Math 301: Homework 4

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Complete the following problems. Fully justify each response.

1. Let A be an $n \times n$ array, in which the numbers 1 through n appear exactly n times each. Prove that there exists either a row or column in A in which appear at least $\sqrt{n}$ distinct numbers.

2. Prove Sperner's Theorem: If $\mathcal{F}$ is a collection of subsets of $[n]$, such that no set in $\mathcal{F}$ is contained in any other set in S, then $|\mathcal{F}| \leq \binom{m}{\lfloor m/2 \rfloor}$.

3. Let G be a triangle-free graph having n vertices and m edges. Given a vertex v, define the closed neighborhood

$$N[v] = \{u \in V(G) \mid u = v \text{ or } u \sim v\}.$$

Prove that there exists a vertex v such that the $V(G) \backslash N[v]$ contains at most $m - \frac{4m^2}{n^2}$ edges[1].

[1] Note that the average degree in G is $\frac{2m}{n}$, so a subgraph containing $m - (\frac{2m}{n})^2$ edges must be substantially sparser than the graph G itself.