Math 301 Homework

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Complete the following problems. Fully justify each response.

1. A group of n people is to be divided into committees. From each committee, an even number of people will be chosen to serve on a subcommittee. Find the exponential generating function for the number of ways to do this.

2. For even n, let e_n be the number of permutations with all cycles even, o_n the number of permutations with all cycles odd, and $p_n = n!$ the total number of permutations. Let $E(x), O(x)$, and $P(x)$ denote the corresponding exponential generating functions. Prove the following:

   (a) $P(t) = (1 - t^2)^{-1}$

   (b) $E(t) = (1-t^2)^{-1/2}$ (A suggested strategy: first, show that the number of matchings in a graph on $2n$ vertices is $\frac{(2n)!}{2^n n!}$ (compare to problem 1 from Homework 2). Show, also that e_n is equal to the number of *p*airs of matchings in a graph on $2n$ vertices, and use this to show the desired result.)

   (c) $E(t)O(t) = P(t)$

   (d) $e_n = o_n$ for all even n.

3. Prove the following theorem for exponential generating functions.

   Let a_n be the number of ways to build a structure of type A on an n-element set, where $a_0 = 0$, and b_n the number of ways to build a structure of type B on an n-element set, where $b_0 = 1$. Let c_n denote the number of ways to partition the set $[n]$ into an unspecified number of nonempty subsets, build a structure of type A on each subset, and then a structure of type B on the set of subsets themselves.
   If $A(x) = \sum_{n\geq 1} \frac{a_n}{n!}x^n$, $B(x) = \sum_{n\geq 0} \frac{b_n}{n!}x^n$, and $C(x) = \sum_{n\geq 0} \frac{c_n}{n!}x^n$, then
   $$C(x) = B(A(x)).$$