Turán number of theta graphs

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Abstract

The theta graph $\Theta_{\ell,t}$ consists of two vertices joined by $t$ vertex-disjoint paths of length $\ell$ each. For fixed odd $\ell$ and large $t$, we show that the largest graph not containing $\Theta_{\ell,t}$ has at most $c\ell t^{1-1/\ell}n^{1+1/\ell}$ edges and that this is tight apart from the value of $c\ell$.

1 Introduction

Given a graph $F$, the Turán number for $F$, denoted by $\text{ex}(n, F)$ is the maximum number of edges in an $n$-vertex graph that contains no subgraph isomorphic to $F$. Mantel and Turán determined this function exactly when $F$ is a complete graph, and the study of Turán numbers has become a fundamental problem in combinatorics (see [20, 22, 26] for surveys). The Erdős–Stone theorem [13] determines the asymptotic behavior of $\text{ex}(n, F)$ whenever $\chi(F) \geq 3$, and so the most interesting Turán-type problems are when the forbidden graph is bipartite.

One of the most well-studied bipartite Turán problems is the even cycle problem: the study of $\text{ex}(n, C_{2\ell})$. Erdős initiated the study of this problem when he needed an upper bound on $\text{ex}(n, C_4)$ in order to prove a theorem in combinatorial number theory [10]. The combination of the upper bounds by Kővari, Sós and Turán [23] and the lower bounds by Brown [5] and Erdős, Rényi and Sós [12] gave the asymptotic formula

$$\text{ex}(n, C_4) \sim \frac{1}{2} n^{3/2}.$$

It is now known that for certain values of $n$ the extremal graphs must come from projective planes [16, 18, 15] and this is conjectured to be the case for all $n$ (see [17]).

A general upper bound of $\text{ex}(n, C_{2\ell})$ of $c\ell n^{1+1/\ell}$ for sufficiently large $n$ was originally claimed by Erdős [11] and first published by Bondy and Simonovits [4] who showed that one can take $c\ell = 20\ell$. Subsequent improvements of the best constant $c\ell$ to $8(\ell - 1)$ by Verstraëte [28], to $(\ell - 1)$ by Pikhurko [25], and to $80\sqrt{\ell}\log \ell$ by Bukh and Jiang [8] were made, and this final bound is the current record.

As stated above, we have an asymptotic formula for $\text{ex}(n, C_4)$. Additionally, the upper bound on $\text{ex}(n, C_{2\ell})$ is of correct order of magnitude for $\ell \in \{3, 5\}$ [2, 30], i.e., $\text{ex}(n, C_{2\ell}) = \Theta(n^{1+1/\ell})$ for these values of $\ell$. However, unlike the case of $C_4$, the sharp multiplicative constant is not known; see [19] for the best bounds on $\text{ex}(n, C_6)$. The order of magnitude for $\text{ex}(n, C_{2\ell})$ is unknown for any $\ell \notin \{2, 3, 5\}$. The best known general lower bounds are given by Lazebnik, Ustimenko and Woldar [24] (but see [27] for a better bound for the $\text{ex}(n, C_{14})$ case).

Although it is unclear whether $\text{ex}(n, C_{2\ell}) = \Omega(n^{1+1/\ell})$ holds in general, more is known if instead of forbidding a pair of internally disjoint paths of length $\ell$ between pairs of vertices (that is, a $C_{2\ell}$) one forbids several paths of length $\ell$ between pairs of vertices. For $t \in \mathbb{N}$, let $\Theta_{t,\ell}$ be the graph made of $t$ internally disjoint paths of
length \( \ell \) connecting two endpoints. The study of \( \text{ex}(n, \Theta_{\ell,t}) \) generalizes the even cycle problem as \( \Theta_{\ell,2} = C_{2\ell} \). Faudree and Simonovits showed [14] that

\[
\text{ex}(n, \Theta_{\ell,t}) = O_{\ell,t} \left( n^{1+1/\ell} \right).
\]

More recently, Conlon showed that this upper bound gives the correct order of magnitude if the number of paths is a large enough constant [9]. That is, there exists a constant \( c_{\ell} \) such that \( \text{ex}(n, \Theta_{\ell,c_{\ell}}) = \Theta_{\ell}(n^{1+1/\ell}) \). Verstraëte and Williford [29] constructed graphs with no \( \Theta_c \) paths is a large enough constant [9]. That is, there exists a constant \( L \) that by Lemma 4 each vertex in \( G \) has at least \( d \) neighbors in \( G \), and since every graph of average degree 2 contains a bipartite subgraph of minimum degree \( 1+1/\ell \), we henceforth assume that the graph is bipartite of minimum degree \( 1+1/\ell \).

In this paper, we are interested in the behavior of \( \text{ex}(n, \Theta_{\ell,t}) \) when \( \ell \) is fixed and \( t \) is large. When \( \ell = 2 \), the result of Füredi [21] shows that \( \text{ex}(n, \Theta_{2,t}) \sim \frac{1}{2} \sqrt{n^{3/2}} \). For general \( \ell \), the result of Faudree and Simonovits gives that \( \text{ex}(n, \Theta_{\ell,t}) \leq c_{\ell} t^2 n^{1+1/\ell} \). We improve this bound as follows.

**Theorem 1.** For fixed \( \ell \geq 2 \), we have

\[
\text{ex}(n, \Theta_{\ell,t}) = O_{\ell,t} \left( n^{1+1/\ell} \right).
\]

When \( \ell \) is odd, we show that the dependence on \( t \) in Theorem 1 is correct.

**Theorem 2.** Let \( \ell \geq 3 \) be a fixed odd integer. Then

\[
\text{ex}(n, \Theta_{\ell,t}) = \Omega_{\ell} \left( t^{1-1/\ell} n^{1+1/\ell} \right).
\]

We do not know if Theorem 1 is tight when \( \ell \) is even. In this case, our best lower bound is the following.

**Theorem 3.** Let \( \ell \geq 2 \) be a fixed even integer. Then

\[
\text{ex}(n, \Theta_{\ell,t}) = \Omega_{\ell} \left( t^{1/\ell} n^{1+1/\ell} \right).
\]

It would be interesting to close the gap between Theorems 1 and 3 for even \( \ell \).

Since the proof of Theorem 1 is relatively involved, we begin by introducing the main ideas in Section 2 where we prove the theorem in the case \( \ell = 3 \). Then in Sections 3 and 4 we extend this argument to prove the general upper bound. In Sections 5 and 6 we give constructions for odd and even values of \( \ell \) respectively.

## 2 Case \( \ell = 3 \)

In this section we present the proof of Theorem 1, dealing with the case \( \ell = 3 \). As every graph of average degree 4d contains a bipartite subgraph of average degree 2d, and since every graph of average degree 2d contains a subgraph of minimum degree d, we henceforth assume that the graph is bipartite of minimum degree d.

**Lemma 4.** Let \( r \) be any vertex of \( G \). Call a vertex \( u \) bad if \( u \neq r \) and \( u \) has more than \( t \) common neighbors with \( r \). If \( G \) is \( \Theta_{3,t} \)-free, then no neighbor of \( r \) is adjacent to \( t \) bad vertices.

**Proof.** Suppose \( w \) is adjacent to bad vertices \( u_1, \ldots, u_t \). Define a sequence of vertices \( z_1, \ldots, z_t \) as follows. We let \( z_i \) be any common neighbor of \( r \) and \( u_i \) other than \( w, z_1, \ldots, z_{i-1} \). It exists since there are more than \( t \) common neighbors between \( r \) and \( u_i \). Then \((wu_1z_ir)_{i=1}^t\) is a collection of \( t \) disjoint paths of length 3 from \( w \) to \( r \).

**Proof of Theorem 1 for \( \ell = 3 \)** Let \( r \) be any vertex of \( G \). Let \( \{r\} \). Let \( L_1 \) be the set of all the neighbors of \( r \). Let \( L_2 \) be the set of all vertices at distance 2 from \( r \) that have at most \( t \) common neighbors with \( r \). Note that by Lemma 4 each vertex in \( L_1 \) has at least \( d-t \) neighbors in \( L_2 \). Call a vertex \( v_1 \in L_1 \) a parent of \( v_2 \in L_2 \) if \( v_1 \) and \( v_2 \) are adjacent. Note that a vertex in \( L_2 \) can have at most \( t \) parents. Hence, each vertex in \( L_2 \) has at least \( d-t \) neighbors in \( V(G) \setminus L_1 \).
Let $L_3$ be all vertices in $V(G) \setminus L_1$ that are adjacent to some $L_2$. Call $v_3 \in L_3$ a descendant of $v_1 \in L_1$ if there is a path of the form $v_1 v_2 v_3$ with $v_2 \in L_2$.

Let $B(v_1) \subset L_3$ be the set of all the descendants of $v_1$ that have more than $t$ common neighbors with $v_1$. By Lemma 4, each $v_2 \in N(v_1)$ has fewer than $t$ neighbors in $B(v_1)$.

Let $H$ be the subgraph of $G$ obtained from $G$ by removing all edges between $B(v_1)$ and $N(v_1)$ for all $v_1 \in L_1$. Since each $v_2 \in L_2$ has at most $t$ parents, each vertex in $L_2$ has at least $d - t - t(t - 1) = d - t^2$ neighbors in $L_3$.

For a vertex $v_3 \in L_3$, let $p(v_3)$ be the number of paths (in $H$) of the form $rv_1v_2v_3$ with $v_i \in L_i$. We claim that $p(v_3) \leq 2t(t - 1)$ for every $v_3 \in L_3$. Indeed, suppose the contrary. We will construct a $\Theta_{3,4}$ subgraph as follows. First, we pick any path $rv_1^{(1)}v_2^{(1)}v_3$ counted by $p(v_3)$. Since $v_3$ and $v_1^{(1)}$ have at most $t$ common neighbors, and since $r$ and $v_2^{(1)}$ have at most $t$ common neighbors, at most $2t$ paths counted by $p(v_3)$ intersect $\{v_1^{(1)}, v_2^{(1)}\}$. So, we can pick another path $rv_1^{(2)}v_2^{(2)}v_3$ that is disjoint from $\{v_1^{(1)}, v_2^{(1)}\}$. We can repeat this, at each step selecting path $rv_1^{(i)}v_2^{(i)}v_3$ that is disjoint from $\bigcup_{j<i} \{v_1^{(j)}, v_2^{(j)}\}$ for $i = 1, \ldots, t$. The paths $rv_1^{(i)}v_2^{(i)}v_3$ together form a $\Theta_{3,4}$. So, $p(v_3) \leq 2t(t - 1)$ after all.

Since each vertex in $L_1$ has at least $d - t$ neighbors in $L_2$ and each vertex in $L_2$ has at least $d - t^2$ neighbors in $L_3$, it follows that

$$|L_3| \geq \frac{d(d - t)(d - t^2)}{2t(t - 1)}.$$

Since $|L_3| \leq n$ the result follows. \qed

### 3 General case

**Outline.** The case of general $\ell$ is similar to the case $\ell = 3$. Starting with a root vertex, we build a sequence of layers $L_1, L_2, \ldots, L_\ell$ such that each next layer is about $d$ times larger than the preceding. The condition of being $\Theta_{\ell,t}$-free is used to ensure that a vertex in $L_j$ descends from a vertex in $L_i$ in at most $O(t^{j-i-1})$ ways. However, there are two complications that are not present in the proof of the $\ell = 3$ case.

First, in the definition of $L_2$ we excluded vertices that have too many neighbors back. Doing so affects degrees of yet-unexplored vertices, such as those in $L_3$. That was not important for the $\ell = 3$ case because the $L_3$ was the final layer. In general, though, we will maintain a set of ‘bad’ vertices and will control how removal of these vertices affects subsequent layers.

Second, removing vertices from later layers reduces degrees of the vertices in the preceding layers. So, instead of trying to ensure that each vertex has large degree, we will maintain a weaker condition that there are many paths from the root to the leaves of the tree.

**Minimum and maximum degree control.** As in the proof of the case $\ell = 3$, we will need to ensure that all vertices are of large degree. For technical reasons, which will become apparent in Section 4 we need to control not only the minimum, but also the maximum degrees. This is done with the help of the following lemma:

**Lemma 5** (Theorem 12 in [8], only in arXiv version). Every $n$-vertex graph with $\geq 4cn^{1+1/\ell}$ edges contains a subgraph $G$ such that

- The graph $G'$ has at least $cn^{1/2\ell}$ vertices, and
- Degree of each vertex of $G'$ is between $cv(G')^{1/\ell}$ and $\Delta cv(G')^{1/\ell}$ where $\Delta = (20t)^{2\ell}$.

Henceforth we assume that our graph is bipartite, and that each vertex has degree between $d$ and $\Delta d$, where $\Delta$ is as above. We will show that $d^\ell \leq (8t)^{\ell-1} n(1 + o_r(1))$, and hence that every $\Theta_{\ell,t}$-free graph has at most

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For this notation is that we imagine $\deg(v_i, v_j)$ denote the number of linear paths from $v_i$ to $v_j$. For sets $A \subseteq L_i$ and $B \subseteq L_j$, we denote by $P(A, B)$ the number of linear paths going from a vertex in $A$ to a vertex in $B$.

In addition to sets $L_0, L_1, \ldots, L_k$ we will also maintain a sets $B_1, \ldots, B_{k-1}$ of bad vertices. All the sets $L_0, L_1, \ldots, L_k, B_1, \ldots, B_{k-1}$ will be disjoint. We shall say that we are at stage $k$ if the sets $L_0, L_1, \ldots, L_k$ and $B_1, \ldots, B_{k-1}$ have been defined, but the sets $L_{k+1}$ and $B_k$ have not yet been defined. We denote by $U \defeq V(G) \setminus (L_0 \cup \cdots \cup L_k \cup B_1 \cup \cdots \cup B_{k-1})$ the set of unexplored vertices.

For $v \in L_i$, let $\overrightarrow{N}(v)$ be the set of children of $v$, and let $\overleftarrow{N}(v)$ be the set of parents of $v$. We define $\deg(v) \defeq |\overrightarrow{N}(v)|$ and $\deg(v) \defeq |\overleftarrow{N}(v)|$ to be the number of children and parents of $v$, respectively. The reason for this notation is that we imagine $L_0, \ldots, L_k$ grow from left to right.

Let

$$R_m \defeq \frac{(2\ell)^m}{m + 1} \left(\begin{array}{c}2m \\ m \end{array}\right) \ell^m.$$ 

Note that $\frac{1}{m + 1} \left(\begin{array}{c}2m \\ m \end{array}\right)$ is the $m$'th Catalan number. We call a pair of layers $(L_i, L_j)$ with $i < j$ regular if for every pair of vertices $(v_i, v_j) \in L_i \times L_j$ the number of linear paths from $v_i$ to $v_j$ is $P(v_i, v_j) \leq R_{j-i-1}$.

We start the exploration process by picking a root vertex $r$, and setting $L_0 = \{r\}$ and $L_1 = N(r)$. At the $k$'th stage of the sets $B_1, B_2, \ldots, B_{k-1}, L_0, L_1, \ldots, L_k$ satisfy the following properties:

P1. The root is preserved: $L_0 = \{r\}$.

P2. No orphans: every vertex of $L_i$ for $i = 1, 2, \ldots, k$ has at least one parent.

P3. The explored part is tree-like: every pair of layers $(L_i, L_j)$ with $0 \leq i < j < k$ is regular. (Note that pairs of the form $(L_i, L_k)$ might be irregular.)

P4. Bad sets are small: $|B_j| \leq \tau_j d^{j-1}$ for all $1 \leq j < k$, where $\tau_j \defeq 2\ell \sum_{i=0}^{j-1} (i + 1)\Delta^i$.

P5. The ‘tree’ is growing: there are at least $d^{k-1}(d - \eta_k)$ linear paths from root $r$ to layer $L_k$, where $\eta_k \defeq \sum_{i=0}^{k-2} ((\Delta + 1)R_i + 2(i + 1)\ell \Delta^i + \tau_i)$.

P6. There are many children: each vertex in $L_k$ has at least $d$ neighbors in $L_{k-1} \cup B_{k-1} \cup U$, and each vertex in $U$ has at least $d$ neighbors in $L_k \cup U$.

The main step in going from $k$'th stage to $k+1$'st is to make P3 hold with $j = k$. To that end, we rely on the following lemma showing that only a few $v_k \in L_k$ are in pairs $(v_i, v_k)$ that violate P3.

Lemma 6 (Proof is in Section 4). Let $B' \defeq \{v_k \in L_k : \exists i < k \ \exists v_i \in L_i \ P(v_i, v_k) > R_{k-i-1}\}$. Then $P(r, B') \leq 2k\ell \Delta d^{k-1}$.
Assuming the lemma, we show next how to go from stage \( k \) to stage \( k + 1 \), for \( k < \ell \).

Because of \( P2 \), \( P(r, v) \geq \deg(v) \) and so \( \deg(v_k) \leq R_{k-1} \) for every \( v_k \in L_k \setminus B' \). Since degree of every vertex of \( L_k \) is at least \( d \), by \( P6 \) this implies that every vertex in \( L_k \setminus B' \) has at least \( d - R_{k-1} \) neighbors in \( U \cup B_{k-1} \). Let \( B'' \) consist of those vertices in \( L_k \setminus B' \) that have at least \( \Delta R_{k-1} \) neighbors in \( B_{k-1} \). Note that

\[
\Delta R_{k-1}|B''| \leq d|B_{k-1}|,
\]

and hence \(|B''| \leq d|B_{k-1}| \leq \frac{\tau_{k-1}}{R_{k-1}}d^{k-1} \).

Let \( L_{k+1} \) be all vertices in \( U \) that are adjacent to some vertex in \( L_k \setminus (B' \cup B'') \), replace \( L_k \) by \( L_k \setminus (B' \cup B'') \), and set \( B_k \equiv B' \cup B'' \). That way, each linear path from \( r \) to \( L_k \) can be extended to a path to \( L_{k+1} \) in at least \( d - R_{k-1} - \Delta R_{k-1} = d - (\Delta + 1)R_{k-1} \) ways. So, since the number of linear paths from \( r \) to the new \( L_k \) is at least

\[
d^{k-1}(d - \eta_k) - P(r, B') - P(r, B'') \geq d^{k-1}(d - \eta_k - 2k\ell t \Delta^{k-1} - R_{k-1}|B''|) \\
\geq d^{k-1}(d - \eta_k - 2k\ell t \Delta^{k-1} - \tau_{k-1}),
\]

it follows that the number of linear paths from \( r \) to \( L_{k+1} \) is at least

\[
(d - (\Delta + 1)R_{k-1})P(r, L_k) \geq d^\ell(d - \eta_k - 2k\ell t \Delta^{k-1} - \tau_{k-1} - (\Delta + 1)R_{k-1}) \\
= d^\ell(d - \eta_{k+1})
\]

This shows that \( P5 \) holds at stage \( k + 1 \). Since \( P2 \) held at stage \( k \), it follows that \(|B'| \leq P(r, B')\), implying

\[
|B_k| = |B'| + |B''| \leq 2k\ell t(\Delta d)^{k-1} + \tau_{k-1}d^{k-1} = \tau_kd^{k-1},
\]

and so Property \( P4 \) holds at stage \( k + 1 \). Property \( P6 \) holds at stage \( k + 1 \) because it held at stage \( k \) and the graph is bipartite. The other properties are immediate.

At \( \ell \)th stage, the number of linear paths from \( r \) to \( L_\ell \) is at least \( d^{\ell-1}(d - \eta_k) = d^\ell(1 + o(1)) \). On the other hand, it is at most \(|L_\ell|R_{\ell-1} \leq (8\ell t)^{\ell-1}n \). The result then follows.

4 Embedding \( \Theta_{\ell,t} \)

In this section we prove Lemma 6 that controls the number of linear paths in a \( \Theta_{\ell,t} \)-free graph. For that we show that if there are many linear paths from some vertex \( v \) to its descendants, then we can embed a subdivision of a star so that its leaves are mapped to the children of \( v \). Adding vertex \( v \) to the subdivision of the star yield a copy of \( \Theta_{\ell,t} \).

The standard method for embedding trees is to find a substructure of large ‘minimum degree’ (in a suitable sense), and then embed vertex-by-vertex avoiding already-embedded vertices. For us the relevant notion of a degree is the number of linear paths.

**Definition 7.** A pair of layers \((L_i, L_j)\) with \( i < j \) is almost-regular if every pair of layers \((L_{i'}, L_{j'})\), with \( i \leq i' < j' \leq j \) and \((i', j') \neq (i, j)\), is regular.

**Lemma 8.** Suppose \( i < k \) and pair \((L_{i-1}, L_k)\) is almost-regular, and we are given a vertex \( v_{i-1} \in L_{i-1} \) and subsets \( A \subseteq N(v_{i-1}) \), \( B \subseteq L_k \). Suppose that the number of linear paths between \( A \) and \( B \) satisfies

\[
P(A, B)/|A| > 2\ell t(\Delta d)^{k-i-1}, \tag{1}
\]

\[
P(A, B)/|B| > R_{k-i}. \tag{2}
\]

Then \( G \) contains \( \Theta_{\ell,t} \).

**Proof.** The proof naturally breaks into three parts: finding a substructure of a large minimum degree, using that substructure to locate many disjoint paths, and then joining these paths to form a copy of \( \Theta_{\ell,t} \).
Part 1 (large minimum degree substructure): We will select a subset \( B' \subset B \) that is well connected by linear paths to the preceding layer \( L_{k-1} \). We use a modification of the standard proof that a graph of average degree \( 2d \) contains a subgraph of minimum degree \( d \).

At start, set \( B' = B \). To each pair \((a, b) \in A \times B'\) we associate a set \( \mathcal{P}(a, b) \) of linear paths between \( a \) and \( b \). At start, \( \mathcal{P}(a, b) \) is the set of all linear paths from \( a \) to \( b \). For brevity we use notations \( \mathcal{P}(\cdot, b) \) and \( \mathcal{P}(a, \cdot) \) where the last equality relies on the convolution identity for the Catalan numbers.

Part 2 (many disjoint paths from vertices of \( B \)): Let \( \mathcal{P}(\cdot, b) \) be arbitrary. Then \( (\mathcal{L}_i - 1, L_k) \) is almost-regular the number of paths in \( \mathcal{P}(\cdot, b) \) that intersect \( \{v_i, v_{i+1}, \ldots, v_{k-1}\} \) is at most

\[
\sum_{j=i}^{k-1} P(v_i, v_j) P(v_j, v_k) \leq \sum_{j=i}^{k-1} R_{j-i} \cdot R_{k-j-1} = \sum_{u+v=k-i-1} R_u R_v = \frac{1}{2\ell t} R_{k-i},
\]

where the last equality relies on the convolution identity for the Catalan numbers.

From \( |\mathcal{P}(\cdot, b)| > R_{k-i}/2 \) it follows that as long we have picked fewer than \( \ell t \) paths, there is another path in \( \mathcal{P}(\cdot, b) \) that is disjoint from the already-picked.

Part 3 (embedding): Let

\[
S \overset{\text{def}}{=} \{v_{k-1} \in L_{k-1} : v_{k-1} \text{ is on some path in } \mathcal{P}(\cdot, \cdot)\}.
\]

Consider the subgraph \( H \) of \( G \) that is induced by \( S \cup B' \). This is a bipartite graph with parts \( S \) and \( B' \). The vertex-disjoint paths found in the previous step show that degree of each vertex in \( B' \) is at least \( \ell t \). We claim that vertices of \( S \) are also of degree at least \( \ell t \). Indeed, let \( s \in S \) be arbitrary. Then \( \mathcal{P}(\cdot, s) \) contains a linear path of the form \( av_{i+1} v_{i+2} \cdots v_{k-2} s \). Since it was not removed by operation 2, there are at least \( \ell t \) linear paths having \( av_{i+1} v_{i+2} \cdots v_{k-2} s \) as a prefix. Therefore, \( s \) is adjacent to at least \( \ell t \) vertices of \( B' \).

Because the minimum degree of \( H \) is at least \( \ell t \), it is possible to embed any rooted tree on at most \( \ell t \) vertices into \( V(H) = S \cup B' \) with the root as any prescribed vertex of \( S \cup B' \). In particular, we can find a
vertex $u \in S \cup B'$ and $t$ vertex-disjoint paths from $u$ to $B'$ of length $t - k + i - 1$ each. Note that the choice of whether $u \in S$ or $u \in B'$ depends on the parity of $t - k + i - 1$.

Let $b_1, \ldots, b_t \in B'$ be the endpoints of these paths, and let $T$ be all the vertices in the union of the paths. Since $|T| < tt$, at least one of the $tt$ vertex-disjoint paths from $b_1$ to $A$ misses $T$. We then join this path to $b_1$. We can extend paths ending at $b_2, b_3, \ldots, b_t$ in turn in a similar way. We obtain an embedding of $\Theta_{t, t}$ minus one vertex. Adding $v_{i-1}$ we obtain an embedding of $\Theta_{t, t}$ into $G$. 

We are now ready to prove Lemma 6 that controls the number of bad vertices.

**Proof of Lemma 6.** Inductively define sets $B'_{k-1}, B'_{k-2}, \ldots, B'_1$ (in that order) by

$$B'_i \overset{\text{def}}{=} \{ v_k \in L_k : \exists v_{i-1} \in L_{i-1} \text{ s.t. } P(v_{i-1}, v_k) > R_{k-i} \} \setminus (B'_{i+1} \cup \cdots \cup B'_{k-1}).$$

Note that $B' = \bigcup_i B'_i$. We will prove that $P(r, B'_i) \leq 2t t(\Delta d)^{k-1}$, from which the lemma would follow.

Decompose $B'_i$ further into sets

$$B'(v_{i-1}) \overset{\text{def}}{=} \{ v_k \in L_k : P(v_{i-1}, v_k) > R_{k-i} \} \setminus (B'_{i+1} \cup \cdots \cup B'_{k-1}).$$

Clearly $B'_i = \bigcup_{v_{i-1} \in L_{i-1}} B'(v_{i-1})$.

To start, observe that if we remove $B'_{i+1} \cup \cdots \cup B'_{k-1}$ from $L_k$ then the pair of layers $(L_{i-1}, L_k)$ is almost-regular. Therefore, for every $v_{i-1} \in L_{i-1}$, since

$$P(\vec{N}(v_{i-1}), B'(v_{i-1})) > R_{k-i}|B'(v_{i-1})|,$$

it follows from Lemma 8 that

$$P(\vec{N}(v_{i-1}), B'(v_{i-1})) \leq 2 \deg(v_{i-1}) t t(\Delta d)^{k-i-1}.$$

In particular,

$$P(r, B'_i) = \sum_{v_{i-1} \in L_{i-1}} P(r, v_{i-1}) P(\vec{N}(v_{i-1}), B'(v_{i-1})) \leq P(r, L_{i-1}) 2 t t(\Delta d)^{k-i}$$

$$\leq 2 t t(\Delta d)^{k-1}$$

since degree of every vertex is at most $d \Delta$. Adding these over all $i$ completes the proof. 

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**5 Lower bound for odd length paths**

In this section we will construct graphs on $n$ vertices that do not contain a $\Theta_{t, t}$ with $\Omega_t (t^{1-1/\ell} n^{1+1/\ell})$ edges when $t$ is odd, showing that Theorem 1 has the correct dependence on $t$ for odd $t$.

We will use the random polynomial method [3, 6]. Our construction is in two stages. First we use random polynomials to construct graphs with only few short cycles. In the second stage we blow up the graph by replacing vertices by large independent sets. We will show that the resulting graph is $\Theta_{t, t}$-free.

Let $q$ be a prime power and let $P_d$ be the set of polynomials in $s$ variables of degree at most $d$ over $\mathbb{F}_q$. That is, $P_d$ is the set of linear combinations over $\mathbb{F}_q$ of monomials $x_1^{a_1} \cdots x_s^{a_s}$ with $\sum_{i=1}^s a_i \leq d$.

We reserve the term *random polynomial* to mean a polynomial chosen uniformly from $P_d$. We note that a random polynomial can equivalently be obtained by choosing the coefficients of each monomial $X_1^{a_1} \cdots X_s^{a_s}$ uniformly and independently from $\mathbb{F}_q$. In particular, because the constant term of a random polynomial is chosen uniformly from $\mathbb{F}_q$, it follows that

$$\Pr[f(x) = 0] = \frac{1}{q} \tag{3}$$

for a random polynomial $f$ and any fixed $x \in \mathbb{F}_q^s$.

We now define a random graph model that we use in our constructions.
**Definition 9** (Random algebraic graphs). Set $d \equiv 2\ell^2$. Let $U$ and $V$ be disjoint copies of $F^\ell$, and consider the following random bipartite graph with parts $U$ and $V$. We pick $\ell - 1$ independent random polynomials $f_1, \ldots, f_{\ell - 1}$ from $P_d^\ell$, and declare $uv$ to be an edge of $G$ if and only if
\[
f_1(u, v) = f_2(u, v) = \cdots = f_{\ell - 1}(u, v) = 0.
\]
We call the resulting graph a random algebraic graph.

Note that in this definition we fixed the degree and number of polynomials, to suit our particular application. More general random algebraic graphs have been been used for instance in [7].

Let $G$ be a random algebraic graph. For $T \in \mathbb{N}$, we say that a pair of vertices $x, y$ is $T$-bad if there are at least $T$ paths of length at most $\ell$ between $x$ and $y$. Define $B_T$ to be the set of $T$-bad pairs of vertices in $G$.

**Proposition 10.** (Case $h = 1$ of Proposition 12) There exists a constant $T = T(\ell)$ depending only on $\ell$ such that
\[
\mathbb{E}[|B_T|] = O(1).
\]

The proof of Proposition 12 is similar to arguments in [7] and [9], and we defer it to Section 7. We use Proposition 10 to make a graph with $\Omega(n^{1+1/\ell})$ edges where each pair of vertices is joined by only a few short paths.

**Theorem 11.** There exists a constant $T$ such that, for all $n$ large enough, there is a bipartite graph on $n$ vertices with at least $\frac{1}{4}n^{1+1/\ell}$ edges and no $T$-bad pair.

**Proof.** Let $q$ be the largest prime power with $2q^\ell \leq n$. Note that $2q^\ell \sim n$, as there is a prime between $x$ and $x + x^{0.525}$ for all large $x$ [1]. Let $G$ be a random algebraic graph as in Definition 9. Let $T$ be the constant from Proposition 10. Remove all $T$-bad pairs from $G$ to obtain a subgraph $G'$ of $G$. Note that for each pair in $B_T$ which is removed from $G$, at most $2n$ edges are removed.

Since $f_1, \ldots, f_{\ell - 1}$ are chosen independently, (3) implies that the expected number of edges in $G$ is
\[
q^{\ell} \cdot q^{\ell} \cdot \left(\frac{1}{q}\right)^{\ell - 1} = q^{\ell + 1}.
\]

Therefore by Proposition 10, we have
\[
\mathbb{E}[e(G')] \geq q^{\ell + 1} - 2n\mathbb{E}[|B_T|] = q^{\ell + 1} - O(n).
\]

Since $2q^\ell \sim n$, for $n$ large enough we have $\mathbb{E}[e(G')] \geq \frac{1}{4}n^{1+1/\ell}$, and so a graph with the desired properties exists. \qed

We now construct our $\Theta_{\ell, T}$-free graphs. Given a graph $H$, an $m$-blowup of $H$ is obtained by replacing every vertex of $H$ with an independent set of size $m$ and replacing each edge of $H$ with a copy of $K_{m, m}$. Note that an $m$-blowup of $H$ has $m^2e(H)$ edges. If $H'$ is a blowup of $H$, for $u \in V(H')$ and $v \in V(H)$, we say that $v$ is a supernode of $u$ if $u$ is in the independent set which replaced $v$.

**Proof of Theorem 2.** Let $\ell \geq 3$ be odd, and let $T$ be as above. With foresight, set $m \equiv \lceil \frac{\ell - 1}{2} \rceil$. Let $G'$ be the graph on $\frac{n}{m}$ vertices whose existence is guaranteed by Theorem 11. So $G'$ has at least $\frac{1}{2} \left(\frac{n}{m}\right) q^{\ell + 1/\ell}$ edges and no $T$-bad pair. Let $G$ be an $m$-blowup of $G'$. To show that $G$ is $\Theta_{\ell, T}$-free, let $x$ and $y$ be vertices in $G$ and let
\[
P_1 = xu_1^1 \cdots u_{\ell - 1}^1 y \quad ; \quad P_R = xu_1^R \cdots u_{\ell - 1}^R y
\]
be $R$ internally disjoint paths of length $\ell$ from $x$ to $y$. Since $\ell$ is odd and $G'$ is bipartite, $x$ and $y$ have distinct supervertices in $G'$, call them $x'$ and $y'$. For $1 \leq i \leq \ell - 1$ and $1 \leq j \leq R$, let $v_i^j \in V(G')$ be the supervertex of $u_i^j \in V(G)$. Now consider the multiset

$$P_1' = x'v_1^1 \cdots v_{\ell-1}^1y'$$

$$
\vdots
$$

$$P_R' = x'v_1^R \cdots v_{\ell-1}^Ry'.
$$

This is a multiset of $R$ not necessarily disjoint or distinct walks of length $\ell$ from $x'$ to $y'$ in $G'$. Removing cycles from these walks, we obtain a multiset of $R$ paths of length at most $\ell$ between $x'$ and $y'$ in $G'$. Although these paths are not necessarily disjoint or distinct, since $G$ is an $m$-blowup of $G'$ and since $P_1, \ldots, P_R$ are internally disjoint, each vertex besides $x'$ and $y'$ may appear in the multiset of $G'$-paths at most $m$ times. In particular, each distinct $G'$-path may appear at most $m$ times. Since $G'$ has at most $T$ paths of length at most $\ell$ between $x'$ and $y'$, we have that

$$R \leq Tm < t$$

by the choice of $m$.

So, $G$ is a graph on $n$ vertices with no $\Theta_{\ell,t}$ and at least

$$\frac{1}{4} \left(\frac{n}{m}\right)^{1+1/\ell} m^2 = \frac{1}{4} t^{1+1/\ell} m^{1-1/\ell} = \Omega_{\ell} \left(t^{1-1/\ell} n^{1+1/\ell}\right)$$

edges. $\square$

6 Lower bound for even length paths

Let $h$ be a parameter to be chosen later. Let $G_1, \ldots, G_h$ be $h$ independent random algebraic graphs with parts $U = V = \mathbb{F}_q^\ell$, chosen as in Definition 9. Consider the multigraph $\overline{G}$ which is the union of all the $G_i$'s. Call a pair of vertices $T$-bad if they are joined by at least $T$ paths of length at most $\ell$ in $\overline{G}$. By Proposition 12 (proved in Section 7) there are constants $T = T(\ell)$ and $C = C(\ell)$ such that the expected number of $Th^{\ell}$-bad pairs is at most $Ch^{\ell}$. Let $G$ be obtained from $\overline{G}$ by removing the multiple edges.

The expected number of edges in the multigraph $\overline{G}$ is $h \cdot q^{\ell+1} = h \left(\frac{n}{2}\right)^{1+1/\ell}$. Let $M$ be the number of multiple edges. Then

$$\mathbb{E}[M] \leq n^2 h^2 \left(q^{-\ell-1}\right)^2 = o(n).$$

Remove from $G$ all $Th^{\ell}$-bad pairs of vertices. Doing this removes at most $2n$ edges per pair removed. The expected number of edges in the obtained graph is at least

$$h \left(\frac{n}{2}\right)^{1+1/\ell} - 2Ch^{\ell}n - o(n).$$

Choosing $h = \left(\frac{4}{7}\right)^{1/\ell}$ shows that there is a $\Theta_{\ell,t}$-free graph with $\Omega_{\ell} \left(t^{1/\ell} n^{1+1/\ell}\right)$ edges and at most $n$ vertices.

7 Analysis of the random algebraic construction

Here we prove the bound, whose proof we deferred, on the number of $T$-bad pairs. Recall that $G_1, \ldots, G_h$ are independent random algebraic graphs with parts $U = V = \mathbb{F}_q^\ell$, and $\overline{G}$ is the multigraph which is the union of the $G_i$'s. As before, a pair of vertices is $T$-bad if it is joined by at least $T$ paths of length at most $\ell$.

Let $B_T$ be the set of all $T$-bad pairs in $\overline{G}$.
Proposition 12. There exist constants $T = T(\ell)$ and $C = C(\ell)$ such that
\[ \mathbb{E}[|B_{Tk}|] \leq C\ell^k. \]

Proof of Proposition 12. Let $r \leq \ell$ and $(i_1, \ldots, i_r) \in [h]^r$ be fixed. A path made of edges $e_1, \ldots, e_r$ (in order) is of type $(i_1, \ldots, i_r)$ if $e_j \in E(G_{j,i})$. For a type $I$, a pair of vertices $x, y$ is $(T, I)$-bad if there are $T$ paths of type $I$ between $x$ and $y$. We will show that there is a constant $T = T(\ell)$ such that, for each fixed type $I$, the expected number of $(T/\ell, I)$-bad pairs is $O_\ell(1)$. Since the total number of types is $\sum_{r \leq \ell} h^r \leq \ell h^r$, the proposition will follow by the linearity of expectation.

We will need the fact that if the degrees of random polynomials are large enough, then the values of these polynomials in a small set are independent. Specifically, because of the way we defined graphs $G_i$, we are interested in the probabilities that the polynomials vanish on a given set.

Lemma 13. [Lemma 2.3 in [7] and Lemma 2 in [9]] Suppose that $q \geq \binom{m}{2}$ and $d \geq m - 1$. Then if $f$ is a random polynomial from $\mathcal{P}_d$ and $x_1, \ldots, x_m$ are fixed distinct points in $\mathbb{F}_q$, then
\[ \Pr[f(x_1) = \cdots = f(x_m) = 0] = \frac{1}{q^m}. \]

We need to estimate the expected number of short paths between pairs of vertices. To this end, let $x$ and $y$ be fixed vertices in $G$, let $I = (i_1, \ldots, i_r)$ be fixed, and let $S_r$ be the set of paths of type $I$ between $x$ and $y$. We use an argument of Conlon [9], to estimate the $2\ell$th moment of $S_I$.

The $|S_I|^{2\ell}$ counts ordered collections of $2\ell$ paths of type $I$ from $x$ to $y$. Let $P_{m,r}$ be all such ordered collections of paths in $K_{q^d}$ whose union has exactly $m$ edges. Note that $m \leq 2\ell \cdot r \leq 2\ell^2 \leq d$. Conlon showed [9, p.5] that every collection in $P_{m,r}$ spans at least $(r - 1)m/r$ vertices other than $x$ and $y$.

By Lemma 13 and independence between different $G_i$’s, the probability that a given collection in $P_{m,r}$ is contained in $G$ is $q^{-(t-1)m}$. From Conlon’s bound on the number of internal vertices it follows that
\[ |P_{m,j}| \leq q^{\ell m(j-1)/j}. \]

Therefore,
\[ \mathbb{E}[|S_I|^{2\ell}] = \sum_{m=1}^{2\ell^2} |P_{m,r}| q^{-(t-1)m} \leq \sum_{m=1}^{2\ell^2} q^{\ell m - \ell m + m} \leq \sum_{m=1}^{2\ell^2} 1, \]
where the last inequality uses $r \leq \ell$.

We next show that $|S_I|$ is either bounded or is of order at least $q$. To do this, we must describe the paths as a points on appropriate varieties. We write $\overline{\mathbb{F}_q}$ for the algebraic closure of $\mathbb{F}_q$. A variety over $\overline{\mathbb{F}_q}$ is a set
\[ W = \{ x \in \overline{\mathbb{F}_q} : f_1(x) = \cdots = f_s(x) = 0 \} \]
where $f_1, \ldots, f_s : \overline{\mathbb{F}_q} \to \overline{\mathbb{F}_q}$ are polynomials. We say that $W$ is defined over $\mathbb{F}_q$ if the coefficients of the polynomials are in $\mathbb{F}_q$ and we let $W(\mathbb{F}_q) = W \cap \mathbb{F}_q$. We say $W$ has complexity at most $M$ if $s$, $t$, and the degree of each polynomial are at most $M$. We need the following lemma of Bukh and Conlon [7].

Lemma 14. [Lemma 2.7 in [7]] Suppose $W$ and $D$ are varieties over $\overline{\mathbb{F}_q}$ of complexity at most $M$ which are defined over $\mathbb{F}_q$. Then one of the following holds:
- $|W(\mathbb{F}_q) \setminus D(\mathbb{F}_q)| \leq c_M$, where $c_M$ depends only on $M$, or
- $|W(\mathbb{F}_q) \setminus D(\mathbb{F}_q)| \geq q \left( 1 - O_M(q^{-1/2}) \right)$. 

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Note that $S_I$ is a subset of a variety. Indeed, suppose $x \in U$ and $y \in V$ (if $r$ is odd) or $y \in U$ (if $r$ is even) be the two endpoints. Let

$$W \overset{\text{def}}{=} \{(u_0, \ldots, u_r) \in (\mathbb{F}_q)^{r+1}: u_0 = x, u_r = y, f_k^{u_1}(u_0, u_1) = \cdots = f_k^{u_{r-1}}(u_{r-1}, u_r) = 0, 1 \leq k \leq \ell - 1\},$$

where $f_k^{i}$ is the $k$'th random polynomial used to define the random graph $G_i$.

The set $W(\mathbb{F}_q)$ is nothing but the set of walks of type $I$ from $x$ to $y$. To obtain $S_I$ we need to exclude the walks that are not paths. To that end, define

$$D_{a,b} \overset{\text{def}}{=} W \cap \{(u_0, \ldots, u_r) : u_a = u_b\} \quad \text{for} \quad 0 \leq a < b \leq r,$$

and set $D = \bigcup_{a,b} D_{a,b}$, which is a variety since the union of varieties is a variety. Furthermore, its complexity is bounded since it is defined by polynomials that are products of polynomials defining $D_{a,b}$'s.

We then have that

$$S_I = W(\mathbb{F}_q) \setminus D(\mathbb{F}_q).$$

Since complexity of both $W$ and $D$ is bounded, Lemma 14 implies that either $|W(\mathbb{F}_q) \setminus D(\mathbb{F}_q)| \leq c_j$ or $|W(\mathbb{F}_q) \setminus D(\mathbb{F}_q)| \geq q \left(1 - O_r(q^{-1/2})\right)$ where $c_j$ is a constant depending only on $r$. In particular, there is a constant $T_r$ such that, for $q$ large enough, we have either $|S_I| \leq T_r$ or $|S_I| \geq \frac{q}{2}$. Since $\mathbb{E}[|S_I|^{2\ell}] \leq 2\ell^2$, Markov’s inequality gives that

$$\Pr[|S_I| > T_r] = \Pr\left[|S_I| \geq \frac{q}{2}\right] = \Pr\left[|S_I|^{2\ell} \geq (q/2)^{2\ell}\right] \leq \frac{\mathbb{E}[|S_I|^{2\ell}]}{(q/2)^{2\ell}} = O_r\left(q^{-2\ell}\right). \quad (4)$$

Upon letting $T \overset{\text{def}}{=} \ell \cdot \max_{r \leq \ell} T_r$, inequality (4) implies that the expected number of $(T/\ell, I)$-bad pairs is at most $O_r\left(|V||U|q^{-2\ell}\right) = O_r(1)$. \qed

References


