Exponential Generating Functions: why does it work?

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In class we looked at exponential generating functions, and we came to this conclusion that if we wanted to do something on n letters, in which the order matters, we could look individually at each of the letters, and take the product of the generating function. In this short note, I'll explain why that works.

Let's suppose I want to look at sequences of length n in an alphabet of length k (or any other similar counting problem). I'll say the number of such sequences is s_n.

Let's divide up the alphabet into two, one of size k_1 and another of size k_2 (so $k_1 + k_2 = k$). Let's take a_n to be the number of sequences over k_1 and b_n to be the number of sequences over k_2, of length n.

First, I'd like to claim that

$$s_n = \sum_{j=0}^{n} \binom{n}{j} a_j b_{n-j}.$$

Let's explain this formula combinatorially. I can think of a sequence over the length k alphabet as really two different sequences, one over the first k_1 letters, and the other one over the last k_2 letters, that are mixed up together (but not permuted; order matters). For example, I could take one sequence $1, 2, 1, 1, 2$ and a second sequence $3, 4, 3, 3$ over $[4]$. This could produce many different sequences, such as $1, 2, 1, 1, 2, 3, 4, 3, 3$, or $1, 3, 2, 4, 1, 3, 1, 3, 2$, but notice that in either case, the subsequences are in the correct order.

Ok, so how many ways could I put two subsequences together? Well, if the first subsequence is of length j, then I need to choose the positions to put those j elements into. Hence, we end up with the formula above, that the number of sequences of length n over the alphabet $[k]$ will satisfy. We have the $\binom{n}{j}$ to decide positions for the sequences over the first k_1 letters, and then a_j and b_{n-j} to pick which sequences to use.

Let's bring this back now to generating functions. In class we claimed (without proof) that it should be the case that

$$\left(\sum_{n\geq 0} \frac{a_n}{n!} x^n\right) \left(\sum_{n\geq 0} \frac{b_n}{n!} x^n\right) = \left(\sum_{n\geq 0} \frac{s_n}{n!} x^n\right).$$

That is, we claimed that it was sufficient to just take a product of the generating functions for the smaller problems to get the generating function for the larger problem. So let's prove here that this really comes down to the formula for s_n we proved above.

Let's calculate:

$$
\begin{aligned}
\left(\sum_{n\geq 0}\frac{a_n}{n!}x^n\right)\left(\sum_{n\geq 0}\frac{b_n}{n!}x^n\right) &= \sum_{n\geq 0}\left(\sum_{k=0}^{n}\frac{a_k b_{n-k}}{k!(n-k)!}\right)x^n \text{ (this is just a Cauchy product)}\\
&= \sum_{n\geq 0}\frac{1}{n!}\left(\sum_{k=0}^{n}\frac{n!a_k b_{n-k}}{k!(n-k)!}\right)x^n \text{(need } \tfrac{1}{n!} \text{ since this is an EGF)}\\
&= \sum_{n\geq 0}\frac{1}{n!}\left(\sum_{k=0}^{n}\binom{n}{k}a_k b_{n-k}\right)x^n\\
&= \sum_{n\geq 0}\frac{1}{n!}s_n x^n,
\end{aligned}
$$

as desired.

Recall from class that I told you that exponential generating functions were most useful in the case that order matters? That's exactly what the $\binom{n}{k}$ term is accounting for in the summation. Since the order matters in the sequence, it's important that we choose which positions will perform which role. This is, of course, different from an OGF, in which the product looks the same in every way but WITHOUT any factorials, so that position is irrelevant.

Hope this helps clear up some of the differences between OGF and EGF! :)