Math 301: Homework 2

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Complete the following problems. Fully justify each response.

1. Use generating functions to find a closed form for a_n for each of the following recurrence relations.

   - $a_{n+1} = \alpha a_n + \beta$ for $n \geq 0$, $a_0 = 0$.
   - $a_{n+2} = 2a_{n+1} - a_n$ for $n \geq 0$, $a_0 = 0$, $a_1 = 1$

2. Merge sorting is an algorithm for sorting a list S of numbers. It works as follows:

   (a) If S contains just one element, it is sorted.

   (b) If S contains $n > 1$ elements, choose some $k < n$. Divide the list into two parts, one having k elements and one having $n - k$ elements. Recursively apply a merge sort to each part.

   (c) Once every part is sorted, merge the parts by comparing successive elements.

   For example, if we had two sorted parts $\{1, 2, 5, 7\}$ and $\{3, 6, 8\}$, our final step would look as follows: Compare $1, 3$, and take 1 to be first. Compare $2, 3$, and take 2 next. Compare $5, 3$, and take 3 next. This continues until all elements from both sets have been added to a new, totally sorted array.

   Given a set S containing 2^k elements, define m_k to be the maximum number of comparisons that must be made to perform a merge sort on S, in which each time we divide the list into two parts, they are equal in size (this is actually the optimum procedure). That is to say, m_k is an approximation of the longest running time of a merge sort on a set of 2^k elements. Find a recurrence relation for m_k, and solve it.

3. For even n, let e_n be the number of permutations with all cycles even, o_n the number of permutations with all cycles odd, and $p_n = n!$ the total number of permutations. Let $E(x), O(x)$, and $P(x)$ denote the corresponding exponential generating functions. Prove the following:

   (a) $P(t) = (1 - t^2)^{-1}$

   (b) $E(t) = (1 - t^2)^{-1/2}$

   (c) $E(t)O(t) = P(t)$

   (d) $e_n = o_n$ for all even n.