Math 241: Exam 2 Review

Key Concepts

- Definition of matrix product AB, basic properties
- Definition of invertible linear function, matrix, understanding how to compute inverse of an $n \times n$ matrix A.
- Definitions of linearly dependent, independent, basis
- Know how to construct a basis from a spanning set for a space
- Change of basis: how to compute a matrix that performs change of basis, what the matrix actually does, and how to represent vectors with respect to various bases.
- Understand why two bases for a space have the same size, and the theorems used to build that theorem
- Know what dimension of a space is, how to determine it, and related theory
- Rank-Nullity Theorem, for matrices and for linear transformations
- Determinant: Definition, how to compute, purpose
- Eigenvalues/eigenvectors: definition, how to compute, purpose
- Diagonalization: how to determine if a matrix is diagonalizable, and how to diagonalize it if so
- Norm, inner product
- Projection onto a subspace; you need only be able to compute projection onto a one-dimensional space for this exam.

Terms

- Inverse (for a function or a matrix)
- Linearly independent/dependent
- Basis
- Dimension
- Rank (of a matrix, or a set S)
- Nullity
- Determinant
- Eigenvalue, eigenvector, eigenspace

- Characteristic equation
- Algebraic multiplicity, geometric multiplicity
- Diagonalizable
- Similar
- Norm
- Inner product, associated norm ($\|v\| = \sqrt{\langle v, v\rangle}$)
- Orthogonal
- Projection onto a subpsace

Key Theorems

- The inverse of a linear function is linear.
- If A, B are invertible, then AB is invertible and $(AB)^{-1} = B^{-1}A^{-1}$.
- If $S = \{v_1, v_2, \ldots, v_n\}$ is a spanning set for V, then there exists $\mathcal{B} \subseteq S$ that is a basis for V.
- If S is a generating set for a vector space V and $\mathcal{B}$ is a finite linearly independent set in V, then $|\mathcal{B}| \leq |S|$.
- If $\mathcal{B}_1$ and $\mathcal{B}_2$ are finite bases for a vector space V, then $|\mathcal{B}_1| = |\mathcal{B}_2|$.
- if S is a generating set for V, and $|S|$ is smallest among all possible generating sets for V, then S is a basis for V.
- If V is a finite-dimensional vector space, with $\dim V = n$, and $A \subseteq V$ is linearly independent, then there exists a basis $\mathcal{B}$ for V with $A \subseteq \mathcal{B}$.
- If V is a finite-dimensional vector space, with $\dim V = n$, and $A \subseteq V$ has $|A| = n$ and A is linearly independent, then A is a basis for V.
- (Rank Nullity Theorem v1) If $f : V \to W$ is a linear function, then

$$\dim \operatorname{Ker}(f) + \dim \operatorname{Im}(f) = \dim V$$

- (Rank Nullity Theorem v2) If A is an $m \times n$ matrix, then

$$\operatorname{rank} A + \operatorname{nullity} A = n.$$

Practice Problems If unspecified, you may assume that any vectors mentioned are members of a vector space V over $\mathbb{R}$. If unspecified, you may assume that any matrices mentioned have all entries from $\mathbb{R}$.

1. Let A be an $m \times n$ matrix, B an $n \times k$ matrix. Prove that the i^{th} row of AB is the i^{th} row of A times B.

2. Prove that if A and B are both symmetric $n \times n$ matrices, then AB is symmetric if and only if $AB = BA$.

3. Let A be an $n \times n$ matrix. Show that if there exists some $\mathbf{b}$ for which $A\mathbf{x} = \mathbf{b}$ has a unique solution, then A is invertible.

4. Suppose A is an $n \times n$ invertible matrix, and B is obtained from A by adding 2 times the first row of A to the second row of A. Show that B^{-1} is obtained from A^{-1} by adding -2 times the first row of A^{-1} to the second row of A^{-1}.

5. Is the matrix $A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix}$ invertible? How do you know?

6. Determine if each of the following sets of vectors are linearly independent or dependent; prove that your answer is correct. If they are dependent, find a maximum subset that is independent.

   (a) $\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} \right\}$

   (b) $\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix} \right\}$

   (c) $\left\{ \begin{bmatrix} a \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} b \\ c \\ 0 \end{bmatrix}, \begin{bmatrix} d \\ e \\ f \end{bmatrix} \right\}$, where a, b, c, d, e, f are all nonzero constants.

7. Suppose $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ are linearly independent, and that $\mathbf{v}_{k+1} \notin \text{Span}\{\mathbf{v}_i\}$ for any i. Must the set $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k, \mathbf{v}_{k+1}\}$ also be linearly independent? Explain.

8. Let $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ be a spanning set for a vector space V. Explain why the following two statements are equivalent:

   - For all $\mathbf{v} \in V$, there is a unique set of coefficients $\alpha_1, \alpha_2, \ldots, \alpha_k$ such that $\mathbf{v} = \alpha_1\mathbf{v}_1 + \alpha_2\mathbf{v}_2 + \cdots + \alpha_k\mathbf{v}_k$.
   - S is a basis for V.

9. Suppose $\mathcal{B}$ and $\mathcal{C}$ are both bases for a vector space V.

   (a) How do you compute the change of basis matrix A from representations with respect to $\mathcal{B}$ to representations with respect to $\mathcal{C}$?

   (b) Explain why the matrix A from part (a) will always be invertible.

10. Give an example of four vectors that are linearly dependent, but any subset of three of them is linearly independent.

11. Prove that for any set S of vectors, $\operatorname{rank} S \leq |S|$.

12. Suppose $W \subseteq V$ are both vector spaces. Prove that $\dim W \leq \dim V$, and that $\dim W = \dim V$ if and only if $W = V$.

13. Suppose A and B are $n \times n$ matrices such that $AB = I_n$. Prove that A and B must both be invertible matrices, and that $B = A^{-1}$. That is to say, the only matrix by which we can multiply A to get the identity is, in fact, its inverse.

14. Show that the dot product over $\mathbb{Z}_2$ is not an inner product.

15. Suppose V is a vector space with an inner product and associated norm. Prove that if $\mathbf{u}$ is orthogonal to $\mathbf{v}$, then

$$\|\alpha\mathbf{u} + \beta\mathbf{v}\|^2 = \alpha^2\|\mathbf{u}\|^2 + \beta^2\|\mathbf{v}\|^2.$$

Give an example to show that the condition of orthogonality is necessary.

16. Given a subspace W and a vector $\mathbf{v}$, we say that $\mathbf{z} \in W$ is the "closest point in W to $\mathbf{v}$" if $\mathbf{z} = \arg\min_{\mathbf{w}\in W} \|\mathbf{w} - \mathbf{v}\|$. That is to say, $\mathbf{w}$ is the vector in W with the smallest distance to $\mathbf{v}$.

Prove that if $W = \operatorname{Span}\{\mathbf{w}\}$, then the closest point in W to $\mathbf{v}$ is $\operatorname{Proj}_{\mathbf{w}} \mathbf{v}$.

17. Determine if the matrix $A = \begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix}$ can be diagonalized. If so, diagonalize it.

18. Prove that if A is similar to B, then the eigenvalues of A are the same as the eigenvalues of B, with the same algebraic and geometric multiplicities.

19. Consider the matrix $A = \begin{bmatrix} 0 & a \\ a & 0 \end{bmatrix}$ for some constant a.

Given $\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \in \mathbb{R}^2$, write v as a linear combination of eigenvectors for A. Use this to calculate $A^n\mathbf{v}$ for any $n \geq 1$.

20. Suppose 0 is an eigenvalue for a matrix A. What is the corresponding eigenspace?