Math 241: Exam 2 Review Solutions

Practice Problems If unspecified, you may assume that any vectors mentioned are members of a vector space V over $\mathbb{R}$. If unspecified, you may assume that any matrices mentioned have all entries from $\mathbb{R}$.

1. Let A be an $m \times n$ matrix, B an $n \times k$ matrix. Prove that the i^{th} row of AB is the i^{th} row of A times B.

   Solution: Write $A = [a_{ij}]$ and $B = [b_{ij}]$. By definition, we have that the ij^{th} position of AB is given by $\sum_{k=1}^{n} a_{ik}b_{kj}$.
   Now, let $\mathbf{a}_i = [a_{i1}, a_{i2}, \ldots, a_{in}]$ be the i^{th} row of A. Then $\mathbf{a}_i B$ is a $1 \times k$ matrix, with the j^{th} position given by $\sum_{k=1}^{n} a_{ik}b_{kj}$. Note that this is the same as the ij^{th} position of AB. Hence, the i^{th} row of AB is equal to the i^{th} row of A times B.

2. Prove that if A and B are both symmetric $n \times n$ matrices, then AB is symmetric if and only if $AB = BA$.

   Solution: Suppose A and B are both symmetric $n \times n$ matrices, and suppose further that $AB = BA$. Then $(AB)^T = B^T A^T = BA = AB$, so AB is symmetric.
   Suppose A and B are both symmetric $n \times n$ matrices, and suppose further that AB is symmetric. Then $AB = (AB)^T = B^T A^T = BA$, so $AB = BA$.

3. Let A be an $n \times n$ matrix. Show that if there exists some $\mathbf{b}$ for which $A\mathbf{x} = \mathbf{b}$ has a unique solution, then A is invertible.

   Solution: Suppose there exists $\mathbf{b}$ such that $A\mathbf{x} = \mathbf{b}$ has a unique solution, say $\mathbf{x}_0$.
   Now, suppose that $A\mathbf{y} = \mathbf{0}$. Then it must be the case that $\mathbf{x}_0 + \mathbf{y}$ is also a solution to $A\mathbf{x} = \mathbf{b}$, and hence since $\mathbf{x}_0$ is the only such solution, we must have that $\mathbf{y} = \mathbf{0}$.
   Therefore, the only solution to $A\mathbf{y} = \mathbf{0}$ is the 0 vector, and hence $\text{Null}\,(A)$ is trivial. Thus, A is invertible.

4. Suppose A is an $n \times n$ invertible matrix, and B is obtained from A by adding 2 times the first row of A to the second row of A. Show that B^{-1} is obtained from A^{-1} by adding -2 times the first row of A^{-1} to the second row of A^{-1}.

   Solution: Consider the matrix

$$E = \begin{bmatrix} 1 & 0 & 0 & \ldots & 0 \\ 2 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & 1 \end{bmatrix}.$$

Notice that EA performs the operation of adding two times the first row of A to the second row of A, so $EA = B$. Then $B^{-1} = A^{-1}E^{-1}$, and by noting that E^{-1} is identical to E, with 2 replaced with -2, the result follows.

5. Is the matrix $A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix}$ invertible? How do you know?

   Solution: Yes, this matrix is invertible. This can be seen because the columns of the matrix are linearly independent, and hence the matrix is invertible.

6. Determine if each of the following sets of vectors are linearly independent or dependent; prove that your answer is correct. If they are dependent, find a maximum subset that is independent.

   (a) $\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} \right\}$

   Solution: No, these are not linearly independent, as

   $$\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} - 2 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

   A maximum linearly independent subset is any subset of size 2.

   (b) $\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix} \right\}$

   Solution: No, these are not linearly independent, as there are 4 vectors in a subspace of dimension 3, which cannot ever be LI. Note, however, that the first three vectors are LI, since if $\alpha_1 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + \alpha_2 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + \alpha_3 \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$, we must have $2\alpha_2 = -\alpha_3$, and $3\alpha_1 = -\alpha_3$, but then $\alpha_1 + \alpha_2 + \alpha_3 \neq 0$. Hence, the first three vectors form a maximum subset that is linearly independent.

   (c) $\left\{ \begin{bmatrix} a \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} b \\ c \\ 0 \end{bmatrix}, \begin{bmatrix} d \\ e \\ f \end{bmatrix} \right\}$, where a, b, c, d, e, f are all nonzero constants.

   Solution: Yes, these three vectors are linearly independent. If $\alpha_1 \begin{bmatrix} a \\ 0 \\ 0 \end{bmatrix} + \alpha_2 \begin{bmatrix} b \\ c \\ 0 \end{bmatrix} + \alpha_3 \begin{bmatrix} d \\ e \\ f \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$, then it must be the case that $f\alpha_3 = 0$, so $\alpha_3 = 0$, and then $c\alpha_2 = 0$, so $\alpha_2 = 0$, and then $a\alpha_1 = 0$, so $\alpha_1 = 0$.

7. Suppose $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ are linearly independent, and that $\mathbf{v}_{k+1} \notin \operatorname{Span}\{\mathbf{v}_i\}$ for any i. Must the set $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k, \mathbf{v}_{k+1}\}$ also be linearly independent? Explain.

   Solution: No. Imagine that $k = 2$, and that $\{\mathbf{v}_1, \mathbf{v}_2\}$ span a plane. Select any vector $\mathbf{v}_3$ on that plane that is not parallel to $\mathbf{v}_1$ or $\mathbf{v}_2$. Then this vector satisfies the condition that $\mathbf{v}_3 \notin \operatorname{Span}\{\mathbf{v}_i\}$ for $i = 1, 2$, but $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is linearly dependent.

8. Let $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ be a spanning set for a vector space V. Explain why the following two statements are equivalent:

   - For all $\mathbf{v} \in V$, there is a unique set of coefficients $\alpha_1, \alpha_2, \ldots, \alpha_k$ such that $\mathbf{v} = \alpha_1\mathbf{v}_1 + \alpha_2\mathbf{v}_2 + \cdots + \alpha_k\mathbf{v}_k$.
   - S is a basis for V.

   Solution: Recall that S is a basis for V if and only if the only solution to the equation $a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + \cdots + a_k\mathbf{v}_k = \mathbf{0}$ is the trivial one, i.e., all $a_i = 0$. Moreover, if $\mathbf{v} = \alpha_1\mathbf{v}_1 + \alpha_2\mathbf{v}_2 + \cdots + \alpha_k\mathbf{v}_k = \beta_1\mathbf{v}_1 + \beta_2\mathbf{v}_2 + \cdots + \beta_k\mathbf{v}_k$, then $a_1 = \alpha_1 - \beta_1, a_2 = \alpha_2 - \beta_2, \ldots, a_k = \alpha_k - \beta_k$ is a solution to the homogeneous linear combination above. Hence S is a basis for V if and only if for every $v \in V$, if there at most one set of coefficients for which the above linear combination works. Note that as S is a spanning set, this is true if and only if for every v, there is exactly one unique set of coefficients.

9. Suppose $\mathcal{B}$ and $\mathcal{C}$ are both bases for a vector space V.

   (a) How do you compute the change of basis matrix A from representations with respect to $\mathcal{B}$ to representations with respect to $\mathcal{C}$?

   Solution: The columns of A are obtained by writing the vectors from $\mathcal{B}$ in their $\mathcal{C}$-coordinate representation.

   (b) Explain why the matrix A from part (a) will always be invertible.

   Solution: Since a basis always consists of linearly independent vectors, the columns of A will be linearly independent. Moreover, as $|\mathcal{B}| = |\mathcal{C}|$, the number of columns of A is the same as the number of rows of A. These two things together ensure that A is invertible.

10. Give an example of four vectors that are linearly dependent, but any subset of three of them is linearly independent.

    Solution: $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$. (or, you know, any 4 that work)

11. Prove that for any set S of vectors, $\operatorname{rank} S \leq |S|$.

Solution: By definition, we have that $\operatorname{rank} S = \dim \operatorname{Span}\{S\}$. Let $\mathcal{B}$ be a basis for $\operatorname{Span}\{S\}$. Then $\mathcal{B}$ is linearly independent in $\operatorname{Span}\{S\}$, and S is spanning for $\operatorname{Span}\{S\}$, so by the lemma that spanning sets are at least as large as linearly independent sets, we have $|S| \geq |\mathcal{B}| = \dim \operatorname{Span}\{S\} = \operatorname{rank} S$.

12. Suppose $W \subseteq V$ are both vector spaces. Prove that $\dim W \leq \dim V$, and that $\dim W = \dim V$ if and only if $W = V$.

Solution: Let $\mathcal{B}$ be a basis for W. Then $\mathcal{B} \subseteq V$ is a subset of V, and is linearly independent in V, and hence $|\mathcal{B}| = \dim W \leq \dim V$.

On the other hand, if $|\mathcal{B}| = \dim V$, then $\mathcal{B}$ is a $\dim V$ sized linearly independent set in V, and hence is a basis for V. But $\mathcal{B}$ is assumed to be a basis for W, so $W = V$.

13. Suppose A and B are $n \times n$ matrices such that $AB = I_n$. Prove that A and B must both be invertible matrices, and that $B = A^{-1}$. That is to say, the only matrix by which we can multiply A to get the identity is, in fact, its inverse.

Solution: Define $f : \mathbb{R}^n \to \mathbb{R}^n$ by $f(x) = Ax$, and define $g : \mathbb{R}^n \to \mathbb{R}^n$ by $g(x) = Bx$. Note that $f \circ g$ is the identity function. In particular, $f \circ g$ is injective and surjective. Note that by injectivity of $f \circ g$, we must have that g is injective, since $\operatorname{Ker}(f \circ g) \supseteq \operatorname{Ker}(g)$. Therefore, by the TFAE theorem, we have that B is invertible. But then as $AB = I_n$, multiplying by B^{-1} on the right yields $A = B^{-1}$, allowing us to conclude also that A is invertible and that A and B are inverse to each other.

14. Show that the dot product over $\mathbb{Z}_2$ is not an inner product.

Solution: Consider $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = 1 + 1 + 0 = 0$, which violates positivity for an inner product. Hence the dot product over $\mathbb{Z}_2$ is not an inner product.

15. Suppose V is a vector space with an inner product and associated norm. Prove that if $\mathbf{u}$ is orthogonal to $\mathbf{v}$, then

$$\|\alpha\mathbf{u} + \beta\mathbf{v}\|^2 = \alpha^2\|\mathbf{u}\|^2 + \beta^2\|\mathbf{v}\|^2.$$

Give an example to show that the condition of orthogonality is necessary.

Solution:

$$\begin{aligned} \|\alpha\mathbf{u} + \beta\mathbf{v}\|^2 &= \langle \alpha\mathbf{u} + \beta\mathbf{v}, \alpha\mathbf{u} + \beta\mathbf{v} \rangle \\ &= \alpha^2\langle \mathbf{u}, \mathbf{u} \rangle + 2\alpha\beta\langle \mathbf{u}, \mathbf{v} \rangle + \beta^2\langle \mathbf{v}, \mathbf{v} \rangle \\ &= \alpha^2\|\mathbf{u}\|^2 + 0 + \beta^2\|\mathbf{v}\|^2. \end{aligned}$$

Literally every example of nonorthogonal vectors will fail.

16. Given a subspace W and a vector $\mathbf{v}$, we say that $\mathbf{z} \in W$ is the "closest point in W to $\mathbf{v}$" if $\mathbf{z} = \arg\min_{\mathbf{w}\in W} \|\mathbf{w} - \mathbf{v}\|$. That is to say, $\mathbf{w}$ is the vector in W with the smallest distance to $\mathbf{v}$.

    Prove that if $W = \text{Span}\{\mathbf{w}\}$, then the closest point in W to $\mathbf{v}$ is $\text{Proj}_{\mathbf{w}}\,\mathbf{v}$.

    Solution: Let $\mathbf{a} = \text{Proj}_{\mathbf{w}}\,\mathbf{v}$, so $\mathbf{a} = \frac{\langle \mathbf{v},\mathbf{w}\rangle}{\langle \mathbf{w},\mathbf{w}\rangle}\mathbf{w}$. Write $\alpha = \frac{\langle \mathbf{v},\mathbf{w}\rangle}{\langle \mathbf{w},\mathbf{w}\rangle}$. Now, let $\beta\mathbf{w}$ be any other point in W. Write $\beta\mathbf{w} = \mathbf{a} + (\mathbf{a} - \beta\mathbf{w}) = \alpha\mathbf{w} + (\beta - \alpha)\mathbf{w}$. Then

$$\begin{aligned}\|\beta\mathbf{w} - \mathbf{v}\| &= \|\alpha\mathbf{w} + (\beta-\alpha)\mathbf{w} - \mathbf{v}\| \\ &= \|(\alpha\mathbf{w} - \mathbf{v}) + (\beta-\alpha)\mathbf{w}\|.\end{aligned}$$

    Now, by definition, $(\alpha\mathbf{w} - \mathbf{v}) \perp \mathbf{w}$, so by the previous problem, we therefore have

$$\begin{aligned}\|\beta\mathbf{w} - \mathbf{v}\|^2 &= \|(\alpha\mathbf{w} - \mathbf{v}) + (\beta-\alpha)\mathbf{w}\|^2 \\ &= \|(\alpha\mathbf{w} - \mathbf{v})\|^2 + (\beta-\alpha)^2\|\mathbf{w}\|^2 \\ &\geq \|(\alpha\mathbf{w} - \mathbf{v})\|^2,\end{aligned}$$

    with equality only if $\beta = \alpha$. Hence, for any $\beta\mathbf{w} \in W$ with $\beta\mathbf{w} \neq \text{Proj}_{\mathbf{w}}\,\mathbf{v}$, we have $\|\beta\mathbf{w} - \mathbf{v}\| \geq \|\mathbf{a} - \mathbf{v}\|$, and hence $\mathbf{a} = \arg\min_{w\in W} \|\mathbf{w} - \mathbf{v}\|$.

17. Determine if the matrix $A = \begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix}$ can be diagonalized. If so, diagonalize it.

    Solution: First, consider

$$\det(A - \lambda I) = -\lambda(1-\lambda) - 2 = \lambda^2 - \lambda - 2 = (\lambda - 2)(\lambda + 1).$$

    Hence, each eigenvalue of A has algebraic multiplicity 1, and thus has geometric multiplicity 1. Therefore, A can be diagonalized.

    Note that the nullspace of $A - 2I = \begin{bmatrix} -2 & 1 \\ 2 & -1 \end{bmatrix}$ is $\text{Span}\left\{\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right\}$, and the nullspace of $A + I = \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}$ is $\text{Span}\left\{\begin{bmatrix} 1 \\ -1 \end{bmatrix}\right\}$. Hence, a diagonalization of A as $S\Lambda S^{-1}$ can be obtained by

$$S = \begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix}, \quad \Lambda = \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix}.$$

18. Prove that if A is similar to B, then the eigenvalues of A are the same as the eigenvalues of B, with the same algebraic and geometric multiplicities.

    Solution: Recall from Klein that for any two matrices C, D, we have $\det(CD) = \det(C)\det(D)$.

Now, suppose that $A = SBS^{-1}$. Then,

$$\begin{aligned}
\det(A - \lambda I) &= \det(SBS^{-1} - \lambda I) \\
&= \det(SBS^{-1} - \lambda SIS^{-1}) \\
&= \det(S(B - \lambda I)S^{-1}) \\
&= \det S \det(B - \lambda I) \det(S^{-1}) \\
&= \det S \det(B - \lambda I) \frac{1}{\det S} \\
&= \det(B - \lambda I).
\end{aligned}$$

Therefore, as the characteristic equations of A and B are identical, we have that the eigenvalues of A and B are identical, with the same algebraic multiplicities.

Now, suppose that $\mathbf{v}$ is an eigenvector for B with corresponding eigenvalue λ, and let $\mathbf{w} = S\mathbf{v}$. Then $A\mathbf{w} = SBS^{-1}\mathbf{w} = SBS^{-1}S\mathbf{v} = SB\mathbf{v} = \lambda S\mathbf{v} = \lambda\mathbf{w}$, so $S\mathbf{v}$ is an eigenvector for A corresponding to eigenvalue λ. Moreover, as S is invertible, any basis for the eigenspace for λ as an eigenvalue of B will be linearly independent under the transformation S, and hence the eigenspace for λ as an eigenvalue of A has dimension at least as large as that for B. Running this same argument backwards yields the opposite result, and hence the eigenspaces have the same dimensions. Therefore, the eigenvalues have the same geometric multiplicities as well.

19. Consider the matrix $A = \begin{bmatrix} 0 & a \\ a & 0 \end{bmatrix}$ for some constant a.

Given $\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \in \mathbb{R}^2$, write v as a linear combination of eigenvectors for A. Use this to calculate $A^n\mathbf{v}$ for any $n \geq 1$.

Solution: First, note that $\det(A - \lambda I) = (-\lambda)^2 - a^2 = (\lambda - a)(\lambda + a)$, so the eigenvalues of A are $\pm a$. For the eigenvalue a, we have eigenvector $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$, and for eigenvalue $-a$ we have eigenvector $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

Hence we may write $\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \frac{v_1+v_2}{2} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \frac{v_1-v_2}{2} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$. Therefore, we have

$$A^n\mathbf{v} = A^n \left(\frac{v_1 + v_2}{2} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \frac{v_1 - v_2}{2} \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right) = a^n \frac{v_1 + v_2}{2} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + (-a)^n \frac{v_1 - v_2}{2} \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

20. Suppose 0 is an eigenvalue for a matrix A. What is the corresponding eigenspace?

Solution: The eigenspace for 0 is the null space for A.