Homework Set 5

1) Let $A$ be a nonzero hermitian matrix. Show that $\text{tr}(AA^*) > 0$.

2 a) Show that the absolute value of the determinant of a real unitary matrix is 1.
   b) If $A$ is a complex square matrix, then show that $\text{Det}(\overline{A}) = \overline{\text{Det}(A)}$. Conclude that the absolute value of the determinant of a complex unitary matrix is 1.

3) Let $A : V \to V$ be a symmetric linear map. Show that the index of nullity of the form
   $$(v, w) \to \langle Av, w \rangle$$
   is equal to the dimension of the kernel of $A$. Show that the index of positivity is equal to the number of eigenvectors in a spectral basis having a positive eigenvalue.

4) If $A$ and $B$ are submodules of $M$, then show that:
   (a) $A \cap B$ is a submodule of $M$
   (b) $A + B = \{a + b : a \in A, b \in B\}$ is a submodule of $M$
   (c) $(A + B)/B$ is isomorphic to $A/(A \cap B)$. 