \cup \{v\}$ must be missing from H , otherwise these two sets together with e form a 2-regular subgraph of H . Consequently, $|H^*| \geq \binom{n-k-1}{k/2-1}$.

Claim 4. $|G| > \frac{99}{100}|H^*|$.

Proof. We show $|H_{k-1}| + |H_k| < \frac{1}{100}|H^*|$. By Theorem 3, there is a smallest integer $n_0 = n_0(k)$ such every k -graph on n vertices with no 2-regular subgraph and with $n > n_0$ has at most $2 \binom{n_0-1}{k-1}$ edges. Assume also that $n_0 > 3k^2$. If $|W| < n_0$, then $|H_k| + |H_{k-1}| < |W|^k < n_0^k$. If n

is large enough then, by Claim 3, this is less than $\frac{|H^*|}{100}$, as required. So we assume $|W| > n_0$. Since the k -graph H_k itself contains no 2-regular subgraph, $|H_k| \leq 2 \binom{|W|-1}{k-1}$. Recall that

$$|W| = \lceil k(n-s) \rceil = \left\lceil k \left(k-1 + \frac{2k|H^*|}{\binom{n-1}{k-2}} \right) \right\rceil < k^2 + \frac{2k^2|H^*|}{\binom{n-1}{k-2}}.$$

Using this and $|W| > n_0 > 3k^2$, we obtain

$$|W| < \frac{3 \cdot 2k^2|H^*|}{2 \binom{n-1}{k-2}} = \frac{3k^2|H^*|}{\binom{n-1}{k-2}}.$$

Now suppose, for a contradiction, that $|H_k| + |H_{k-1}| > \frac{|H^*|}{100}$. By Claim 2,

$$\frac{|H^*|}{100} < |H_k| + |H_{k-1}| < 2 \binom{|W|-1}{k-1} + \binom{|W|}{k-1} < \frac{3|W|^{k-1}}{(k-1)!} < \frac{k^{2k}|H^*|^{k-1}}{\binom{n-1}{k-2}^{k-1}}.$$

Simplifying,

$$|H^*|^{k-2} > \binom{n-1}{k-2}^{k-1} \frac{1}{100k^{2k}} > \binom{n-1}{k-2}^{(k-2)(k-1)} \frac{1}{100k^{2k}} > \frac{(n-1)^{(k-2)(k-1)}}{100k^{(k-2)(k-1)+2k}}.$$

This implies that $|H^*| > \frac{(n-1)^{k-1}}{100k^{k-1+2k/(k-2)}} > \varepsilon n^{k-1}$, which contradicts (14). This completes the proof of Claim 4.

Let p be the number of pairs (e, f) such that

- (1) $v \notin e \in H$ and $|e \cap W| \leq k-2$ (i.e. $e \in G = \cup_{i=0}^{k-2} H_i$)
- (2) $v \in f \notin H$ and $|f| = k$ (so the number of such f s is $|H^*|$)
- (3) $|e \cap f| = \frac{k}{2}$
- (4) $e \cap f$ and $e \setminus f$ (which are both $\frac{k}{2}$ -sets) have a point outside W .

Fix $e \in H$ as in (1) above. Since $|e \setminus W| \geq 2$, there is a $\frac{k}{2}$ -subset $g \subset e$ such that neither g nor $e \setminus g$ lies within W . Let h be a $(\frac{k}{2}-1)$ -subset of $V(H) \setminus (W \cup e \cup \{v\})$ and let $f = g \cup h \cup \{v\}$. Then the three sets $e, f, (e \setminus g) \cup h \cup \{v\}$ form a 2-regular subgraph. Consequently, either $g \cup h$ or $(e \setminus g) \cup h$ is not in H_v . The number of pairs $\{g, e \setminus g\}$ that we can take in this argument is at least

$$\frac{1}{2} \binom{k}{k/2} - \binom{|W \cap e|}{k/2} \geq \frac{1}{2} \binom{k}{k/2} - \binom{k-2}{k/2}.$$

Therefore, counting p from the e 's we have

$$\begin{aligned} p &\geq (0.99)|H^*| \left(\frac{1}{2} \binom{k}{k/2} - \binom{k-2}{k/2} \right) \binom{n-|W|-k-1}{k/2-1} \\ &> (0.98)|H^*| \left(\frac{1}{2} \binom{k}{k/2} - \binom{k-2}{k/2} \right) \binom{n}{k/2-1}, \end{aligned}$$

where the last inequality holds since $|W| < \varepsilon k^{4k} n$ and n is sufficiently large.

On the other hand, counting p from the f s we have $p \leq |H^*| \binom{k-1}{k/2} q$, where q is the number of times the $\frac{k}{2}$ -sets $g \subset f \setminus \{v\}$ can extend to e , where $e \cap f = g$. Let F be the $\frac{k}{2}$ -graph of these possible extensions of g to e . Let $F_0 \subset F$ be the $\frac{k}{2}$ -graph whose edges have no points in W and $F_1 \subset F$ be the $\frac{k}{2}$ -graph whose edges have at least one point in W .

Claim 5. $|F_0| < 2 \binom{n}{k/2-1} / k$ and $|F_1| \leq \varepsilon k^{2k} \binom{n}{k/2-1}$.

Proof. We start with F_1 : to each $\frac{k}{2}$ -set $h \in F_1$ associate a $(\frac{k}{2} - 1)$ -set $h' \subset h$ such that $h' \cap W \neq \emptyset$ and $W \cap h \subset h'$. Such an h' exists by the definition of F_1 . If there are distinct $h_1, h_2 \in F_1$ with $h'_1 = h'_2$, then there are distinct vertices $y, z \notin W$ such that $h_1 = h'_1 \cup \{y\}$ and $h_2 = h'_2 \cup \{z\}$. By Claim 1, there exists i for which e_i has no point of $W \cap (g \cup h_1 \cup h_2)$. Now the four sets $g \cup h_1, g \cup h_2, e_i \cup \{v, y\}, e_i \cup \{v, z\}$ form a 2-regular subgraph of H , contradicting that H has no 2-regular subgraph. Consequently, $|F_1|$ is at most the number of $(\frac{k}{2} - 1)$ -sets of $V(H)$ that contain at least one point of W . This is at most

$$|W| \binom{n}{k/2-2} < \frac{3k^2 \varepsilon n^{k-1}}{\binom{n-1}{k-2}} \binom{n}{k/2-2} < \varepsilon k^{2k} \binom{n}{k/2-1}.$$

This gives the bound on $|F_1|$. If there are distinct $h_1, h_2 \in F_0$ with $|h_1 \cap h_2| = k/2 - 1$, then arguing as above we find a 2-regular subgraph of H . Consequently, $|F_0| < \binom{n}{k/2-1} / \binom{k/2}{k/2-1}$.

Putting these bounds together we have $q \leq \varepsilon k^{2k} \binom{n}{k/2-1} + 2 \binom{n}{k/2-1} / k$, and this gives

$$p \leq (1 + \varepsilon k^{4k}) |H^*| \binom{k-1}{k/2} \frac{2}{k} \binom{n}{k/2-1}.$$

Comparing the upper and lower bounds for p and dividing by $|H^*| \binom{n}{k/2-1}$ yields

$$(0.98) \left(\frac{1}{2} \binom{k}{k/2} - \binom{k-2}{k/2} \right) < (1 + \varepsilon k^{4k}) \frac{2}{k} \binom{k-1}{k/2}.$$

Since $\varepsilon < k^{4k} / 100$ this implies that

$$(0.97) \left(\frac{1}{2} \binom{k}{k/2} - \binom{k-2}{k/2} \right) < \frac{2}{k} \binom{k-1}{k/2}.$$

A short calculation shows that this is equivalent to $(0.97k - 2)(k - 1) < 0.97k(\frac{k}{2} - 1)$, and it is easily verified that this is false for $k \geq 4$. This contradiction completes the proof of the theorem. \square

7 Concluding Remarks

A k -graph is r -regular if all its vertices have degree r . In contrast to Theorem 1, if the degrees in a k -graph are all the same, then a linear number of edges already forces a 2-regular subgraph. Precisely, let ϕ_k denote the maximum number ϕ such that there exists a ϕ -regular k -graph

containing no 2-regular subgraph. Then Lemma 1 immediately implies that $\phi_k \leq (6k)^k$. On the other hand, we have a lower bound of $\binom{3k/2-1}{k-1}$ when k is even and $\binom{2k-1}{k-1}$ when k is odd, by taking complete k -graphs of the appropriate size (these contain no 2-regular subgraphs because every 2-regular subgraph has at least $3k/2$ vertices when k is even and at least $2k$ vertices when k is odd). The lower bounds are of order c^k , so there is a substantial gap in the bounds for ϕ_k . We leave the open problem of determining ϕ_k and, in particular, ϕ_3 . It is expected that if a k -graph is ϕ -regular and ϕ is a large enough constant depending on k and r , then every ϕ -regular k -graph has an r -regular subgraph (a subgraph in which every vertex has degree r). In fact, this should hold for multi- k -graphs – instead of a set of edges a multiset of edges is allowed. Therefore we make the following conjecture:

Conjecture 2. *Let $k, r \geq 2$. There exists an integer $\phi_k(r)$ such that for $\phi \gg \phi_k(r)$, every ϕ -regular multi- k -graph contains an r -regular subgraph.*

This conjecture is wide open for $k, r \geq 3$. If r is a prime not dividing k and we superimpose $r - 1$ copies of the complete k -graph on $k + 1$ vertices, namely K_{k+1}^k , then we obtain a multi- k -graph $H_{r,k}$ containing no r -regular subgraph. To check this, let J be the all-one matrix, so that $J - I$ is the incidence matrix of K_{k+1}^k , with rows indexed by edges and columns by vertices. Then over \mathbb{Z}_r , the field of integers mod r , we have $(r - 1)(J - I) = I - J$, and this matrix has full rank over \mathbb{Z}_r , since r does not divide k . Therefore no set of rows of $(r - 1)(J - I)$ is linearly dependent over \mathbb{Z}_r , which means $H_{r,k}$ has no non-empty subgraph in which all vertices have degree zero modulo r . This simple construction shows that if $\phi_k(r)$ exists, then $\phi_k(r) \geq k(r - 1)$. For $k = 2$, in other words, for multigraphs, Tâskinov [15] completely determined $\phi_k(r)$ using Tutte’s f -Factor Theorem. Unfortunately, no analogous theorem for k -graphs is known when $k \geq 3$. The following positive evidence for Conjecture 2 follows immediately by extending the proof of Alon, Friedland, Kalai [1] and uses Chevalley’s theorem:

Theorem 5. *Let H be an n -vertex multi- k -graph, such that H is $k(r - 1) + 1$ -regular, where r is a prime number. Then H has a subgraph all of whose vertex degrees are elements of $\{r, 2r, \dots, (k - 1)r\}$.*

The multi- k -graph $H_{r,k}$ shows that Theorem 5 is tight. Further evidence for Conjecture 2 comes from Rödl’s packing method [13]. A k -graph is linear if no two of its edges intersect in two or more points. If M is a matching in a k -graph H , let $\text{ex}(M)$ denote the number of vertices not covered by M . Rödl’s Theorem [13] says that every linear n -vertex d -regular k -graph contains a matching M such that $\text{ex}(M) \leq d^{-\varepsilon}n$, for some constant $\varepsilon > 0$ depending only on k . In fact, the degrees of the vertices in the hypergraph are allowed to be between $(1 - \delta)d$ and $(1 + \delta)d$ for the same conclusion, provided $\delta > 0$ is a sufficiently small constant depending on ε . By repeatedly removing r such matchings from a linear d -regular k -graph, we see that for any fixed r , we obtain a subgraph in which all vertices have degree at most r , and at most $rd^{-\varepsilon}n$ vertices have degree less than r . In other words, we find an “almost r -regular” subgraph. On the other hand, we do not even have a verification that every large enough Steiner triple system has a three-regular subgraph.

References

- [1] N. Alon, S. Friedland, G. Kalai. Every 4-regular graph plus an edge contains a 3-regular subgraph. *J. Combin. Theory Ser. B* 37 (1984), no. 1, 92–93.
- [2] Zs. Baranyai. On the factorization of the complete uniform hypergraph, *Infinite and finite sets (Colloq., Keszthely, 1973)*, *Colloq. Math. Soc. Janos Bolyai* 10 North-Holland, Amsterdam, (1975) 91–108.
- [3] C. Berge. *Hypergraphs: The Theory of Finite Sets*. Amsterdam, Netherlands: North-Holland, 1989.
- [4] Z. Füredi. Hypergraphs in which all disjoint pairs have distinct unions. *Combinatorica* 4 no. 2-3 (1984) 161–168.
- [5] Z. Füredi, O. Pikhurko and M. Simonovits. 4-Books of three pages, *J. Combin. Theory, Ser. A* 113 (2006), 882–891.
- [6] Z. Füredi and M. Simonovits. Triple systems not containing a Fano configuration, *Combin. Probab. Comput.* no. 14 (2005), 467–484.
- [7] P. Keevash and B. Sudakov, The Turán number of the Fano plane, *Combinatorica* no. 25 (2005), 561–574.
- [8] P. Keevash and B. Sudakov, On a hypergraph Turán problem of Frankl, *Combinatorica* no. 25 (2005), 673–706.
- [9] D. Mubayi. Structure and Stability of Triangle-free set systems, *Trans. Amer. Math. Soc.* 359 (2007), 275–291.
- [10] D. Mubayi. An intersection theorem for four sets , *Advances in Mathematics*, 215 (2007) no. 2, 601–615.
- [11] D. Mubayi and O. Pikhurko. A new generalization of Mantel’s theorem to k -graphs, *J. Combin. Theory Ser. B*, to appear.
- [12] D. Mubayi, J. A. Verstraëte. hypergraph extension of the bipartite Turán problem. *J. Combin. Theory Ser. A* 106 (2004) no. 2 237–253.
- [13] V. Rödl. On a Packing and Covering Problem, *European Journal of Combinatorics* 5 (1985) 69–78
- [14] M. Simonovits. A method for solving extremal problems in graph theory, stability problems. 1968 *Theory of Graphs (Proc. Colloq., Tihany, 1966)* 279–319, Academic Press, New York.
- [15] V. A. Tâskinov, Regular subgraphs of regular graphs, *Soviet Math. Dokl.* 26 (1982) 37–38.