Math 241: Exam 1 Review Practice Problems Solutions

NOTE. For some of these problems, you would likely be expected to offer a longer explanation. There are some problems for which only a numerical solution is given; a full exam solution would include a demonstration of how that numerical solution was determined.

1. Let $\mathbf{u} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$, and $\mathbf{v} = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$. Write an expression for the line that goes through these two vectors in $\mathbb{R}^3$.

   Solution: A line through two points is given by the affine combinations:

   $$\{\alpha\mathbf{u} + \beta\mathbf{v} \mid \alpha, \beta \in \mathbb{R}, \alpha + \beta = 1\}$$

   This can also be presented in several other forms, such as

   $$\{\alpha\mathbf{u} + (1-\alpha)\mathbf{v} \mid \alpha \in \mathbb{R}\}, \text{ or}$$

   $$\{\mathbf{v} + \alpha(\mathbf{u} - \mathbf{v}) \mid \alpha \in \mathbb{R}\},$$

   or by interchanging the roles of $\mathbf{u}, \mathbf{v}$ in any of the above.

2. Let $\mathbf{u}, \mathbf{v}$ be vectors over $\mathbb{Z}_2$. Explain why $\mathbf{u} + \mathbf{v} = \mathbf{u} - \mathbf{v}$.

   Solution: In $\mathbb{Z}_2$, we have $1 = -1$ and $0 = -0$, and therefore $x = -x$ for every $x \in \mathbb{Z}_2$. Thus, $\mathbf{v} = -\mathbf{v}$, so $\mathbf{u} + \mathbf{v} = \mathbf{u} - \mathbf{v}$.

3. Here is a piece of code used to solve a triangular system of n equations in n variables.

```
def triangular_solve(coeff, b):
    n=len(b)
    x=zero_vec(n)
    for i in reversed(range(n)):
        x[i] = (b[i]-dot_product(coeff[i], x))/coeff[i][i]
    return x
```

   In this code, the first input is a list of lists representing coefficients for the equations, and the second input is the right hand side of the system. So, for example, if you had a system

   $$\begin{aligned} 2x_1 + 3x_2 &= 3 \\ 4x_2 &= 1 \end{aligned}$$

   then the inputs would take the form

   `coeff=[[2,3],[0,4]], b=[3,1].`

   You may assume here that zero_vec and dot_product are correctly defined helper functions that return the length n zero vector (as a list) and the dot product of two vectors (represented as lists), respectively.

(a) Explain (perhaps by showing a small example) how this algorithm works to solve a triangular system.

(b) Explain (perhaps by showing a small example) when this code will fail.

Solution:

(a) The code implements the back-substitution method we have discussed in class to solve an upper triangular system. Starting from n, it calculates a value for each x_i, and then stores these values in the list x. To calculate the value for $x[i]$, the code computes

$$x[i] = (b[i] - coeff[i] \cdot x)/coeff[i][i].$$

If the i^{th} equation in the system takes the form $a_{i,i}x_i + a_{i,i+1}x_{i+1} + \cdots + a_{i,n}x_n = b_i$, then $coeff[i] = [0, 0, \ldots, 0, a_{i,i}, a_{i,i+1}, \ldots, a_{i,n}]$. At the time of the computation, the list x takes the form $[0, 0, \ldots, 0, x_{i+1}, x_{i+2}, \ldots, x_n]$. Hence, the variable x_i is calculated to be $b_i - (x_{i+1}a_{i,i+1} + \cdots + a_{i,n}x_n)/a_{i,i}$, just as it should be.

(b) This code will fail anytime that $a_{i,i}$ is 0. This is because there will be no way to calculate a value for x_i.

4. Show that the set

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$$

is a generating set for $\mathbb{R}^3$.

Solution: Let $\mathbf{v} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ be an arbitrary vector in $\mathbb{R}^3$. Notice that we can write

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = (x - y) \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + (y - z) \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + z \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix},$$

and hence $\mathbf{v}$ is a linear combination of the three listed vectors. By definition, then as any vector in $\mathbb{R}^3$ can be written as a linear combination of the three listed vectors, these vectors are a generating set for $\mathbb{R}^3$.

Alternative Approach: Show that $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ can all be written as linear combinations of the three listed vectors.

5. Show that the set

$$\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} \right\}$$

is not a generating set for $\mathbb{R}^3$.

Solution: Consider the vector $\mathbf{v} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$. Suppose that there are constants α, β, γ such that

$$\alpha \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + \beta \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + \gamma \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

By considering the third coordinate, we note then that $\beta + \gamma = 1$, and hence $\beta = 1 - \gamma$. By considering the second coordinate, we note that $\alpha + \gamma = 0$, and hence $\alpha = -\gamma$. By considering the first coordinate, we therefore have that $\alpha + \beta + 2\gamma = 0$. However, $\alpha + \beta + 2\gamma = -\gamma + (1 - \gamma) + 2\gamma = 1$, a contradiction.

Therefore, no such constants can exist, so the three listed vectors do not generate $\mathbb{R}^3$.

Alternative Approach: Label these vectors as $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$, respectively. Note that $\mathbf{v}_3 = \mathbf{v}_1 + \mathbf{v}_2$, and hence any linear combination of the three vectors can be written as a linear combination of just $\mathbf{v}_1$ and $\mathbf{v}_2$. Thus, the three listed vectors generate only a plane in $\mathbb{R}^3$.

6. Consider the plane in $\mathbb{R}^3$ given by the solutions to the equation $4x - 3y + z = 0$. Write this plane as the span of some vectors in $\mathbb{R}^3$.

   Solution: Notice that we can write $z = -4x + 3y$. Hence, we have

$$\begin{aligned}
\left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \mid x, y, z \in \mathbb{R},\ 4x - 3y + z = 0 \right\} &= \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \mid x, y, z \in \mathbb{R},\ z = -4x + 3y \right\} \\
&= \left\{ \begin{bmatrix} x \\ y \\ -4x + 3y \end{bmatrix} \mid x, y \in \mathbb{R} \right\} \\
&= \left\{ x \begin{bmatrix} 1 \\ 0 \\ -4 \end{bmatrix} + y \begin{bmatrix} 0 \\ 1 \\ 3 \end{bmatrix} \mid x, y \in \mathbb{R} \right\} \\
&= \text{Span} \left\{ \begin{bmatrix} 1 \\ 0 \\ -4 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 3 \end{bmatrix} \right\}
\end{aligned}$$

7. Determine whether each of the following sets is a vector space in $\mathbb{R}^4$. Justify your response.

   (a) $\left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \in \mathbb{R}^4 \ :\ x_1 + x_2 = 3 \right\}$

Solution: No. $\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$ is not a member of this set.

(b) $\left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \in \mathbb{R}^4 \;:\; x_1 + x_2 + x_3 + x_4 = 0 \right\}$

Solution: Yes. Letting A be the 1×4 matrix $\begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix}$, we have that this set is the null space of A, and null spaces are always vector spaces.

(c) $\left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \in \mathbb{R}^4 \;:\; x_1 x_2 x_3 x_4 = 0 \right\}$

Solution: No. Note that $\begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}$ are both members of the set, but $\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$ is not. Hence, this set is not closed under addition.

8. Let $\mathcal{P}_n$ denote the vector space of polynomials having degree at most n, with coefficients from $\mathbb{R}$. Show that the set of polynomials having degree at most k, with $k < n$, is a vector subspace of $\mathcal{P}_n$.

   Solution: Notice that the described set can be written as Span $\left\{1, x, x^2, \ldots x^k\right\}$. As spans of any elements are vector spaces, the described set is therefore a vector subspace of $\mathcal{P}_n$.

9. Explain how to represent a system of equations with a matrix equation. You may want to use an example to help illustrate your points.

   Solution: Suppose we have a system of linear equations

$$\begin{array}{c} \mathbf{a}_1 \cdot \mathbf{x} = b_1 \\ \mathbf{a}_2 \cdot \mathbf{x} = b_2 \\ \vdots \\ \mathbf{a}_m \cdot \mathbf{x} = b_m \end{array},$$

   where each $\mathbf{a}_i$ is a vector of coefficients, $\mathbf{x}$ is a vector of variables, and each b_i is a constant. By definition, the matrix product $A\mathbf{x}$ takes the dot product of each row of A with the vector $\mathbf{x}$, and hence we can represent this list of dot products as a matrix-vector product, taking the matrix A to have rows $\mathbf{a}_1^T, \ldots, \mathbf{a}_m^T$. The equation then takes the form $A\mathbf{x} = \mathbf{b}$, where $\mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$.

10. Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ be a real-valued matrix. Show that Null (A) is nontrivial if and only if $ad - bc = 0$.

Solution: We consider two cases, according to whether $a = 0$ or $a \neq 0$.

Case 1: If $a = 0$, then we have $A = \begin{bmatrix} 0 & b \\ c & d \end{bmatrix}$. The null space consists of all vectors $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ satisfying $bx_2 = 0$ and $cx_1 + dx_2 = 0$. If $b \neq 0$, then the first equation implies that $x_2 = 0$. The second equation, then implies that $cx_1 = 0$, and hence either $c = 0$ or $x_1 = 0$. If $c = 0$, then we may choose any value for x_1, yielding that any vector of the form $\begin{bmatrix} x_1 \\ 0 \end{bmatrix}$ is a member of the null space. We note in this case that $ad - bc = 0$, since both $a = 0$ and $c = 0$. Otherwise, if $c \neq 0$, then the only member of the null space is $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$. Moreover, b, c are both nonzero, and $a = 0$, so $ad - bc \neq 0$.

If $b = 0$, then $ad - bc = 0$. Moreover, the only equation necessary to be satisfied in order that a vector be a member of the null space is $cx_1 + dx_2 = 0$, which always has infinitely many solutions for x_1, x_2.

Thus, in the case that $a = 0$, the nullspace is nontrivial if and only if $ad - bc = 0$.

Case 2: If $a \neq 0$, then we consider the two equations $ax_1 + bx_2 = 0$ and $cx_1 + dx_2 = 0$. We add $-c/a$ times the first equation to the second, yielding the two new equations

$$\begin{array}{rcrcl} ax_1 & + & bx_2 & = & 0 \\ & & (d - \frac{cb}{a})x_2 & = & 0 \end{array}$$

As $a \neq 0$, we note that this has a unique solution (namely $x_1 = x_2 = 0$) if and only if $d - \frac{cb}{a} \neq 0$, if and only if $ad - bc \neq 0$.

11. Let $A\mathbf{x} = \mathbf{b}$ be a matrix equation, and let $\mathbf{x}_1$ be a solution to this equation. How do you find other solutions? Why does this technique work?

Solution: To find the remaining solutions, we add any vector that is a member of the null space of A. This technique works because if we have any other solution $\mathbf{x}_2$, we have that $A(\mathbf{x}_2 - \mathbf{x}_1) = \mathbf{b} - \mathbf{b} = 0$, so it must be that $\mathbf{x}_2 - \mathbf{x}_1$ is a member of the null space of A. Since their difference lies in the null space, we must have that $\mathbf{x}_2$ is obtained from $\mathbf{x}_1$ by adding a null space vector.

Similarly, if $\mathbf{y}$ is any null space vector, we have that $A(\mathbf{x}+\mathbf{y}) = \mathbf{b}+\mathbf{0} = \mathbf{b}$, and hence we may always add null space vectors to yield additional solutions.

12. Let

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Write the null space of A as the span of some vector(s).

Solution:

$$\text{Null}\,(A) = \text{Span}\left\{ \begin{bmatrix} -5 \\ 1 \\ 1 \\ 0 \end{bmatrix} \right\}$$

13. Define a function $f : \mathbb{R}^4 \to \mathbb{R}^4$ by $f(\mathbf{x}) = A\mathbf{x}$, where

$$A = \begin{bmatrix} 1 & 0 & -1 & 4 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

(a) Write $\text{Ker}\,(f)$ as the span of some vector(s).

Solution:

$$\text{Ker}\,(f) = \text{Span}\left\{ \begin{bmatrix} -3 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \right\}$$

(b) Write $\text{Im}\,(f)$ as the span of some vector(s).

Solution:

$$\text{Im}\,(f) = \text{Span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} \right\}$$

(c) Is f injective? Why/why not?

Solution: No, because $\text{Ker}\,(f) \neq \{\mathbf{0}\}$.

(d) Is f surjective? Why/why not?

Solution: No, because $\text{Im}\,(f) \neq \mathbb{R}^4$.

14. Suppose $f : \mathbb{R}^2 \to \mathbb{R}^2$ is a linear function, and you know that

$$f(\begin{bmatrix} 1 \\ 0 \end{bmatrix}) = \begin{bmatrix} 2 \\ 3 \end{bmatrix}, \text{ and } f(\begin{bmatrix} 1 \\ 1 \end{bmatrix}) = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$$

Find a matrix A so that $f(\mathbf{x}) = A\mathbf{x}$ for all $\mathbf{x} \in \mathbb{R}^2$.

Solution: Note that

$$f\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = f\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix} - \begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = f\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) - f\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} -1 \\ 2 \end{bmatrix} - \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} -3 \\ -1 \end{bmatrix}.$$

Thus, as the columns of A are $f(\mathbf{e}_1)$ and $f(\mathbf{e}_2)$, we have

$$A = \begin{bmatrix} 2 & -3 \\ 3 & -1 \end{bmatrix}.$$

15. Let $f : V \to W$ be a linear function on vector spaces V and W. Explain how $\text{Ker}\,(f)$ relates to injectivity. Why is this property true?

Solution: We know that f is injective if and only if $\text{Ker}\,(f) = \{\mathbf{0}\}$. This is true because $f(\mathbf{v}_1) = f(\mathbf{v}_2)$ if and only if $\mathbf{v}_1 - \mathbf{v}_2$ is a member of the kernel of f, and hence we can have two different vectors having the same function value if and only if there is a nonzero member of the kernel.

16. Let $f(\mathbf{x}) = A\mathbf{x}$ be a linear function from $\mathbb{R}^n$ to $\mathbb{R}^m$, where A is a matrix. Explain how surjectivity of f relates to systems of linear equations with coefficients given by A. You may wish to use an example to help illustrate your points.

Solution: Consider the linear system $A\mathbf{x} = \mathbf{b}$, where $\mathbf{b}$ is a vector in $\mathbb{R}^m$. If f is surjective, then this system has a solution for $\mathbf{x}$ regardless of the choice of $\mathbf{b}$. If it is not surjective, then there are choices of $\mathbf{b}$ for which the system has no solution.

17. Can you construct a 3×3 matrix A so that

$$A \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} z + x \\ y \\ x \end{bmatrix}$$

for all $\begin{bmatrix} x \\ y \\ z \end{bmatrix} \in \mathbb{R}^3$? If so, do it. If not, explain why not.

Solution: Yes.

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}.$$

18. Can you construct a 3×3 matrix A so that

$$A \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} z + x + 1 \\ y \\ x \end{bmatrix}$$

for all $\begin{bmatrix} x \\ y \\ z \end{bmatrix} \in \mathbb{R}^3$? If so, do it. If not, explain why not.

Solution: No. Recall that the matrix vector product can be represented as linear combination of the columns of A; that is, if the columns of A are $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3$, then we have $A \begin{bmatrix} x \\ y \\ z \end{bmatrix} = x\mathbf{a}_1 + y\mathbf{a}_2 + z\mathbf{a}_3$. However, the vector $\begin{bmatrix} z+x+1 \\ y \\ x \end{bmatrix}$ cannot be represented as a linear combination of three vectors with weights x, y, z, as the constant term would not be present in such a combination. Hence, this is impossible.

19. Suppose that $\mathbf{v} \in \text{Span}\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k\}$. Show that

$$\text{Span}\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k\} = \text{Span}\{\mathbf{v}, \mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k\}.$$

Solution: Certainly it is true that $\text{Span}\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k\} \subseteq \text{Span}\{\mathbf{v}, \mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k\}$, as anything written as a linear combination of $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k$ can also be written as a linear combination of $\mathbf{v}, \mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k$ by taking the coefficient of $\mathbf{v}$ to be 0.

For the opposite containment, note that as $\mathbf{v} \in \text{Span}\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k\}$, we have that there are coefficients $a_1, a_2, \dots, a_k$ such that $\mathbf{v} = a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + \dots a_k\mathbf{v}_k$.

Suppose $\mathbf{w} \in \text{Span}\{\mathbf{v}, \mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k\}$. Then we have that there are coefficients $\beta, b_1, b_2, \dots, b_k$ such that

$$\begin{aligned} \mathbf{w} &= \beta\mathbf{v} + b_1\mathbf{v}_1 + b_2\mathbf{v}_2 + \dots b_k\mathbf{v}_k \\ &= \beta(a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + \dots a_k\mathbf{v}_k) + b_1\mathbf{v}_1 + b_2\mathbf{v}_2 + \dots b_k\mathbf{v}_k \\ &= (\beta a_1 + b_1)\mathbf{v}_1 + (\beta a_2 + b_2)\mathbf{v}_2 + \dots + (\beta a_k + b_k)\mathbf{v}_k. \end{aligned}$$

Hence $\mathbf{w} \in \text{Span}\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k\}$.

20. Let W_1, W_2 be vector subspaces of a vector space V. Define $U = \{w_1 + w_2 : w_1 \in W_1, w_2 \in W_2\}$. Is this a vector subspace of V? Why/why not?

Solution: Yes. Consider the necessary properties:

1. $0 \in W_1$ and $0 \in W_2$, hence $0 + 0 = 0 \in U$.
2. Let $w_1 + w_2, v_1 + v_2 \in U$, where $w_1, v_1 \in W_1$ and $w_2, v_2 \in W_2$. Then $(w_1 + w_2) + (v_1 + v_2) = (w_1 + v_1) + (w_2 + v_2)$. But since W_1 and W_2 are vector spaces, $w_1 + v_1 \in W_1$ and $w_2 + v_2 \in W_2$. Hence $(w_1 + v_1) + (w_2 + v_2) \in U$, so $(w_1 + w_2) + (v_1 + v_2) \in U$ and U is closed under addition.
3. Let $w_1 + w_2 \in U$, where $w_1 \in W_1$ and $w_2 \in W_2$, and let α be a scalar. Then as $\alpha w_1 \in W_1$ and $\alpha w_2 \in W_2$, we have that $\alpha(w_1 + w_2) = \alpha w_1 + \alpha w_2 \in U$, so U is closed under scalar multiplication.

21. Let $D = \text{diag}(d_1, d_2, \dots, d_n)$ be a diagonal matrix over $\mathbb{R}$, and let $f(\mathbf{x}) = D\mathbf{x}$ be a linear transformation from $\mathbb{R}^n$ to $\mathbb{R}^n$. Show that $\text{Ker}(f)$ is the span of all $\mathbf{e}_i$ such that $d_i = 0$, and $\text{Im}(f)$ is the span of all $\mathbf{e}_i$ such that $d_i \neq 0$.

Solution: First, note that for any vector $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$, we have that $D\mathbf{x} =$

$$\begin{bmatrix} d_1 x_1 \\ d_2 x_2 \\ \vdots \\ d_n x_n \end{bmatrix}.$$

Suppose $\mathbf{x}$ is such that $D\mathbf{x} = 0$. Note then that $d_i x_i = 0$ for all i, so for any x_i with $x_i \neq 0$, we must have $d_i = 0$. Hence, x_i can be written as a linear combination of $\mathbf{e}_i$, over only those $\mathbf{e}_i$ having $d_i = 0$. Likewise, any such linear combination will clearly have $D\mathbf{x} = 0$. Therefore, $\mathrm{Ker}\,(f)$ is precisely the span of all $\mathbf{e}_i$ such that $d_i = 0$.

For the other part, consider the span of the columns of D. Note that any column for which $d_i = 0$ is an all zeros column, and hence can be removed from the generating set for the span of the columns. Thus, the only columns that contribute to the span are those for which $d_i \neq 0$.