21-241 Homework 6 Solutions

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Recall the following *extremely* useful lemmas given in class:

Lemma 1 (Basis completion)**.** *If V is a finite-dimensional vector space and $S \subseteq V$ is linearly independent, then there exists a basis $\mathcal{B}$ for V with $\mathcal{B} \supseteq S$.*

Lemma 2 (Basis extraction)**.** *If S is finite and $\operatorname{span}(S) = V$, then there exists a basis $\mathcal{B} \subseteq S$ for V.*

Here are two *extremely* useful corollaries. Try first to prove them on your own! They are excellent practice problems and the proofs are very simple. I've also provided a proof at the end of these notes for your reference.

Corollary 3. *If U is a subspace of V, then $\dim U \leq \dim V$.*

Corollary 4. *Let V be a vector space of dimension n and let $S \subseteq V$ be a linearly independent set of size n. Then S is a basis of V. In particular, $\operatorname{span}(S) = V$.*

1. Suppose for contradiction that S is linearly independent. Then by lemma 1, we can find a basis $\mathcal{B} \supseteq S$ for V, so the dimension of V is

$$\dim V = |\mathcal{B}| \geq |S| = n + 1$$

which contradicts that $\dim V = n$.

(5 pts for a correct proof)

2. Let $\mathbf{v} \in \operatorname{null}(B)$, so $B\mathbf{v} = 0$. Then by multiplying both sides by A, we see that $AB\mathbf{v} = 0$ so $\mathbf{v} \in \operatorname{null}(AB)$. Thus, $\operatorname{null}(B) \subseteq \operatorname{null}(AB)$. By lemma 3, $\dim(\operatorname{null}(B)) \leq \dim(\operatorname{null}(AB))$. Now by the rank-nullity theorem, we have

$$\operatorname{rank}(B) + \dim(\operatorname{null}(B)) = \operatorname{rank}(AB) + \dim(\operatorname{null}(AB)) \geq \operatorname{rank}(AB) + \dim(\operatorname{null}(B))$$

so $\operatorname{rank}(B) \geq \operatorname{rank}(AB)$. Equality is achieved for example by setting $A = B = 1$ in which case

$$\operatorname{rank}(1) = \operatorname{rank}(AB) = \operatorname{rank}(B).$$

However, equality is not achieved if we set

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \qquad B = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

in which case

$$\operatorname{rank}\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = \operatorname{rank}(AB) = 0 < 1 = \dim\left(\operatorname{span}\left(\begin{pmatrix} 0 \\ 1 \end{pmatrix}\right)\right) = \dim(\operatorname{col}(B)) = \operatorname{rank}(B).$$

(3 pts for correct proof, 1 pt for example of equality, 1 pt for example of equality not achieved)

3. (a) True. We have 5 vectors in 3 dimensions, so homework 6 problem 1 tells us that they are linearly dependent.

   (b) True, since we may find a basis $\mathcal{B} \subseteq S$ by lemma 2, which means $\dim V = |\mathcal{B}| \leq |S| = n$.

(c) False. Consider $V = \mathbb{R}^2$ and $B = \{\mathbf{e}_1\}$. Certainly, B is linearly independent with $n = 1$, but $\dim V = 2$.

(d) True, since we then have more column vectors than there are dimensions of the column vectors, so homework 6 problem 1 again applies.

(e) False. Consider
$$A = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$
Then if $x \in \text{null}(A) \subseteq \mathbb{R}$, then
$$Ax = \begin{pmatrix} 1 \\ 0 \end{pmatrix} x = \begin{pmatrix} x \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$
so $x = 0$ and thus the null space is trivial.

(1 pt for each correct justification)

4. We will explicitly construct one such f. Let $\mathcal{V} := \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ and $\mathcal{W} := \{\mathbf{w}_1, \ldots, \mathbf{w}_n\}$ be bases for V and W, respectively. Then we define the image of $\mathbf{v} \in V$ under f as follows. Let $\mathbf{v} = \sum_{i=1}^n \alpha_i \mathbf{v}_i$ be the unique expansion of $\mathbf{v}$ in the basis $\mathcal{V}$. Then, we define $f(\mathbf{v}) = \sum_{i=1}^n \alpha_i \mathbf{w}_i$. Note that this map is well-defined since α is well-defined (uniquely determined).

   We first show that f is injective. Let $f(\mathbf{u}_1) = f(\mathbf{u}_2)$. Then, letting $\mathbf{u}_1 = \sum_{i=1}^n \alpha_i \mathbf{v}_i$ and $\mathbf{u}_2 = \sum_{i=1}^n \beta_i \mathbf{v}_i$
$$\sum_{i=1}^n \alpha_i \mathbf{w}_i = \sum_{i=1}^n \beta_i \mathbf{w}_i \implies \sum_{i=1}^n (\alpha_i - \beta_i)\mathbf{w}_i = 0 \implies \alpha - \beta = 0 \implies \alpha = \beta$$
$$\implies \mathbf{u}_1 = \sum_{i=1}^n \alpha_i \mathbf{v}_i = \sum_{i=1}^n \beta_i \mathbf{v}_i = \mathbf{u}_2.$$
   Next, we show that f is surjective. Let $\mathbf{w} = \sum_{i=1}^n \alpha_i \mathbf{w}_i \in W$ be an arbitrary vector of W uniquely expanded in the basis $\mathcal{W}$. Then clearly, $\mathbf{v} := \sum_{i=1}^n \alpha_i \mathbf{v}_i$ is a vector in V that maps to $\mathbf{w}$ under f. Thus, we conclude that f is invertible.

   Finally, we show that f is linear. If we represent $u_1, u_2 \in V$ uniquely as
$$\mathbf{u}_1 = \sum_{i=1}^n \alpha_i \mathbf{v}_i, \qquad \mathbf{u}_2 = \sum_{i=1}^n \beta_i \mathbf{v}_i,$$
   then for any scalars $c_1, c_2 \in \mathbb{F}$,
$$f(c_1\mathbf{u}_1 + c_2\mathbf{u}_2) = f\left(c_1 \sum_{i=1}^n \alpha_i \mathbf{v}_i + c_2 \sum_{i=1}^n \beta_i \mathbf{v}_i\right) = f\left(\sum_{i=1}^n (c_1\alpha_i + c_2\beta_i)\mathbf{v}_i\right)$$
$$= \sum_{i=1}^n (c_1\alpha_i + c_2\beta_i)\mathbf{w}_i = \sum_{i=1}^n c_1\alpha_i \mathbf{w}_i + \sum_{i=1}^n c_2\beta_i \mathbf{w}_i = c_1 f(\mathbf{u}_1) + c_1 f(\mathbf{u}_2)$$
   so by homework 3 problem 6, f is linear.

   (2 pts for construction of f, 1 pt for showing injectivity, 1 pt for showing surjectivity, 1 pt for showing linearity)

5. Let $A \in \mathbb{F}^{m \times n}$ be an invertible matrix. Since A is surjective onto $\mathbb{F}^m$, its rank is m. On the other hand, it's injective so its kernel is trivial and thus the nullity is 0. Finally, the dimension of the domain is n, so by the rank-nullity theorem, $m + 0 = n$, as desired.

   (4 pts for any proof, 1 pt for using the rank-nullity theorem)

6. Here are some cool corollaries of the rank-nullity theorem:

- Homework 6 problem 2.
- Consider a k-dimensional hyperplane $\mathcal{H}$ in $\mathbb{R}^n$. Recall that we may view this hyperplane as the solution set to some homogenous linear equations, i.e. $\mathcal{H} = \text{null}(A)$ where A is the matrix of the linear equations. Then by the rank-nullity theorem, we know that if the homogeneous linear equations are linearly independent, then the number of equations is $n - k$.
- Homework 6 problem 5.
- Corollary of homework 6 problem 5: all bases of $\mathbb{F}^n$ have the same cardinality (alternate proof, might be circular but whatever). Indeed, putting the basis vectors in the columns of a matrix A clearly creates an injective and surjective linear map with n rows, since the basis vectors are in $\mathbb{F}^n$. Thus, A is an invertible map with n rows, so A must have n columns and thus there must be n basis vectors.
- Quick proof that a matrix is injective: we can quickly check whether a matrix is injective or not. Before this theorem, we first reduced A to upper triangular form A' and then checked for the dimension of the kernel by solving $A'x = 0$. Now, we just count the number of linearly independent columns of A', subtract from n, and bam we're done.
- A generalization of homework 6 problem 5: any two of the three following conditions implies the remaining condition:
  (a) $A \in \mathbb{R}^{n \times n}$
  (b) the linear map corresponding to A is injective
  (c) the linear map corresponding to A is surjective

  Homework 6 problem 5 shows that (b) and (c) implies (a). If we have (a) and (c), then $n + \dim(\text{null}(A)) = n$ so $\dim(\text{null}(A)) = 0$. Then, the kernel must be trivial and thus we have (b). If we have (a) and (b), then $\text{rank}(A) + 0 = n$. Then by corollary 4, $\text{col}(A) = \mathbb{F}^n$, the entire codomain, so we have (c).

(approximately 1 pt for each reason)

Proof of Corollaries

Proof of corollary 3. Let S be a basis of U. Then by lemma 1, we may find a basis B of V such that $B \supseteq S$. Thus,

$$\dim U = |S| \leq |B| = \dim V$$

as desired. □

Proof of corollary 4. Let $\mathcal{B} \supseteq S$ be the basis given to us by lemma 1. Then,

$$n = |S| \leq |B| = n$$

and $S \subseteq \mathcal{B}$, so it must be that $S = \mathcal{B}$ and thus S is a basis. □