Math 241 Homework

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Complete the following problems. Fully justify each response.

1. Read sections 5.4.1, 5.4.3, and 5.5.3 in Coding the Matrix about Minimum Spanning Forest. Then complete exercises 5.14.4 and 5.14.10 on pages 254-255.

2. For each of the following sets of vectors over $\mathbb{R}$, determine if the set is linearly dependent or independent. If it is linearly dependent, find a maximum subset that is linearly independent.

   (a) $\left\{ \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ -1 \end{bmatrix} \right\}$

   (b) $\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \end{bmatrix} \right\}$

   (c) $\left\{ \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} \right\}$

3. Suppose that $\{\mathbf{a}, \mathbf{b}, \mathbf{c}\}$ is a set of linearly independent vectors.

   (a) Is $\{\mathbf{a}+\mathbf{b}, \mathbf{b}+\mathbf{c}, \mathbf{a}+\mathbf{c}\}$ linearly independent? Either prove that it is, or give a counterexample to demonstrate that it is not.

   (b) Is $\{\mathbf{a}-\mathbf{b}, \mathbf{b}-\mathbf{c}, \mathbf{a}-\mathbf{c}\}$ linearly independent? Either prove that it is, or give a counterexample to demonstrate that it is not.

4. Let S be a set of vectors, and let $A \subseteq S$. Suppose $\mathbf{z} \in \text{Span}\,\{S\}$, and that $A \cup \{\mathbf{z}\}$ is linearly independent. Prove that there is a vector $\mathbf{w} \in S \backslash A$ such that $\text{Span}\,\{S\} = \text{Span}\,\{S \backslash \{\mathbf{w}\} \cup \{\mathbf{z}\}\}$.

   NOTE: Basically, what we are saying in this proposition is that we can exchange vector $\mathbf{z}$ for $\mathbf{w}$ in S, without changing the span. The part about set A is essentially to protect a set of vectors in S from being replaced by this process. So we have a set S, some vectors we want to keep (namely A), and a vector $\mathbf{z}$ we want to include. The lemma says that there's some $\mathbf{w} \in S \backslash A$ such that we can switch out $\mathbf{w}$ for $\mathbf{z}$ under the condition that $A \cup \{\mathbf{z}\}$ is linearly independent. This condition, of course, is not necessary, but it is sufficient.

5. Write a change of basis matrix that will change coordinates from $\mathcal{B}$ to $\mathcal{C}$, two different bases for $\mathbb{R}^3$, listed below.

$$\mathcal{B} = \left\{ \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} \right\}$$

$$\mathcal{C} = \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\}$$

6. Let A be an $n \times n$ matrix over $\mathbb{R}$, and suppose $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k$ are nonzero vectors satisfy the following:

   - For each i, $A\mathbf{v}_i = \lambda_i \mathbf{v}_i$ for some $\lambda_i \in \mathbb{R}$
   - If $i \neq j$, then $\lambda_i \neq \lambda_j$

   Prove that $\{\mathbf{v}_1, \dots, \mathbf{v}_k\}$ is linearly independent.

   (Hint: try induction on k).

7. Complete the fifth problem set found at `autolab.andrew.cmu.edu`. The submission for this is directly on autolab, no need to hand it in on paper.