21-241 Homework 3 Solutions

Taisuke Yasuda

September 26, 2018

1. (a) Note that since $H\mathbf{e} \neq 0$, $\mathbf{e}$ has at least one bit of error, so $\mathbf{e}$ is a vector with exactly one 1 in index i and all other entries 0. Then, $H\mathbf{e}$ is the linear combination of the columns of H where the ith column of H has coefficient 1 and the rest 0s. In other words, $H\mathbf{e}$ is just the ith column of H. Note that $[1, 1, 0]$ is the 6th column of H, so $\boxed{\mathbf{e} = [0, 0, 0, 0, 0, 1, 0]}$. **(1 pt for correct answer)**

   (b) Let $\tilde{\mathbf{c}} = \mathbf{c} + \mathbf{e}$, where $\mathbf{c}$ is the original message and $\mathbf{e}$ is an error vector with at most one 1. Then,

$$H\tilde{\mathbf{c}} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = H(\mathbf{c} + \mathbf{e}) = \mathbf{c} + H\mathbf{e} = 0 + H\mathbf{e}$$

   since $\mathbf{c}$ is a codeword and thus in the null space of H. Then from part (a), we know that $\mathbf{e} = [0, 0, 0, 0, 0, 1, 0]$ so the original message is

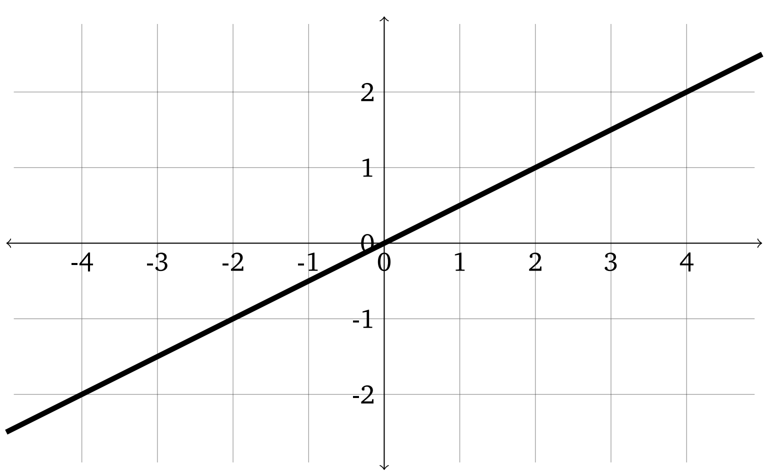
$$\mathbf{c} = \tilde{\mathbf{c}} - \mathbf{e} = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 1 \\ 1 \\ 1 \end{pmatrix} - \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} = \boxed{\begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 1 \\ 0 \\ 1 \end{pmatrix}}.$$

   (1 pt for correctly computing $H\tilde{\mathbf{c}}$, 1 pt for finding $\mathbf{e}$, 1 pt for subtracting $\mathbf{e}$ from $\tilde{\mathbf{c}}$, 1 pt for correct answer)

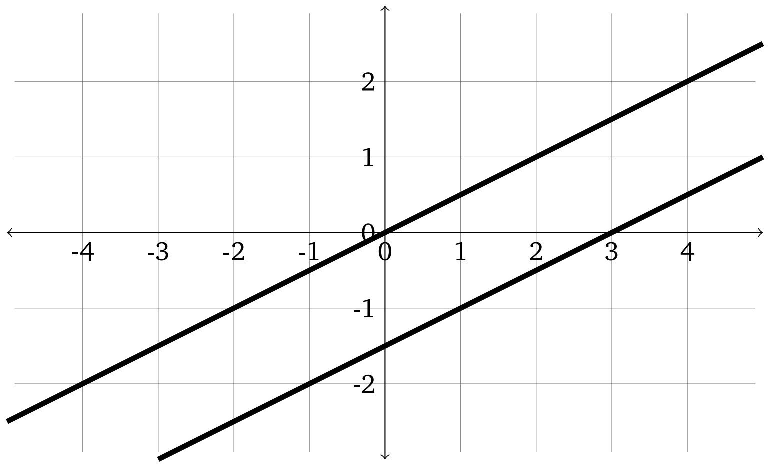
2. Note that for $x, y \in \mathbb{R}$,

$$A\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x - 2y \\ 0 \end{pmatrix}$$

   so $[x, y]$ is in the null space of A if and only if $A[x, y] = [x - 2y, 0] = [0, 0]$, which happens if and only if $x - 2y = 0$. Then, the null space is $\{[x, y] \in \mathbb{R}^2 : x - 2y = 0\}$, which is just the graph of the line $y = x/2$.



Now using the same computation as before, we have that $[x,y]$ is a solution to $A[x,y] = \mathbf{b} = [3,0]$ if and only if $x - 2y = 3$, so the null space is $\{[x,y] \in \mathbb{R}^2 : x - 2y = 3\}$, which is just the graph of the line $y = x/2 - 3/2$.



(1 pt for graph of null space, 2 pts for justification for graph of null space, 1 pt for graph of solution set, 1 pt for justification for graph of solution set)

3. (a) Suppose that $\mathbf{x} = [x_1, x_2, x_3, x_4, x_5, x_6]$ belongs to the null space of A. Then, solving $A\mathbf{x} = 0$ for $x_1, x_2, x_3, x_4, x_5, x_6$ gives us that $x_5 = -x_6$, $x_4 = -x_6$, $x_3 = 0$, and $x_1 = -2x_2$. Thus, we may write

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{pmatrix} = \begin{pmatrix} -2x_2 \\ x_2 \\ 0 \\ -x_6 \\ -x_6 \\ x_6 \end{pmatrix} = x_2 \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} + x_6 \begin{pmatrix} 0 \\ 0 \\ 0 \\ -1 \\ -1 \\ 1 \end{pmatrix}.$$

Thus, the null space of A is contained in the span of $\mathbf{v}_1 := [-2,1,0,0,0,0]$ and $\mathbf{v}_2 := [0,0,0,-1,-1,1]$. On the other hand, note that $A\mathbf{v}_1 = 0$ and $A\mathbf{v}_2 = 0$. Then any vector $\mathbf{v} = \alpha\mathbf{v}_1 + \beta\mathbf{v}_2$ in the span of $\mathbf{v}_1$ and $\mathbf{v}_2$ is in the null space of A since

$$A\mathbf{v} = A(\alpha\mathbf{v}_1 + \beta\mathbf{v}_2) = \alpha A\mathbf{v}_1 + \beta A\mathbf{v}_2 = 0 + 0 = 0.$$

Thus, the null space of A is exactly the span of $\mathbf{v}_1$ and $\mathbf{v}_2$.

(1 pt for finding a valid span, 1 pt for showing that the null space is in the span, 1 pt for showing that the span is in the null space)

(b) By inspection, one solution is $\hat{\mathbf{x}} := [-1,0,0,0,0,1]$. We now work on a expression for all solutions. Suppose $\mathbf{x}$ satisfies $A\mathbf{x} = \mathbf{b}$. Then note that $\mathbf{b} = A\hat{\mathbf{x}}$ and so by plugging into the above and rearranging, the above is true if and only if $A(\mathbf{x} - \hat{\mathbf{x}}) = 0$. Thus, $\mathbf{x}$ solves $A\mathbf{x} = \mathbf{b}$ exactly when $\mathbf{x} - \hat{\mathbf{x}}$ belongs to the null space of A. Thus, if $\mathcal{N}$ is the null space of A, then the set of solutions to $A\mathbf{x} = \mathbf{b}$ is exactly

$$\left\{\mathbf{x} \in \mathbb{R}^6 : A\mathbf{x} = \mathbf{b}\right\} = \left\{\mathbf{x} \in \mathbb{R}^6 : \mathbf{x} - \hat{\mathbf{x}} \in \mathcal{N}\right\} = \{\hat{\mathbf{x}} + \mathbf{x} : \mathbf{x} \in \mathcal{N}\} = \hat{\mathbf{x}} + \mathcal{N}.$$

(1 pt for finding a particular solution, 1 pt for characterizing the solution set)

4. • **Exercise 4.10.7**

We have that $g([0,0]) = [0,0,1] \neq [0,0,0]$ so g is not linear. **(2 pts for a correct counterexample)**

- **Exercise 4.10.8**

  The reflection about the y-axis preserves the y component and negates the x component, so we may write $h([x,y]) = [-x,y]$. This function is linear since for a scalar $\alpha \in \mathbb{R}$ and a vector $[x,y] \in \mathbb{R}^2$

  $$h(\alpha[x,y]) = h([\alpha x, \alpha y]) = [-\alpha x, \alpha y] = \alpha[-x,y] = \alpha h([x,y])$$

  and for two vectors $[x_1,y_1],[x_2,y_2] \in \mathbb{R}^2$,

  $$\begin{aligned} h([x_1,y_1]+[x_2,y_2]) &= h([x_1+x_2, y_1+y_2]) = [-(x_1+x_2), y_1+y_2] \\ &= [-x_1,y_1]+[-x_2,y_2] = h([x_1,y_1]) + h([x_2,y_2]). \end{aligned}$$

  (1 pt for correct h, 1 pt for showing property L1, 1 pt for showing property L2)

5. (a) Let $\alpha \in \mathbb{R}$ and let $p,q \in \mathcal{P}_n(\mathbb{R})$. Then, by basic rules of calculus,

   $$D(\alpha p(x)) = \frac{d}{dx}\alpha p(x) = \alpha\frac{d}{dx}p(x) = \alpha D(p(x))$$

   and

   $$D(p(x)+q(x)) = \frac{d}{dx}p(x)+q(x) = \frac{d}{dx}p(x) + \frac{d}{dx}q(x) = D(p(x)) + D(q(x))$$

   so the function is linear. **(1 pt for showing commutativity with scalar multiplication, 1 pt for showing commutativity with addition)**

   (b) Suppose p is in the kernel of D, i.e. $D(p(x)) = \frac{d}{dx}p(x) = 0$. Then in view of a theorem in calculus, functions whose derivatives are 0 on $\mathbb{R}$ are exactly constant functions, so $D(p(x)) = 0$ if and only if $p(x) = c$ for some constant $c \in \mathbb{R}$. So, the kernel of D is just the set of all constant functions.

   We claim that the range is exactly $\mathcal{P}_{n-1}(\mathbb{R})$, i.e. the map D is surjective. Indeed, suppose $p \in \mathcal{P}_{n-1}(\mathbb{R})$. Then, we may write $p(x)$ as

   $$p(x) = a_0 + a_1x + a_2x^2 + \cdots + a_{n-1}x^{n-1} = \sum_{i=0}^{n-1} a_i x^i.$$

   Now define the polynomial

   $$q(x) := a_0x + \frac{a_1}{2}x^2 + \cdots + a_{n-1}x^n = \sum_{i=0}^{n-1}\frac{a_i}{i+1}x^{i+1} = \sum_{i=1}^{n}\frac{a_{i-1}}{i}x^i.$$

   Clearly, q is a polynomial of degree at most n, so it belongs to $\mathcal{P}_n(\mathbb{R})$. Furthermore, its image under D, i.e. the derivative, is clearly p. We have shown that every element of $\mathcal{P}_{n-1}(\mathbb{R})$ has an element in $\mathcal{P}_n(\mathbb{R})$ that maps to it via D, so D is surjective and thus the range is exactly $\mathcal{P}_{n-1}(\mathbb{R})$. **(1 pt for characterizing the kernel, 2 pts for characterizing the range)**

6. Suppose f is linear. We prove the given formula by inducting on $k \geq 2$. For $k = 2$, the formula holds since

   $$f(a_1v_1 + a_2v_2) = f(a_1v_1) + f(a_2v_2) = a_1f(v_1) + a_2f(v_2)$$

   where in the first equality we use property L2 of linear functions and in the second equality we use property L1 of linear functions. Now assume the formula holds for $k-1$. Then,

   $$\begin{aligned} f(a_1v_1 + a_2v_2 + \cdots + a_kv_k) &= f(a_1v_1 + a_2v_2 + \cdots + a_{k-1}v_{k-1}) + f(a_kv_k) && \text{by property L2} \\ &= f(a_1v_1 + a_2v_2 + \cdots + a_{k-1}v_{k-1}) + a_kf(v_k) && \text{by property L1} \\ &= [a_1f(v_1) + a_2f(v_2) + \cdots + a_{k-1}f(v_{k-1})] + a_kf(v_k) && \text{by inducting} \end{aligned}$$

   so the formula holds for k.

   Now suppose the formula holds for some fixed $k \geq 2$. We need to show that f satisfies properties L1 and L2. Let $v \in \mathbb{R}^n$ and $\alpha \in \mathbb{R}$. Then since the formula holds for any a_i and v_i, we can in particular choose $v_1 \coloneqq v$, $a_1 \coloneqq \alpha$, and $v_i = 0$ and $a_i = 0$ for every $2 \leq i \leq k$, and the formula will still hold. Then,

$$\begin{aligned} f(\alpha v) &= f(a_1 v_1 + 0 + 0 + \cdots + 0) = f(a_1 v_1 + a_2 v_2 + \cdots + a_k v_k) \\ &= a_1 f(v_1) + a_2 f(v_2) + \cdots + a_k f(v_k) = \alpha f(v) + 0 + \cdots + 0 \end{aligned}$$

so f has property L1. Next, let $u, v \in \mathbb{R}^n$. Then, set $v_1 \coloneqq u$, $v_2 \coloneqq v$, $a_1 \coloneqq 1$, $a_2 \coloneqq 1$, and set all the other vectors and coefficients to $a_i = 0$, $v_i = 0$ for $2 < i \leq k$. Then,

$$\begin{aligned} f(u+v) &= f(a_1 v_1 + a_2 v_2 + 0 + 0 + \cdots + 0) = f(a_1 v_1 + a_2 v_2 + a_3 v_3 + \cdots + a_k v_k) \\ &= a_1 f(v_1) + a_2 f(v_2) + a_3 f(v_3) + \cdots + a_k f(v_k) = f(v_1) + f(v_2) + 0 + \cdots + 0 \end{aligned}$$

so f has property L2. Thus, f is linear. **(2 pts for showing that linear implies formula, 3 pts for showing the formula implies linear)**