# Math 228: Solving linear recurrence with eigenvectors

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## 1 Example

I'll begin these notes with an example of the eigenvalue-eigenvector technique used for solving linear recurrence we outlined in class. Since all the recurrences in class had only two terms, I'll do a three-term recurrence here so you can see the similarity.

Let us consider the recurrence $a_n = 2a_{n-1} + a_{n-2} - 2a_{n-3}$, subject to $a_0 = 2, a_1 = 2, a_2 = 4$.

As in class, define the vector $\mathbf{v}_n$ as

$$\mathbf{v}_n = \begin{pmatrix} a_n \\ a_{n-1} \\ a_{n-2} \end{pmatrix}.$$

Notice that

$$\mathbf{v}_{n+1} = \begin{pmatrix} a_{n+1} \\ a_n \\ a_{n-1} \end{pmatrix} = \begin{pmatrix} 2a_n + a_{n-1} - 2a_{n-2} \\ a_n \\ a_{n-1} \end{pmatrix} = \begin{bmatrix} 2 & 1 & -2 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{pmatrix} a_n \\ a_{n-1} \\ a_{n-2} \end{pmatrix}.$$

By defining $A = \begin{bmatrix} 2 & 1 & -2 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$, we thus have that $\mathbf{v}_{n+1} = A\mathbf{v}_n$. Therefore, since $\mathbf{v}_2$ is the first vector, we have $\mathbf{v}_{n+1} = A^{n-1}\mathbf{v}_2$

Hence, we would like to represent $\mathbf{v}_n$ in terms of the eigenvectors of $A$. If we are able to do so, then we can calculate a formula for $\mathbf{v}_n$, and hence for $a_n$.

First, let's calculate eigenvectors of $A$. Note that

$$\left| \begin{bmatrix} 2-\lambda & 1 & -2 \\ 1 & -\lambda & 0 \\ 0 & 1 & -\lambda \end{bmatrix} \right| = (2-\lambda)(\lambda^2) - 1(-\lambda) - 2(1) = -\lambda^3 + 2\lambda^2 + \lambda - 2 = (\lambda^2 - 1)(2 - \lambda).$$

Hence, the eigenvalues of $A$ are $\lambda_1 = 1, \lambda_2 = -1$, and $\lambda_3 = 2$.

Next we calculate eigenvectors of $A$ using standard row reduction techniques:

$$\lambda_1 = 1 : A - \lambda I = \begin{bmatrix} 1 & 1 & -2 \\ 1 & -1 & 0 \\ 0 & 1 & -1 \end{bmatrix} \xrightarrow{R2-R1} \begin{bmatrix} 1 & 1 & -2 \\ 0 & -2 & 2 \\ 0 & 1 & -1 \end{bmatrix}$$

$$\xrightarrow{R3-\frac{1}{2}R2, R2/2} \begin{bmatrix} 1 & 1 & -2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\xrightarrow{R1-R2} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

Examining coefficients, we thus obtain that the first coordinate is the negative of the third, and the second coordinate is also the negative of the third. Hence, the desired eigenvector is $\mathbf{x}_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$.

Without detailing the computations for the remaining eigenvectors, we obtain

$$\text{for } \lambda_2 = -1, \mathbf{x}_2 = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}; \qquad \text{for } \lambda_3 = 2, \mathbf{x}_3 = \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix}.$$

We therefore wish to write the initial vector $\mathbf{v}_2$ in terms of $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3$, as follows:

$$\mathbf{v}_2 = \begin{pmatrix} a_2 \\ a_1 \\ a_0 \end{pmatrix} = \begin{pmatrix} 4 \\ 2 \\ 2 \end{pmatrix} = c_1\mathbf{x}_1 + c_2\mathbf{x}_2 + c_3\mathbf{x}_3 = c_1 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} + c_3 \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix}.$$

Hence, we consider the linear system $c_1 + c_2 + 2c_3 = 4, c_1 - c_2 + 2c_3 = 2, c_1 + c_2 + c_3 = 2$. Without detailing the algebra, we obtain the solution to this system is given by $c_1 = -1$, $c_2 = 1$, $c_3 = 2$.

Therefore, we have

$$\begin{aligned}
\mathbf{v}_n = \begin{pmatrix} a_n \\ a_{n-1} \\ a_{n-2} \end{pmatrix} &= A^{n-2}\mathbf{v}_2 \\
&= A^{n-2}\left(-\mathbf{x}_1 + \mathbf{x}_2 + 2\mathbf{x}_3\right) \\
&= -A^{n-2}\mathbf{x}_1 + A^{n-2}\mathbf{x}_2 + 2A^{n-2}\mathbf{x}_3 \\
&= -(1)^{n-2}\mathbf{x}_1 + (-1)^{n-2}\mathbf{x}_2 + 2(2)^{n-2}\mathbf{x}_3 \\
&= -\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + (-1)^n \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} + 2^{n-1} \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix}
\end{aligned}$$

Noting that the first term of the vector is indeed $a_n$, we thus obtain

$$a_n = -1 + (-1)^n + 2^n.$$

## 2 Theory

In general, this technique will work with any recurrence relation that takes the form

$$a_n = \alpha_1 a_{n-1} + \alpha_2 a_{n-2} + \cdots + \alpha_k a_{n-k} + p(n),$$

where $p(n)$ is a polynomial in $n$. We here sketch the theoretical underpinnings of the technique, in the case that $p(n) = 0$.

Imagine a recurrence relation takin the form $a_n = \alpha_1 a_{n-1} + \alpha_2 a_{n-2} + \cdots + \alpha_k a_{n-k}$, where the $\alpha_i$ are constants and the first $k$ values of the sequence $(a_n)$ are known.

Write $\mathbf{v}_n = \begin{pmatrix} a_n \\ a_{n-1} \\ \vdots \\ a_{n-k+1} \end{pmatrix}$. Note that the first vector defined in this way will always be $\mathbf{v}_{k-1} =$

$\begin{pmatrix} a_{k-1} \\ a_{k-2} \\ \vdots \\ a_0 \end{pmatrix}$, which is known by the initial conditions.

As in the example above, our goal is to construct a matrix $A$ so that $A\mathbf{v}_n = \mathbf{v}_{n+1}$. We note that

$$\mathbf{v}_{n+1} = \begin{pmatrix} a_{n+1} \\ a_n \\ \vdots \\ a_{n-k+2} \end{pmatrix} = \begin{pmatrix} \alpha_n a_n + \alpha_2 a_{n-1} + \cdots + \alpha_k a_{n-k+1} \\ a_n \\ \vdots \\ a_{n-k+2} \end{pmatrix},$$

and hence we may take $A = \begin{bmatrix} \alpha_1 & \alpha_2 & \alpha_3 & \cdots & \alpha_{k-1} & \alpha_k \\ 1 & 0 & 0 & \dots & 0 & 0 \\ 0 & 1 & 0 & \dots & 0 & 0 \\ & & & \vdots & & \\ 0 & 0 & 0 & \cdots & 1 & 0 \end{bmatrix}$.

Suppose that $A$ is diagonalizable, and has the eigenvalue-eigenvector pairs $(\lambda_1, \mathbf{x}_1)$, $(\lambda_2, \mathbf{x}_2), \dots, (\lambda_k, \mathbf{x}_k)$. Write $\mathbf{v}_{k-1}$, our known constant vector, as a linear combination of $\mathbf{x}_1, \dots, \mathbf{x}_k$, in the following form:

$$\mathbf{v}_{k-1} = c_1\mathbf{x}_1 + c_2\mathbf{x}_2 + \cdots + c_k\mathbf{x}_k.$$

Note that since $A$ is diagonalizable, it is necessarily true that there exists constants $c_1, \dots, c_k$ such that this equation holds. Moreover, as in the above example, we have that $\mathbf{v}_n = A^{n-k+1}\mathbf{v}_{k-1}$, and hence

$$\mathbf{v}_n = A^{n-k+1}\mathbf{v}_{k-1} = A^{n-k+1}(c_1\mathbf{x}_1 + c_2\mathbf{x}_2 + \cdots + c_k\mathbf{x}_k) = c_1\lambda_1^{n-k+1}\mathbf{x}_1 + c_2\lambda_2^{n-k+1}\mathbf{x}_2 + \cdots + c_k\lambda_k^{n-k+1}\mathbf{x}_k.$$

Noting that $a_n$ is the first coordinate of $\mathbf{v}_n$, we can then read off the first coordinate of the vector to obtain a formula for $a_n$.