Math 241: Exam 1 Review

Key Concepts

- Understanding of vectors and vector arithmetic, including properties of addition, scalar multiplication, and inner products
- Expressing linear equations/systems in terms of dot products, solutions to triangular systems
- Spans and linear combinations, and how these relate to planes/lines/etc, and to solutions to systems of linear equations
- Vector spaces: what they are, how to determine if a set of vectors is a space or not
- Matrices and basic matrix operations, addition, scalar multiplication, and matrix-vector multiplication
- Interpretations of matrix-vector multiplication: dot products, linear combinations of the columns of A, system of linear equations
- Null space of a matrix A: how to compute, relationship to the matrix equation $A\mathbf{x} = \mathbf{b}$
- Linear functions: what they are, how to determine if a function is linear
- Injectivity and surjectivity, and how to use a matrix representation $f(\mathbf{x}) = A\mathbf{x}$ to determine whether f is injective and/or surjective.

Terms

- Vector (associated terms: elements, entries, length)
- Standard unit vector
- Linear combination (associated terms: affine combination, convex combination)
- System of linear equations
- Span, Spanning set (aka generating set)
- Homogeneous linear system (also nonhomogeneous)
- Vector space, vector subspace
- Matrix (associated terms: dimension, elements, square, column, row, upper triangular, transpose)
- Column space
- Null space
- Linear function (aka linear transformation)

- Kernel
- Injective (aka one-to-one)
- Image (aka range)
- Surjective (aka onto)

Key Theorems

- If V is a vector space and $W \subseteq V$, then W is a vector subspace of V if and only if
  1. W contains $\mathbf{0}$
  2. W is closed under addition
  3. W is closed under scalar multiplication
- If V is a vector space and $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k \in V$, then $\mathrm{Span}\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ is a vector subspace of V.
- If A is an $m \times n$ matrix, then $A\mathbf{e}_k$ is the k^{th} column of A.
- Let A be an $m \times n$ matrix. Then $\mathrm{Null}(A)$ is a vector subspace of $\mathbb{R}^n$.
- Let A be an $m \times n$ matrix, and let $\mathbf{b} \in \mathbb{R}^m$. If $\mathbf{x}_1$ is a particular solution to the equation $A\mathbf{x} = \mathbf{b}$, then the set of solutions to the equation takes the form $\{\mathbf{x}_1 + \mathbf{y} \mid \mathbf{y} \in \mathrm{Null}(A)\}$.
- Let $f : V \to W$ be a linear function on vector spaces V and W. Then $\mathrm{Ker}(f)$ is a vector subpsace of V, and $\mathrm{Im}(f)$ is a vector subspace of W.
- Let $f : V \to W$ be a linear function on vector spaces V and W. Then f is injective if and only if $\mathrm{Ker}(f) = \{\mathbf{0}\}$.
- Let $f : V \to W$ be a linear function on vector spaces V and W. Then f is surjective if and only if $\mathrm{Im}(f) = W$.
- Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be a linear function, represented as $f(\mathbf{x}) = A\mathbf{x}$. Then $\mathrm{Ker}(f) = \mathrm{Null}(A)$, and $\mathrm{Im}\, f = \mathrm{Col}(A)$.

Practice Problems

1. Let $\mathbf{u} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$, and $\mathbf{v} = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$. Write an expression for the line that goes through these two vectors in $\mathbb{R}^3$.

2. Let $\mathbf{u}, \mathbf{v}$ be vectors over $\mathbb{Z}_2$. Explain why $\mathbf{u} + \mathbf{v} = \mathbf{u} - \mathbf{v}$.

3. Here is a piece of code used to solve a triangular system of n equations in n variables.

```
def triangular_solve(coeff, b):
    n=len(b)
    x=zero_vec(n)
    for i in reversed(range(n)):
        x[i] = (b[i]-dot_product(coeff[i], x))/coeff[i][i]
    return x
```

   In this code, the first input is a list of lists representing coefficients for the equations, and the second input is the right hand side of the system. So, for example, if you had a system

$$\begin{aligned} 2x_1 + 3x_2 &= 3 \\ 4x_2 &= 1 \end{aligned}$$

   then the inputs would take the form

```
coeff=[[2,3],[0,4]], b=[3,1].
```

   You may assume here that zero_vec and dot_product are correctly defined helper functions that return the length n zero vector (as a list) and the dot product of two vectors (represented as lists), respectively.

   (a) Explain (perhaps by showing a small example) how this algorithm works to solve a triangular system.

   (b) Explain (perhaps by showing a small example) when this code will fail.

4. Show that the set

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$$

   is a generating set for $\mathbb{R}^3$.

5. Show that the set

$$\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} \right\}$$

   is not a generating set for $\mathbb{R}^3$.

6. Consider the plane in $\mathbb{R}^3$ given by the solutions to the equation $4x - 3y + z = 0$. Write this plane as the span of some vectors in $\mathbb{R}^3$.

7. Determine whether each of the following sets is a vector space in $\mathbb{R}^4$. Justify your response.

(a) $\left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \in \mathbb{R}^4 \ : \ x_1 + x_2 = 3 \right\}$

(b) $\left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \in \mathbb{R}^4 \ : \ x_1 + x_2 + x_3 + x_4 = 0 \right\}$

(c) $\left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \in \mathbb{R}^4 \ : \ x_1 x_2 x_3 x_4 = 0 \right\}$

8. Let $\mathcal{P}_n$ denote the vector space of polynomials having degree at most n, with coefficients from $\mathbb{R}$. Show that the set of polynomials having degree at most k, with $k < n$, is a vector subspace of $\mathcal{P}_n$.

9. Explain how to represent a system of equations with a matrix equation. You may want to use an example to help illustrate your points.

10. Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ be a real-valued matrix. Show that $\text{Null}\,(A)$ is nontrivial if and only if $ad - bc = 0$.

11. Let $A\mathbf{x} = \mathbf{b}$ be a matrix equation, and let $\mathbf{x}_1$ be a solution to this equation. How do you find other solutions? Why does this technique work?

12. Let

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Write the null space of A as the span of some vector(s).

13. Define a function $f : \mathbb{R}^4 \to \mathbb{R}^4$ by $f(\mathbf{x}) = A\mathbf{x}$, where

$$A = \begin{bmatrix} 1 & 0 & -1 & 4 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

(a) Write $\text{Ker}\,(f)$ as the span of some vector(s).

(b) Write $\text{Im}\,(f)$ as the span of some vector(s).

(c) Is f injective? Why/why not?

(d) Is f surjective? Why/why not?

14. Suppose $f : \mathbb{R}^2 \to \mathbb{R}^2$ is a linear function, and you know that

$$f(\begin{bmatrix} 1 \\ 0 \end{bmatrix}) = \begin{bmatrix} 2 \\ 3 \end{bmatrix}, \text{ and } f(\begin{bmatrix} 1 \\ 1 \end{bmatrix}) = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$$

Find a matrix A so that $f(\mathbf{x}) = A\mathbf{x}$ for all $\mathbf{x} \in \mathbb{R}^2$.

15. Let $f : V \to W$ be a linear function on vector spaces V and W. Explain how $\operatorname{Ker}(f)$ relates to injectivity. Why is this property true?

16. Let $f(\mathbf{x}) = A\mathbf{x}$ be a linear function from $\mathbb{R}^n$ to $\mathbb{R}^m$, where A is a matrix. Explain how surjectivity of f relates to systems of linear equations with coefficients given by A. You may wish to use an example to help illustrate your points.

17. Can you construct a 3×3 matrix A so that

$$A \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} z + x \\ y \\ x \end{bmatrix}$$

for all $\begin{bmatrix} x \\ y \\ z \end{bmatrix} \in \mathbb{R}^3$? If so, do it. If not, explain why not.

18. Can you construct a 3×3 matrix A so that

$$A \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} z + x + 1 \\ y \\ x \end{bmatrix}$$

for all $\begin{bmatrix} x \\ y \\ z \end{bmatrix} \in \mathbb{R}^3$? If so, do it. If not, explain why not.

19. Suppose that $\mathbf{v} \in \operatorname{Span}\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$. Show that

$$\operatorname{Span}\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\} = \operatorname{Span}\{v, \mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}.$$

20. Let W_1, W_2 be vector subspaces of a vector space V. Define $U = \{w_1 + w_2 : w_1 \in W_1, w_2 \in W_2\}$. Is this a vector subspace of V? Why/why not?

21. Let $D = \operatorname{diag}(d_1, d_2, \ldots, d_n)$ be a diagonal matrix over $\mathbb{R}$, and let $f(\mathbf{x}) = D\mathbf{x}$ be a linear transformation from $\mathbb{R}^n$ to $\mathbb{R}^n$. Show that $\operatorname{Ker}(f)$ is the span of all $\mathbf{e}_i$ such that $d_i = 0$, and $\operatorname{Im}(f)$ is the span of all $\mathbf{e}_i$ such that $d_i \neq 0$.