Math 241 Homework

Mary Radcliffe

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Complete the following problems. Fully justify each response.

1. Let V be a vector space of dimension n. Prove that if $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_{n+1}\}$, then S is linearly dependent.

2. Let A and B be $n \times n$ matrices. Prove that $\mathrm{rank}\,(AB) \leq \mathrm{rank}\,(B)$. Give an example to show that equality can hold. Give an example to show that equality may not hold.

3. Mark each of the following as true or false. Give a reason for your answer.

   (a) If A is a 3×5 matrix, then the columns of A must be linearly dependent.

   (b) If V is a vector space, and $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ spans V, then $\dim V \leq n$.

   (c) If V is a vector space, and $B = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is a linearly independent subset of V, then $\dim V \leq n$.

   (d) If A has more columns than rows, then $\mathrm{Null}\,(A)$ is nontrivial.

   (e) If A has more rows than columns, then $\mathrm{Null}\,(A)$ is nontrivial.

4. Suppose that V and W are both vector spaces of dimension n over the field F. Show that there exists an invertible linear function $f : V \to W$.

5. Use the Rank-Nullity Theorem to prove that every invertible matrix is square.

6. Spend a few minutes reading back over the Rank-Nullity Theorem, and thinking about why we might care about a theorem like this. Write 5 reasons why the Rank-Nullity Theorem could be useful in terms of deriving practical information about a matrix.