21-241 Homework 5 solutions

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Problem 1

5.14.4

(a) $[0, 0, 1, 1, 0, 0, 0, 0] = v_1 + v_2 + v_3$
(b) $[0, 0, 0, 0, 0, 1, 1, 0] = v_7 + v_8$
(c) $[1, 0, 0, 0, 1, 0, 0, 0] = v_3 + v_6$
(d) $[0, 1, 0, 1, 0, 0, 0, 0] = v_1 + v_3$

5.14.10

(a) $v_2 + v_4 + v_5 = 0$
(b) $v_2 + v_4 + v_5 = 0$
(c) $v_1 + v_3 + v_4 + v_6 = 0$
(d) $v_1 + v_2 + v_3 + v_5 + v_6 = 0$

Problem 2

(a) This set is not linearly independent, since

$$\begin{bmatrix} 3 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} + \begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix} - \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

It is easy to verify that removing this vector makes the set linearly independent.

(b) This set is in fact linearly independent. It is easy to see that the only solution to the system of equations

$$\begin{aligned} x_1 + x_3 &= 0 \\ x_2 + x_4 &= 0 \\ x_1 - x_3 &= 0 \\ x_2 - x_4 &= 0 \end{aligned}$$

is $x_1 = x_2 = x_3 = x_4$.

(c) This set is also linearly independent. Indeed, we only need to verify that the system of equations

$$\begin{aligned} x_1 + x_3 &= 0 \\ x_2 + x_3 &= 0 \\ -x_1 - x_2 &= 0 \\ x_1 + x_2 &= 0 \end{aligned}$$

admits only the trivial solution.

Problem 3

(a) This set is linearly independent. To see this observe that

$$\alpha_1(a+b)+\alpha_2(b+c)+\alpha_3(c+a)=0 \implies (\alpha_1+\alpha_3)a+(\alpha_1+\alpha_2)b+(\alpha_2+\alpha_3)c=0$$

Which yields the system $\alpha_1+\alpha_2=\alpha_2+\alpha_3=\alpha_3+\alpha_1=0$. Since the only solution to this system is $\alpha_1=\alpha_2=\alpha_3=0$, the set is linearly independent.

(b) Observe that

$$-(c-a)=a-c=a-b+b-c=(a-b)+(b-c)$$

And therefore the set is not linearly independent.

Problem 4

We can denote without loss of generality the sets as $S=\{v_1,\cdots v_n\}$ and $A=\{v_1,\cdots v_k\}$ for $k<n$. Then, for any $z\in \text{Span } S$ that satisfies the conditions, we have that since $A\cup\{z\}$ is linearly independent,

$$\text{If } z=\sum_{i=1}^{n}\alpha_i v_i \implies \exists\, m>k \text{ such that } \alpha_j\neq 0$$

We can thus set $w=v_j$, and then we have that

$$x\in \text{Span } S \implies x=\sum_{i=1}^{n}\beta_i v_i \implies x=\sum_{\substack{i=1\\ i\neq j}}^{n}\beta_i v_i+\frac{\beta_j}{\alpha_j}\left(z-\sum_{\substack{i=1\\ i\neq j}}^{n}\alpha_i v_i\right) \implies x\in \text{Span }\{S\setminus\{w\}\cup\{z\}\}$$

And similarly

$$x\in \text{Span }\{S\setminus\{w\}\cup\{z\}\} \implies x=\sum_{\substack{i=1\\ i\neq k}}^{n}\gamma_i v_i+\gamma_0 z=\sum_{\substack{i=1\\ i\neq j}}^{n}\gamma_i v_i+\gamma_0\sum_{i=1}^{n}\alpha_i v_i\in \text{Span } S$$

by the closure of the span.

Problem 5

The required matrix is

$$\begin{bmatrix}-1 & 1 & 1\\ 1 & -1 & 1\\ 1 & 1 & -1\end{bmatrix}\begin{bmatrix}1 & 1 & 0\\ 1 & 0 & 1\\ 0 & 1 & 1\end{bmatrix}^{-1}=\begin{bmatrix}-0.5 & -0.5 & 1.5\\ -0.5 & 1.5 & -0.5\\ 1.5 & -0.5 & -0.5\end{bmatrix}$$

Problem 6

We will prove this by induction. Observe that for $k=1$, we have that the set $\{v_1\}$ is clearly linearly independent.
Suppose the set $\{v_1,v_2,\cdots v_k\}$ is linearly independent. We want to show that the set $\{v_1,\cdots v_{k+1}\}$ is also linearly independent.

Suppose not. Then we have that

$$v_{k+1} = \sum_{i=1}^{k} \alpha_i v_i \implies Av_{k+1} = A\left(\sum_{i=1}^{k} \alpha_i v_i\right) \implies \lambda_{k+1} v_{k+1} = \sum_{i=1}^{k} \lambda_k \alpha_k v_k$$

From the assumption, we have that

$$\lambda \sum_{i=1}^{k} \alpha_i v_i = \sum_{i=1}^{k} \lambda_i \alpha_i v_i \implies \sum_{i=1}^{k} \alpha_i (\lambda_{k+1} - \lambda_i) v_i = 0$$

Therefore, by linear independence, we have that $\alpha_i(\lambda_{k+1} - \lambda_i)$ is zero for all i. To finish of the proof, we observe that since v_{k+1} is non-zero by assumption, we have that there is a non-zero α_j, and therefore, we have $\lambda_j = \lambda_{k+1}$ which is a contradiction.