Math 301: Homework 9

Mary Radcliffe

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Complete the following problems. Fully justify each response. Turn in ONLY problems 1, 2, and 3.

1. Complete problem 14 from Applied Combinatorics on page 125 (figure 6.18 is on page 126).
2. Complete problem 3 from Networks Crowds and Markets on page 687 (literally the last page of the book)
3. Consider the Erdős-Rényi graph $G_{n,p}$, in the regime that $np = C \log n$, for C constant.
   (a) Arbitrarily root G at some vertex $v = v_0$. Consider the Galton-Watson branching process from v_0, allowing repetition of vertices on multiple levels. Define
   $$L_i = \{v \in V(G) \mid v \text{ appears on the } i^{\text{th}} \text{ level of the GW process}\},$$
   so $L_0 = \{v_0\}$, $L_1 = N(v_0)$, $L_2 = \{\text{vertices that can be reached by a walk of length 2 from } v_0\}$, etc.
   Show that for any vertex $v \in V(G)$, $\mathbb{P}(v \in L_k) \geq n^{k-1}p^k$.
   (b) Use the above analysis to show that if $k > \frac{\log n + \log(C-1)}{\log(C)}$, then $\operatorname{diam}(G) \leq 2k$ with probability $1 - o(1)$.
4. Let $w = (w_1, w_2, \ldots, w_n)$ be a degree sequence, and let $G = G(w)$ be the corresponding Chung-Lu random graph. Prove that with probability $1 - o(1)$, we have $\deg(v_i) = w_i(1 + o(1))$ for every vertex $v_i \in V(G)$.
5. Suppose you wish to model a proximity network, in which people whose homes are in geographic proximity are more likely to come into contact with one another. Why might a RGG not quite fulfill your needs? What tweaks could you make to the model to improve its usefulness?
6. Define a random graph as follows.
   - Begin with a fixed graph G, where G is the $n \times n$ integer lattice having $(x, y) \sim (w, z)$ if and only if the Hamming distance between (x, y) and (w, z) is exactly 1. (In other words, draw the integer grid. Treat each intersection as a node, and lines as edges between nodes.)
   - Fix some m, and randomly choose m pairs of vertices in G to add edges.

   This graph, a variant on the Kleinberg Small World model, was originally developed by Watts and Strogatz.

   (a) What is the diameter of the $n \times n$ integer lattice?
   (b) Determine a bound on the diameter of the random graph produced by the above process (which should depend somewhat on m).