Math 301: Homework 5

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Complete the following problems. Fully justify each response.

1. Prove the following generalized Local Lemma. Let $A_1, A_2, \ldots, A_n$ be events in a probability space, and let G be a dependency graph on the A_i. Suppose there exists a sequence of numbers $x_1, x_2, \ldots, x_n \in [0, 1)$ such that for all i,
$$\mathbb{P}(A_i) \leq x_i \prod_{A_i \sim A_j} (1 - x_j).$$
Then
$$\mathbb{P}(\bigcap A_i^c) \geq \prod_{i=1}^{n} (1 - x_i) > 0.$$

2. Use the Lovasz Local Lemma to find a lower bound on $R(3, k)$.

3. A hypergraph $H = (V, E)$ is defined as
   - a finite set V, called vertices, and
   - E is a collection of subsets of V, called edges.

   A hypergraph is called k-uniform if $|e| = k$ for all $e \in E$, that is, every edge contains exactly k vertices. In this setting, a typical graph is just a 2-uniform hypergraph. The degree of a vertex in H is the number of edges of which it is a member.

   A hypergraph is called 2-colorable if we can assign two colors to the vertices in such a way that every edge includes vertices of both colors. That is, no edge is monochromatic.

   (a) Use the standard probabilistic method to show that if H is a k-uniform hypergraph having at most $2^{k-1} - 1$ edges, then H is 2-colorable.

   (b) Use the Lovasz Local Lemma to prove the stronger statement that if H is a k-uniform hypergraph having maximum degree at most $2^{k-3} - 1$, then H is 2-colorable.

The following is NOT part of the homework set. Just interesting information.

The Lovasz Local Lemma gives us a probabilistic technique for proving the existence of a sometimes rare object. An open question is whether the local lemma actually gives a hint as to a constructive technique; sometimes, you can use the structure of the lemma and its proof to actually construct the rare object. So here is an open question, perhaps simplified a bit.

Open Problem. Let S be a set, and let $S_1, S_2, \ldots, S_n$ be subsets of S. Suppose we wish to color S so that no S_i is monochromatic (many of the problems we study here can be reduced to a problem of this form), and the Lovasz Local Lemma tells us that it is possible to do so with k colors. Is there a *polynomial time* algorithm for finding such a coloring?

Some partial answers to this question are known (see the work of J. Beck and others).