Math 301: Homework 1

Mary Radcliffe

due 12 Sept 2016

Complete the following problems. Fully justify each response.

1. How many integer-valued solutions are there for the following equation, subject to the listed restrictions?

$$x_1 + x_2 + x_3 + x_4 = 152$$

   (a) $x_1, x_2, x_3, x_4 > 0$

   (b) $x_1, x_2, x_3 > 0$ and $x_4 \geq 0$

   (c) $x_1, x_2, x_3 > 0$ and $x_4 \geq 0$ and $x_2 \leq 15$

2. Let k and n be integers such that $0 \leq k \leq n-1$. Provide a combinatorial proof of the identity

$$\sum_{j=0}^{k} \binom{n}{j} = \sum_{j=0}^{k} \binom{n-1-j}{k-j} 2^j.$$

   (Note: by a "combinatorial proof," we mean a proof that is based on counting, rather than an algebraic proof.)

3. For $a, b \in \mathbb{Z}$, let $L(a, b)$ denote the set of lattice paths from $(0, 0)$ to (a, b). Fix $n \in \mathbb{Z}$, and let $1 \leq k \leq n-1$. Construct an injective function

$$F : L(k-1, n-k+1) \times L(k+1, n-k-1) \to L(k, n-k) \times L(k, n-k).$$

   Use this to prove that for all $k \leq n-1$,

$$\binom{n}{k-1}\binom{n}{k+1} \leq \binom{n}{k}^2.$$

   (This property is called *log-concavity*. That is, a sequence $a_0, a_1, \ldots, a_n$ is called log-concave if for all $1 \leq k \leq n-1$, we have $a_{k-1}a_{k+1} \leq a_k a_k$. Hence, here we are proving that the sequence $\binom{n}{0}, \binom{n}{1}, \ldots, \binom{n}{n}$ is log-concave.)

4. Suppose a class contains $2n$ students, and they have projects to work on in pairs.

   (a) How many different ways can the professor pair the students up?

   (b) After the first project, the professor wishes to create new pairings, so that nobody will be working with the same partner a second time. How many ways can the professor create this second set of pairings?