Math 301 Homework

Mary Radcliffe

Bonus to Posets: problem 5

The problem in your homework asks you to prove that Hall's Theorem, Dilworth's Theorem, and König's Theorem are equivalent. For 10 points each, you may add one or both of the following equivalences to the problem:

- Menger's Theorem. A graph G is called k-connected if there does not exist a set S of $k-1$ vertices such that $G \backslash S$ is disconnected. That is to say, if you wish to disconnect the graph, you must remove at least k vertices to do so.

  Given u and v vertices of G, we say two paths from u to v are internally disjoint if those two paths share no vertices other than the endpoints u and v. A set of paths is called internally disjoint if the paths in the set are pairwise internally disjoint.

  Menger's Theorem states the following:

  > A graph G is k-connected if and only if for every $u, v \in V(G)$, there exist at least k internally disjoint paths between u and v.

- Max-Flow/Min-Cut Theorem. Let G be a directed graph, with a source s and a sink t. (Note: A source is a vertex from which all incident edges direct, and a sink is a vertex to which all incident edges direct). We shall write uv to denote an edge from u toward v, and vu for the opposite direction. Weight each edge uv with positive weight w_{uv}. A flow on G is an assignment f_{uv} to each edge uv such that the following two conditions hold:

  1. $f_{uv} \leq w_{uv}$ for all uv
  2. For all $v \in V(G)$ other than s, t, $\sum_{uv \in E} f_{uv} = \sum_{vu \in E} f_{vu}$, and $\sum_{su \in E} f_{su} = \sum_{ut \in E} f_{ut}$

  That is, the flow in to a vertex is equal to the flow out, and the flow is bounded on each edge by its weight (sometimes called capacity), and all flow originates at s and ends at t.

  A maximum flow from s to t is a flow for which $\sum_{su \in E} f_{su}$ is maximized, and this maximum value is called the weight of the flow.

  A cut in G is a set of edges S such that $G \backslash S$ is disconnected. The weight of a cut S is $\sum_{e \in S} w_e$. A minimum cut is one for which the weight of the cut is minimum among all cuts.

  The Max Flow/Min Cut Theorem states the following:

  > For any directed graph with source and sink as described above, the weight of the maximum flow from s to t is equal to the weight of the minimum cut in G.