Kempe's Proof

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Here is a full write-up of Kempe's "Proof" of the 4-color conjecture. We shall use the following lemma, whose proof is a quick application of Euler's Formula.

Lemma 1. *Every planar graph has a vertex of degree at most 5.*

Kempe's Proof. Suppose that the 4-color conjecture is false, so that there exists a planar graph having no 4-coloring. Let G be such a graph, having minimal order (that is, the smallest number of vertices). Embed G in the plane, and by possibly adding edges produce a graph T that has every region bounded by exactly three edges. Note that since $G \subset T$, it must be that $\chi(G) \leq \chi(T)$. Hence, $\chi(T) \geq 5$.

Let $v \in V(T)$ be a vertex having degree at most 5. Note that as every region in T is bounded by a triangle, it must be that the degree of v is at least 2. Moreover, by minimality, we have that $T \backslash \{v\}$ is 4-colorable.

Case 1: $\deg v = 2$ **or** $\deg v = 3$

Color $T \backslash \{v\}$ with 4 colors. Note that v can be adjacent to at most 3 distinct such colors, and hence this coloring can be extended to a 4-coloring of T, a contradiction.

Case 2: $\deg v = 4$

Color $T \backslash \{v\}$ with 4 colors. If v is adjacent to at most 3 distinct such colors, we can extend this coloring to T. Hence we may assume that v is adjacent to one vertex of each color, as shown in Figure 1.

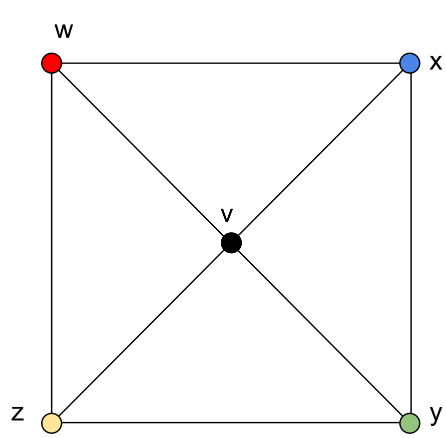

Figure 1

Let T_{RG} be the subgraph of $T\backslash\{v\}$ induced on those vertices colored Red or Green. If w and y are in different components of T_{RG}, we may select one such component, and exchange the roles of Red and Green. In this way, we force w and y to take the same color, and thus we have only 3 distinct colors adjacent to v, and can extend the 4-coloring to T.

If w and y are in the same component of T_{RG}, then there exists a path connecting w to y consisting only of Red and Green vertices, as shown in Figure 2. Note that in this case, we cannot have a path connecting x to z consisting only of Blue and Yellow vertices, due to planarity. Hence, if we consider T_{BY}, the subgraph of $T\backslash\{v\}$ induced on only Blue and Yellow vertices, it must be the case that z and x are in different components of T_{BY}. We may therefore select one such component and switch colors, as in the case described above. Hence, we will have only 3 distinct colors adjacent to v, and thus we can extend the 4-coloring to T.

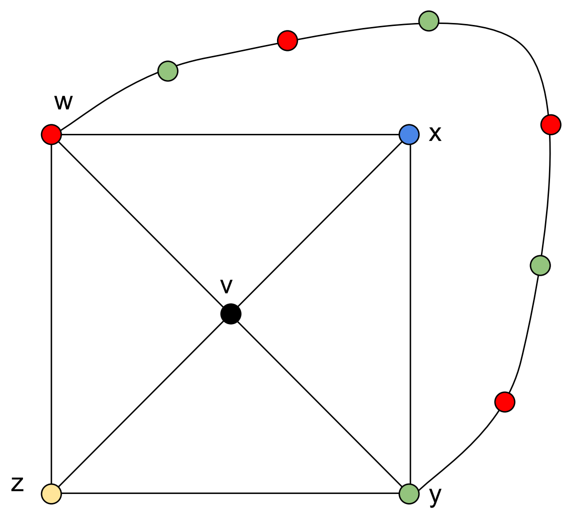

Figure 2

In any case, if $\deg v = 4$, we can extend a 4-coloring on $T\backslash\{v\}$ to a 4-coloring on T, a contradiction.

Case 3: $\deg v = 5$

As above, if v is adjacent to at most 3 distinct colors in a 4-coloring of $T\backslash\{v\}$, we may extend such a coloring immediately to T. Hence, we may assume that v is adjacent to all 4 colors. Note that this will occur as shown in Figure 3, without loss of generality.

As before, let us consider T_{RG}. Note that if vertices u and x are in different components, we may perform the same component color-switching as in the previous case, in order to extend to a 4-coloring on T. Likewise, if u and y are in different components of T_{RY}, we can do the same. Hence, it suffices to consider the case that u and x are connected by a path of only Green and Red vertices, and u and x are connected by a path of only Yellow and Red vertices. This is shown in Figure 4.

Now, let us consider T_{BG}. Note that z and x must be in different components in this graph, as the path between u and y is only in red and yellow, and it is blocking any Blue-Green path between z and x. Hence, we may select the

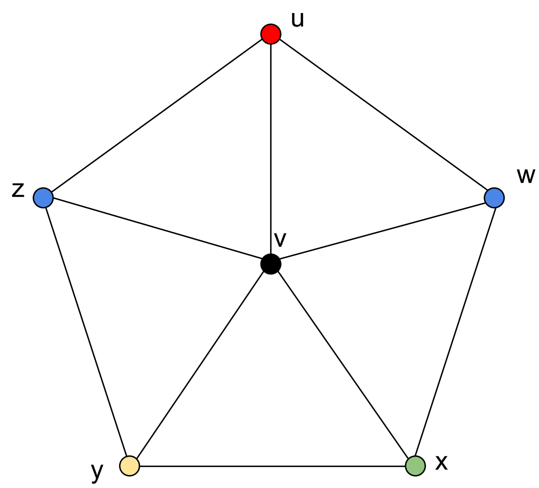

Figure 3

component in T_{BG} containing z and perform the color-switching, which then turns z Green.

Performing the same operation on T_{BY}, we can turn w Yellow.

Upon making these changes, we therefore have the colors of the vertices adjacent to v do not include Blue, and therefore we can extend the 4-coloring on $T \backslash \{v\}$ to one on T by coloring v Blue.

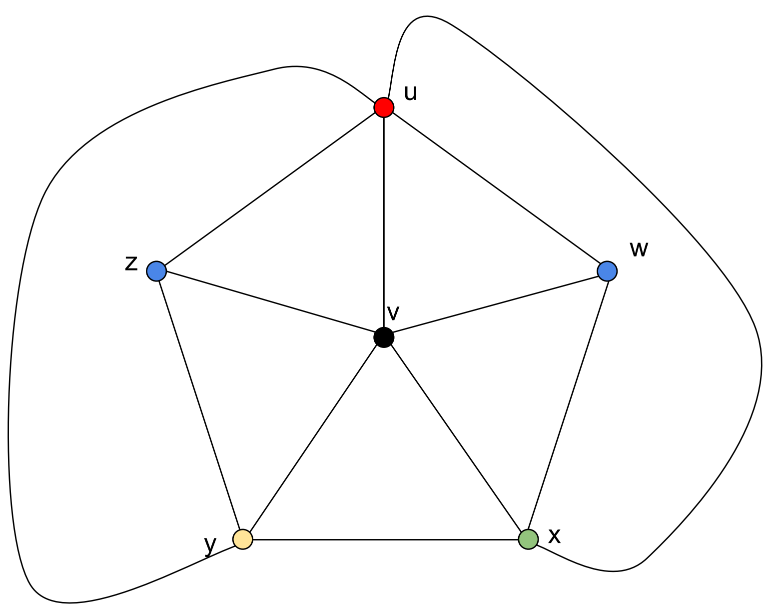

Figure 4

Therefore, in any case, we can produce a 4-coloring on T, and hence $\chi(T) \leq 4$. But as observed above, $\chi(T) \geq \chi(G)$, and therefore $\chi(G) \leq 4$, a contradiction.

Therefore, no minimal counterexample exists.

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