Math 301: Homework 6

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Complete the following problems. Fully justify each response.

1. Suppose you have random variables $X_1, X_2, \ldots, X_n$, each of which is drawn independently, such that $\mathbb{P}(X_i = 0) = \mathbb{P}(X_i = 1) = \mathbb{P}(X_i = 2) = \frac{1}{3}$ for all i. Let $X = \sum X_i$. Derive a Chernoff-type bound for $\mathbb{P}(|X - \mathbb{E}(X)| > a)$.

2. Let $G = G(n, p)$ be a random graph such that each edge is included independently and with probability p, such that p is a constant. Note that the expected degree of each vertex is np. Let δ be the minimum degree in G. Prove that for any constant $c > 0$,

$$\lim_{n\to\infty} \mathbb{P}(\delta < np - c\sqrt{n \log n}) = 0.$$

   That is to say, the probability that $\delta > np - c\sqrt{n \log n} = np(1 - o(1))$ tends to 1 as $n \to \infty$.

3. We here consider the runtime of Quicksort, a popular sorting algorithm for an array of distinct numbers. Given an array A, having n entries, here is the algorithm.

   1. Choose an element of A at random. Call this element the pivot.
   2. By comparing elements of A to the pivot, divide A into two subarrays, A_{less} and A_{more}, according to whether the elements of A are less or more than the pivot.
   3. Recursively sort A_{less} and A_{more} using the same algorithm.

   We shall say that the runtime of Quicksort is X, where X is the number of times we must compare two elements. Hence, when we choose the first pivot, we must make $n - 1$ comparisons.

   (a) Show that the worst case runtime of Quicksort is on order n^2.

   (b) Show that $\mathbb{P}(X > 8n \log n) \leq \frac{1}{n^2}$.

   Hence, although the worst case runtime is quadratic, with high probability, Quicksort runs in subquadratic time.