Homework Set 2

1) Let \( R \) be the ring of \( 2 \times 2 \) matrices with rational entries. Prove that the only ideals of \( R \) are \((0)\) and \( R \).

2) Let \( R \) be the ring of all real valued continuous functions on \([0,1] \). Let \( M \) be a maximal ideal of \( R \). Prove that there is a real number \( \gamma \in [0,1] \) such that \( M = \{ f(x) \in R : f(\gamma) = 0 \} \). Hint: Proceed by contradiction. Use the fact that \([0,1]\) is compact, so every open cover of it has a finite subcover.

3) Let \( R \) be a Euclidean ring and \( a, b \in R \). The least common multiple \( c \) of \( a \) and \( b \) is an element of \( R \) such that \( a|c \) and \( b|c \) and such that whenever \( a|x \) and \( b|x \) for \( x \in R \), then \( c|x \). Prove that \( c \) exists and that \( c \times (a, b) = ab \), where \( (a, b) \) is the gcd of \( a \) and \( b \).

4) Define the derivative \( f'(x) \) of the polynomial \( f(x) = \sum_{i=0}^{n} a_{i}x^{i} \) as \( f'(x) = \sum_{i=1}^{n} ia_{i}x^{i-1} \). Prove that if \( f(x) \in F[x] \), where \( F \) is the field of rational numbers, then \( f(x) \) is divisible by the square of a polynomial if and only if \( f(x) \) and \( f'(x) \) have a gcd \( d(x) \) of positive degree.