21-241 Midterm II Practice Problems

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Note: These problems are relatively difficult and I do think the midterm will be easier than these (I think, sincere apologies if I'm wrong). So, don't be afraid to spend some time on them and build your intuition for these things. Also, please ask for hints before you give up on the problems! I originally had lots of hints built in to the questions, but chose to take some of them out to avoid spoiling some the problems: if you ask, I'll be more than happy to give you a version of this problem set with all the hints put back in.

Finally, for some later problems, results from earlier problems will be useful.

1. **Efficient representation.**
   (a) Suppose that $A \in \mathbb{R}^{n\times n}$ is diagonalizable and has rank k. Show that A can be written as $A = BC$ for $B \in \mathbb{R}^{n\times k}$ and $C \in \mathbb{R}^{k\times n}$. Note that naïvely, A takes n^2 memory cells to store. Explain how we can store A in less than n^2 memory cells if $n > 2k$.
   (b) Now show that for any $A \in \mathbb{R}^{m\times n}$, A can be written as $A = BC$ for $B \in \mathbb{R}^{m\times k}$ and $C \in \mathbb{R}^{k\times n}$. (Hint: use a different decomposition, but still get inspired by part (a).)
2. **Fun with orthogonal projections.**
   (a) Let $P \in \mathbb{R}^{n\times n}$ be an orthogonal projection matrix. Show that $I - P$ is an orthogonal projection matrix.
   (b) Again, let $P \in \mathbb{R}^{n\times n}$ be an orthogonal projection matrix. Show that $\|v\|^2 = \|Pv\|^2 + \|(I - P)v\|^2$ for all $v \in \mathbb{R}^n$.
3. **Inner product matrices.** Let $v_1, v_2, \ldots, v_n \in \mathbb{R}^d$. We define the inner product matrix G to be the matrix with $G_{i,j} = v_i \cdot v_j$. We will prove some useful properties of inner product matrices.
   (a) Show that there exists a matrix $B \in \mathbb{R}^{d\times n}$ such that $G = B^\top B$.
   (b) Show that $x^\top Gx \geq 0$ for all $x \in \mathbb{R}^n$.
   (c) Show that $\operatorname{rank}(G) \leq d$.
   (d) Show that the eigenvalues of G are all nonnegative.
   (e) Show that $v_1, v_2, \ldots, v_n$ are linearly independent if and only if $\det G > 0$.
4. **All ones.** Let $\mathbf{1} \in \mathbb{R}^{n\times n}$ be the matrix of all 1s, i.e. $\mathbf{1}_{i,j} = 1$ for all $i, j \in [n]$.
   (a) Show that $\mathbf{1}$ has at most one nonzero eigenvalue.
   (b) Find the one nonzero eigenvalue and its eigenvector. (Hint: try to be clever and find one by inspection!)
5. **Spread out vectors.** Suppose we want to arrange $n \geq 2$ unit vectors $v_1, v_2, \ldots, v_n \in \mathbb{R}^d$ such that they are as spread out as possible. For example, when $n = 2$, the unit vectors will be pointing in opposite directions and when $n = 3$, the unit vectors will be pointing in directions that are $120°$ apart. It's intuitive that no matter how many vectors you have, the angles between the vectors will all be the same. The question then is, what is this angle?

(a) Recall that the angle θ between two unit vectors $v_i, v_j \in \mathbb{R}^d$ can be retrieved from their dot product via

$$\cos\theta = v_i \cdot v_j,$$

which is all the same for every pair of vectors v_i, v_j with $i \neq j$. Let $\gamma \coloneqq \cos\theta$ and show that the inner product matrix, i.e. the matrix with $G_{i,j} = v_i \cdot v_j$, can be written as

$$G = (1-\gamma)I + \gamma\mathbf{1}$$

where I is the $n \times n$ identity matrix and $\mathbf{1}$ is the $n \times n$ matrix with all 1s.

(b) Find the eigenvalues of G.

(c) Since we want the vectors to be as spread out as possible, we want θ to be as large as possible, which means we want $\cos\theta$ to be as negative as possible. Thus, it will be useful to find a lower bound on $\gamma = \cos\theta$. Show that $\gamma \geq -(n-1)^{-1}$.

From part (c), we find that we can't have γ smaller than $-(n-1)^{-1}$. We will now show that we can explicitly construct n vectors such that they exactly achieve this angle. We will recursively construct these vectors as follows. Let $A_n \in \mathbb{R}^{(n-1)\times n}$ be the matrix with the constructed $v_1, v_2, \ldots, v_n$ in its columns. We first let

$$A_2 \coloneqq \begin{pmatrix} 1 & -1 \end{pmatrix}.$$

Now to construct A_n, we set

$$A_n \coloneqq \begin{pmatrix} 1 & -\frac{1}{n-1} \cdot \mathbf{1}_{1\times(n-1)} \\ 0 \cdot \mathbf{1}_{(n-2)\times 1} & \sqrt{1 - \frac{1}{(n-1)^2}} \cdot A_{n-1} \end{pmatrix}$$

where $\mathbf{1}_{a\times b}$ is the $a \times b$ matrix with all 1s. Essentially, we are setting $v_1 = e_1 \in \mathbb{R}^{n-1}$, setting the first coordinate of $v_2, v_3, \ldots, v_n$ to be $-(n-1)^{-1}$, and setting the rest of the coordinates to be defined by $\sqrt{1-(n-1)^{-2}}$ times the $n-1$ case.

Let $v_1, v_2, \ldots, v_n$ be defined by the columns of A_n.

(d) Show that v_i is a unit vector for all $i \in [n]$.

(e) Show that $v_i \cdot v_j = -(n-1)^{-1}$ for all $i \neq j$.

(f) Conclude that we have indeed constructed a set of n vectors that are spread out as far as possible.

If you took chemistry and learned about VSEPR theory in high school, this is one way to calculate some of those weird angles. That's pretty cool right.