Math 241 Homework

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Due 20 November 2018

Complete the following problems. Fully justify each response.

This homework set can be turned in during lecture on 19 November, or may be turned in to Mary's office between 11:00 and 2:00 on Tuesday, 20 November. If these turn in times are not possible for you, please speak to your TA about an alternative.

1. An unofficial programming exercise: Use any language that has a linear algebra package to look at the singular value decomposition of some matrices. Try a few square matrices, a few $m \times n$ with $m < n$ and a few $m \times n$ with $m > n$. Get a feel for what these things look like. In general they are a lot of work to compute by hand, so try to get a sense of what this is like with computer. Write a few sentences detailing your observations.

2. Let A be a matrix with a SVD $A = U\Sigma V^T$, so Σ is a diagonal matrix containing the singular values of A.

   (a) Recalling that, by definition, V is a matrix with orthogonal columns, show that $V^T = V^{-1}$.

   (b) Use the previous part to show that σ is a singular value of A if and only if σ^2 is an eigenvalue of the square matrix A^TA, and the eigenvector corresponding to σ^2 for A^TA is the singular vector corresponding to σ for A.

3. Use the previous problem to find a singular value decomposition for the matrix $A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$.

4. Suppose that A is a square, invertible matrix, having singular value decomposition $A = \sum_{i=1}^n \sigma_i \mathbf{u}_i \mathbf{v}_i^T$. Prove that $A^{-1} = \sum_{i=1}^n \frac{1}{\sigma_i} \mathbf{v}_i \mathbf{u}_i^T$.

5. Suppose that $A = U\Sigma V^T$, where $\sigma_\ell = 0$ for $\ell \geq k$. Show that $\mathrm{Col}\,(A)$ is equal to the span of the first $\ell - 1$ vectors in U; that is, the span of all left singular vectors corresponding to nonzero singular values.

6. Give at least 3 reasons you might care about singular values of a matrix. What kind of things can you do with them?