21-241 Homework 4

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Problem 1

Problem 4.17.8

1. $AB = \begin{bmatrix} 1 & a+b \\ 0 & 1 \end{bmatrix}$

2. $A^n = \begin{bmatrix} 1 & n \\ 0 & 1 \end{bmatrix}$

Problem 4.17.9

a) $AB = \begin{bmatrix} 0 & 0 & 2 & 0 \\ 0 & 0 & 5 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 6 & 0 \end{bmatrix}$ $\qquad BA = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 4 & 4 & 4 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

b) $AB = \begin{bmatrix} 0 & 2 & -1 & 0 \\ 0 & 5 & 3 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 6 & -5 & 0 \end{bmatrix}$ $\qquad BA = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 5 & -2 & 3 \\ 0 & 0 & 0 & 0 \\ 4 & 4 & 4 & 0 \end{bmatrix}$

c) $AB = \begin{bmatrix} 6 & 0 & 0 & 0 \\ 6 & 0 & 0 & 0 \\ 8 & 0 & 0 & 0 \\ 5 & 0 & 0 & 0 \end{bmatrix}$ $\qquad BA = \begin{bmatrix} 4 & 2 & 1 & -1 \\ 4 & 2 & 1 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

d) $AB = \begin{bmatrix} 0 & 3 & 0 & 4 \\ 0 & 4 & 0 & 1 \\ 0 & 4 & 0 & 4 \\ 0 & -6 & 0 & -1 \end{bmatrix}$ $\qquad BA = \begin{bmatrix} 0 & 11 & 0 & -2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 5 & -2 & 3 \end{bmatrix}$

e) $AB = \begin{bmatrix} 0 & 3 & 0 & 8 \\ 0 & -9 & 0 & 2 \\ 0 & 0 & 0 & 8 \\ 0 & 15 & 0 & -2 \end{bmatrix}$ $\qquad BA = \begin{bmatrix} -2 & 12 & 4 & -10 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -3 & -15 & 6 & -9 \end{bmatrix}$

f) $AB = \begin{bmatrix} -4 & 4 & 2 & -3 \\ -1 & 10 & -4 & 9 \\ -4 & 8 & 8 & 0 \\ 1 & 12 & 4 & -15 \end{bmatrix}$ $\qquad BA = \begin{bmatrix} -4 & -2 & -1 & 1 \\ 2 & 10 & -4 & 6 \\ 8 & 8 & 8 & 0 \\ -3 & 18 & 6 & -15 \end{bmatrix}$

Problem 2

a) $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. This transformation is injective and surjective, since the matrix is invertible (permuting two rows gives I_2. Alternatively, observe that for any reflection, no two points will be mapped to the same point (injective), and that any point is the reflection of its own reflection (surjective).

b) $A = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$. This transformation is injective and surjective, for the same reasons as in part (a).

c) $A = \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix}$. This transformation is not injective; observe that $f([0,0]) = f([1,-1])$. Similary, the transformation is not surjective, as the image only contains points on the line $x = y$.

Problem 3

Recall that $(AB)_{j,i} = \sum_{k=1}^{n} (A)_{jk}(B)_{ki}$. Let C be the j^{th} row of A; then $(C)_{1,i} = (A)_{j,i}$. So $(CB)_{1,i} = \sum_{k=1}^{n} (C)_{1,k}(B)_{k,i} = \sum_{k=1}^{n} (A)_{j,k}(B)_{k,i} = (AB)_{j,i}$. So the CB is in fact the j^{th} row of AB, as desired.

Problem 4

The statement is not true; to see this, take $A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, and let $B = I_2$ and $C = -I_2$. Then clearly $B \neq C$, but $AB = A = AC$, a contradiction.

If we add the hypothesis that A is invertible, then the statement will be true. To see this, observe that

$$AB = AC \implies A^{-1}(AB) = A^{-1}(AC) \implies (A^{-1}A)B = (A^{-1}A)C \implies IB = IC \implies B = C$$

Problem 5

First, recall that $(AB)^T = B^T A^T$. So we have

$$I = I^T = (A^{-1}A)^T = A^T(A^{-1})^T$$

Similarly,

$$I = I^T = (AA^{-1})^T = (A^{-1})^T A^T$$

So we see that $B = (A^{-1})^T$ satisfies both $BA^T = I$ and $A^T B = I$. Therefore, A^T is in fact invertible, and its inverse is $(A^{-1})^T$.

Problem 6

First, if $ad - bc \neq 0$, then observe that $B = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$ is an inverse, so the matrix is invertible.

Now suppose $ad - bc = 0$. By problem 10 on the review sheet, we see that $\text{Null}(A)$ is non-trivial. So the function $f(x) = Mx$ is not one-to-one; hence A is not invertible.

Note: There are many other ways to approach this problem, such as casework on whethere a is 0 or not, and using row-reduction.