Fall 2012 Final

Solutions

1.

$$\text{Volume} = \int_0^1 \pi r^2(x)\,dx$$

$$= \int_0^1 \pi (\sqrt{x})^2\,dx$$

$$= \int_0^1 \pi x\,dx$$

$$= \left.\frac{\pi x^2}{2}\right|_0^1$$

$$= \boxed{\frac{\pi}{2}}$$

2. Solve the following initial value problem.

$$\frac{dy}{dx} = 2x^2 - 3\sin(x) \qquad\qquad y(0) = 0.$$

$$\int \frac{dy}{dx}\, dx = \int 2x^2 - 3\sin(x)\, dx$$

$$y = \frac{2}{3}x^3 + 3\cos(x) + C$$

$$y(0) = 0 + 3 + C = 0$$

$$C = -3$$

$$\boxed{y = \frac{2}{3}x^3 + 3\cos(x) - 3}$$

3. Take the following indefinite integrals. Integration by substitution (u-substitution) is the recommended method.

(a) (5 points) $\displaystyle\int \frac{1}{e^{2x+1}}\,dx.$

$u = 2x+1 \qquad \frac{du}{dx} = 2$

$$= \int \frac{\frac{1}{2}\frac{du}{dx}}{e^{u}}\,dx = \int \frac{1}{2}e^{-u}\,du$$

$$= -\frac{1}{2}e^{-u} + C = \boxed{-\frac{1}{2}e^{-(2x+1)} + C}$$

(b) (5 points) $\displaystyle\int \frac{\sin(\ln(x))}{x}\,dx.$

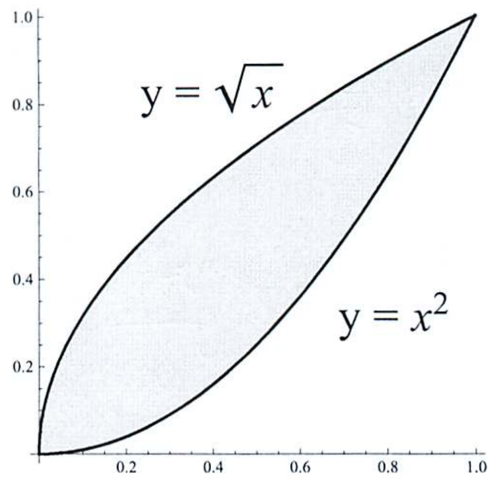
$u = \ln x \qquad \frac{du}{dx} = \frac{1}{x}$

$$= \int \sin(u)\frac{du}{dx}\,dx = \int \sin u\,du$$

$$= -\cos u + C$$

$$= \boxed{-\cos(\ln x) + C}$$

4. (10 points) Find the shaded area (depicted below) between the curves $y = \sqrt{x}$ and $y = x^2$, for $0 \leq x \leq 1$.



$x^2 = \sqrt{x} \Rightarrow x = 0, 1$

Area $= \displaystyle\int_0^1 \sqrt{x} - x^2 \, dx$

$= \left. \frac{2}{3} x^{3/2} - \frac{x^3}{3} \right|_0^1$

$= \frac{2}{3} - \frac{1}{3} - 0 = \boxed{\frac{1}{3}}$

5. Take the following indefinite integrals. Integration by parts is the recommended method.

(a) (5 points) $\int x \ln(2x)\, dx.$

$u = \ln x \qquad v' = x$

$u' = \frac{1}{x} \qquad v = \frac{x^2}{2}$

$= \frac{x^2 \ln x}{2} - \int \frac{x^2}{2} \cdot \frac{1}{x}\, dx$

$= \boxed{\frac{x^2 \ln x}{2} - \frac{x^2}{4} + C}$

(b) (5 points) $\int \frac{x}{e^x}\, dx.$

$u = x \qquad v' = e^{-x}$

$u' = 1 \qquad v = -e^{-x}$

$= -xe^{-x} + \int e^{-x}$

$= \boxed{-xe^{-x} - e^{-x} + C}$

6. Take the following indefinite integrals.

(a) (5 points) $$\int \frac{1}{9+4x^2}\,dx.$$ (hint: trigonometric substitution)

$$= \frac{1}{4}\int \frac{dx}{\frac{9}{4}+x^2}$$

$$x = \frac{3}{2}\tan\theta$$
$$dx = \frac{3}{2}\sec^2\theta$$

$$= \frac{1}{4}\int \frac{\frac{3}{2}\sec^2\theta}{\frac{9}{4}(1+\tan^2\theta)}\,d\theta$$

$$= \frac{1}{6}\int \frac{\sec^2\theta}{\sec^2\theta}\,d\theta =$$

$$\frac{1}{6}\theta + C = \boxed{\frac{1}{6}\arctan\left(\frac{2}{3}x\right) + C}$$

(b) (5 points) $$\int \frac{x^2+1}{x^2(x+1)}\,dx.$$ (hint: partial fractions)

$$\frac{x^2+1}{x^2(x+1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x+1}$$

$$x^2+1 = Ax(x+1) + B(x+1) + Cx^2$$

$x=0$: $\quad B=1$

$x=-1$ $\quad C=2$

coefficient of x^2: $\quad Ax^2 + Cx^2 = x^2$

$$\Rightarrow A = -1$$

$$\boxed{\int \frac{x^2+1}{x^2(x+1)} = -\ln|x| - \frac{1}{x} + 2\ln|x+1| + C}$$

7. (10 points) After finishing your exams, you go out for ice cream in celebration. But when you finally get an ice cream cone and bring it up to your mouth, someone jostles your elbow and your scoop of ice cream falls off the cone. Assuming the ice cream is falling from 4 feet high, and given that the acceleration due to gravity is 32 ft/s^2, how long does it take to hit the ground?

$$\frac{dv}{dt} = -32 \qquad v(0) = 0$$

$$v(t) = -3\cancel{8}2t + 0$$

$$\frac{dp}{dt} = v(t) = -32t \qquad p(0) = 4$$

$$p(t) = \int -32t\,dt$$

$$= -16t^2 + C$$

$$C = 4$$

$$p(t) = -16t^2 + 4$$

$$\cancel{p(t)} \quad -16t^2 + 4 = 0$$

$$t^2 = \frac{1}{4}$$

$$\boxed{t = \tfrac{1}{2} \text{ second}}$$

8. (10 points) Your reaction to try and catch the scoop of ice cream wasn't fast enough, and it hit the ground. Given that your ice cream started at 0°F, and assuming that the temperature of the ice cream $y(t)$ (where t is given in minutes since it hit the ground) is determined by the following differential equation,

$$\frac{dy}{dt} = \frac{64 - y}{16},$$

how many minutes does it take the ice cream to reach 32°F (when it starts to melt)?

$$\frac{dy}{64-y} = \frac{dt}{16}$$

Integrate

$$-\ln|64-y| = \frac{t}{16} + C$$

$$64-y = C_1 e^{-t/16}$$

$$y = 64 - C_1 e^{-t/16}$$

$$y(0) = 0 = 64 - C_1 \qquad C_1 = 64$$

$$y(t) = 64 - 64e^{-t/16} = 32$$

$$64e^{-t/16} = 32$$

$$-\frac{t}{16} = \ln \frac{1}{2}$$

$$\boxed{t = -16 \ln\left(\frac{1}{2}\right)}$$

9. For each of the following integrals, if it converges, write the number it converges to. Otherwise, you may simply write "diverges" as your answer.

(a) (5 points) $\displaystyle\int_1^\infty \frac{1}{x^{4/3}}\,dx.$

$$= \lim_{R\to\infty} \int_1^R \frac{1}{x^{4/3}} = \lim_{R\to\infty} -3x^{-1/3}\Big|_1^R$$

$$= \lim_{R\to\infty} \underbrace{-3R^{-1/3}}_{\to 0} - (-3)$$

$$= \boxed{3}$$

(b) (5 points) $\displaystyle\int_2^\infty \frac{1}{4x+1}\,dx.$

$4x+1 < 5x$ for $x > 2$

$$\int_2^\infty \frac{dx}{4x+1} > \int_2^\infty \frac{dx}{5x} = \frac{1}{5}\int_2^\infty \frac{dx}{x} \text{ which}$$

diverges by p-test.

Thus $\displaystyle\int_2^\infty \frac{dx}{4x+1}$ diverges by the comparison theorem