Math 241 Homework

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Complete the following problems. Fully justify each response.

1. Read Sections 4.7.3-4.7.5 in Coding the Matrix, about error-correcting codes. Using the Hamming Code and matrix H as described in Section 4.7.5, answer the following questions:

   (a) Assuming that $\mathbf{e}$ has at most one bit error, and that $H\mathbf{e} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$, what is $\mathbf{e}$?

   (b) Suppose that the vector
   $$\tilde{\mathbf{c}} = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$
   is received over a noisy channel that introduces at most one bit error, and H is used to decode it. What was the original message intended to be?

2. Consider the matrix
   $$A = \begin{bmatrix} 1 & -2 \\ 0 & 0 \end{bmatrix}.$$
   Draw a picture of the null space of A in $\mathbb{R}^2$. On the same graph, draw a picture of the solution set to the equation $A\mathbf{x} = \mathbf{b}$, where $\mathbf{b} = \begin{bmatrix} 3 \\ 0 \end{bmatrix}$.

3. Consider the matrix
   $$A = \begin{bmatrix} 1 & 2 & -3 & -1 & 2 & 1 \\ 0 & 0 & 1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix}.$$

   (a) Write the null space of A as the span of some list of vectors.

   (b) Consider the equation $A\mathbf{x} = \mathbf{b}$, where $\mathbf{b} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}$. Find one solution to this equation (you may do this by observation, and then demonstrate your solution is correct). Use the null space to write an expression for all solutions.

4. Complete Exercises 4.10.7 and 4.10.8 on pages 176-177 in Coding the Matrix.

5. Recall, from lecture, that the set of polynomials having degree at most n with coefficients from $\mathbb{R}$ is a vector space. Denote this vector space by $\mathcal{P}_n(\mathbb{R})$. Define a function $D : \mathcal{P}_n(\mathbb{R}) \to \mathcal{P}_{n-1}(\mathbb{R})$ by $D(p(x)) = \frac{d}{dx}p(x)$.

   (a) Show that this function is linear.

   (b) Describe the kernel and range of this function.

6. Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be a function, and let k be any (fixed) positive integer with $k \geq 2$. Prove that f is linear if and only if

$$f(a_1v_1 + a_2v_2 + \cdots + a_kv_k) = a_1f(v_1) + a_2f(v_2) + \cdots + a_kf(v_k)$$

   for all $a_1, a_2, \ldots, a_k \in \mathbb{R}$ and $v_1, v_2, \ldots, v_k \in \mathbb{R}^n$.

7. Complete the third problem set found at `autolab.andrew.cmu.edu`. The submission for this is directly on autolab, no need to hand it in on paper.