Mary Radcliffe Math 241 Homework 7 Solution

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Complete the following problems. Fully justify each response.

1. Calculate each of the following determinants:

   (a) $\det\left(\begin{bmatrix} 2 & 3 \\ -1 & 1 \end{bmatrix}\right) = 2*1 - 3*(-1) = 2+3 = 5$

   (b) $\det\left(\begin{bmatrix} 1 & 2 & -1 \\ 2 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix}\right)$
   $= 1*\det\left(\begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}\right) - 2*\det\left(\begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}\right) + (-1)*\det\left(\begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix}\right)$
   $= 1*0 - 2*0 - 1*2 = -2$

   (2.5 each, 5 in total)

2. Prove that the determinant of an upper triangular matrix is equal to the product of the diagonal entries of the matrix.
   Prove by Induction: induce on the dimension of the upper triangular matrix nxn.
   BC: $n = 1$: $\det\left(\begin{bmatrix} a \end{bmatrix}\right) = a$ ✓
   n = 2: $\det\left(\begin{bmatrix} a & b \\ 0 & c \end{bmatrix}\right) = a*c - b*0 = a*c$ ✓
   IH: the determinant of an $(n-1)*(n-1)$ upper triangular matrix is equal to the product of the diagonal entries of the matrix.
   IS: NTS the claim holds for an $n*n$ upper triangular matrix. Let A be an $n*n$ upper triangular matrix:

   $$A = \begin{bmatrix} a_{11} & a_{12} & .. & .. & a_{1n} \\ 0 & a_{22} & .. & .. & a_{2n} \\ 0 & 0 & a_{33} & .. & .. \\ 0 & 0 & .. & .. & a_{nn} \end{bmatrix}$$

   $$\det(A) = a_{11}*\det\left(\begin{bmatrix} a_{22} & a_{23} & .. & a_{2n} \\ 0 & a_{33} & .. & a_{3n} \\ .. & .. & .. & .. \\ 0 & 0 & .. & a_{nn} \end{bmatrix}\right) + a_{12}*\det\left(\begin{bmatrix} 0 & a_{23} & .. & a_{2n} \\ 0 & a_{33} & .. & a_{3n} \\ .. & .. & .. & .. \\ 0 & .. & .. & a_{nn} \end{bmatrix}\right) +$$
   $$... + a_{1n}*\det\left(\begin{bmatrix} 0 & a_{22} & .. & a_{2(n-1)} \\ 0 & 0 & .. & a_{3(n-1)} \\ .. & .. & .. & .. \\ 0 & .. & .. & a_{n(n-1)} \end{bmatrix}\right)$$
   $= a_{11}*(a_{22}*a_{33}*...*a_{nn}) + a_{12}*0 + ... + a_{1n}*0$ by IH
   $= a_{11}*a_{22}*a_{33}*...*a_{nn}$ ✓
   Therefore, the determinant of an upper triangular matrix is equal to the product of the diagonal entries of the matrix.
   (1 for base case, 1 for IH, 3 for IS)

3. Complete problems 12.14.2, 12.14.3, 12.14.9 on pages 483-485 in Coding the Matrix.
12.14.2:
(a) $\begin{bmatrix} 7 & -4 \\ 2 & 1 \end{bmatrix} - \lambda_1 I = \begin{bmatrix} 7-5 & -4 \\ 2 & 1-5 \end{bmatrix} = \begin{bmatrix} 2 & -4 \\ 2 & -4 \end{bmatrix}$
$\Rightarrow 2x_1 - 4x_2 = 0 \Rightarrow x_1 = 2x_2 \Rightarrow v_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$
$\begin{bmatrix} 7 & -4 \\ 2 & 1 \end{bmatrix} - \lambda_2 I = \begin{bmatrix} 7-3 & -4 \\ 2 & 1-3 \end{bmatrix} = \begin{bmatrix} 4 & -4 \\ 2 & -2 \end{bmatrix}$
$\Rightarrow 2x_1 - 2x_2 = 0 \Rightarrow 2x_1 = x_2 \Rightarrow v_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$
(0.5 point for each eigenvector, 1 point in total)
(b) $\begin{bmatrix} 4 & 0 & 0 \\ 2 & 0 & 3 \\ 0 & 1 & 2 \end{bmatrix} - \lambda_1 I = \begin{bmatrix} 4-3 & 0 & 0 \\ 2 & 0-3 & 3 \\ 0 & 1 & 2-3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & -3 & 3 \\ 0 & 1 & -1 \end{bmatrix}$
$\Rightarrow x_1 = 0, 2x_1 - 3x_2 + 3x_3 = 0, x_2 - x_3 = 0$
$\Rightarrow x_1 = 0, x_2 = x_3$
$\Rightarrow v_1 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$
$\begin{bmatrix} 4 & 0 & 0 \\ 2 & 0 & 3 \\ 0 & 1 & 2 \end{bmatrix} - \lambda_2 I = \begin{bmatrix} 4+1 & 0 & 0 \\ 2 & 0+1 & 3 \\ 0 & 1 & 2+1 \end{bmatrix} = \begin{bmatrix} 5 & 0 & 0 \\ 2 & 1 & 3 \\ 0 & 1 & 3 \end{bmatrix}$
$\Rightarrow 5x_1 = 0, 2x_1 + x_2 + 3x_3 = 0, x_2 + 3x_3 = 0$
$\Rightarrow x_1 = 0, x_2 = -3x_3$
$\Rightarrow v_2 = \begin{bmatrix} 0 \\ -3 \\ 1 \end{bmatrix}$
(0.5 point for each eigenvector, 1 point in total)
12.14.3:
(a) $Av_1 = \lambda_1 v_1$
$\Rightarrow \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} = -1 \begin{bmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{bmatrix} \Rightarrow \lambda_1 = -1$
$Av_2 = \lambda_2 v_2$
$\Rightarrow \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 5 \\ 10 \end{bmatrix} = 5 \begin{bmatrix} 1 \\ 2 \end{bmatrix} \Rightarrow \lambda_2 = 5$
(0.5 point for each eigenvalue, 1 point in total)
(b)$Av_1 = \lambda_1 v_1$
$\Rightarrow \begin{bmatrix} 5 & 0 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \end{bmatrix} = 2 \begin{bmatrix} 0 \\ 1 \end{bmatrix} \Rightarrow \lambda_1 = 2$
$Av_2 = \lambda_2 v_2$
$\Rightarrow \begin{bmatrix} 5 & 0 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 15 \\ 5 \end{bmatrix} = 5 \begin{bmatrix} 3 \\ 1 \end{bmatrix} \Rightarrow \lambda_2 = 5$
(0.5 point for each eigenvalue, 1 point in total)
12.14.9
Let v_i be the eigenvector correspond to the eigenvalue λ_i
For an arbitrary v_i
$(A - kI)v_i = Av_i - kIv_i = Av_i - kv_i = \lambda_i v_i - kv_i = (\lambda_i - k)v_i$
Thus the eigenvalues of $A - kI$ are $\lambda_i - k, i = 1, ..., m$
(1 point)

4. Suppose that λ is an eigenvalue of the invertible matrix A having a corresponding eigenvector $\mathbf{v}$. Prove that $\frac{1}{\lambda}$ is an eigenvalue of A^{-1}. What is a

corresponding eigenvector?

$$\begin{aligned} Av &= \lambda v \\ A^{-1}Av &= A^{-1}\lambda v \\ v &= A^{-1}\lambda v \\ \frac{1}{\lambda}v &= A^{-1}v \end{aligned}$$

v is the corresponding eigenvector
(4 points for the proof, 1 points for the corresponding eigenvector)

5. Use the previous problem to prove that a square matrix A is invertible if and only if 0 is not an eigenvalue of A.
   ($\Rightarrow$) AFSOC 0 is an eigenvalue of A
   Since A is invertible, from Problem 4, we know that $\frac{1}{0}$ is an eigenvalue for A^{-1}, which is a contradiction.
   ($\Leftarrow$) If 0 is not an eigenvalue of A, there isn't a trivial solution that $Av = 0v = 0$, so A is invertible
   (2.5 for each direction, 5 in total)

6. Complete the problem set found at `autolab.andrew.cmu.edu`. The submission for this is directly on autolab, no need to hand it in on paper.