Math 228: Homework 1

Mary Radcliffe

due 9 Sept 2016

Complete the following problems. Fully justify each response. You need only turn in those problems marked with a (*).

1. (*) Let $n \in \mathbb{Z}^+$. We call a subset of $[n]$ odd if it has odd size, and even if it has even size.

   Prove that the number of odd subsets of $[n]$ is equal to the number of even subsets of $[n]$.

2. (*) You have 20 different presents to distribute among 10 children. You may distribute presents in any way; it is even permissible that all the presents go to the same child. How many ways can the presents be distributed?

3. Repeat problem 2, but add the restriction that each child must get at least one present.

4. n boys and n girls all go to a dance. In how many ways can they partner up to dance together, assuming that each partnership is of one boy and one girl?

5. (*) Prove each of the following binomial identities in two ways; one using algebra, and one using combinatorics.

   (a) $\binom{n}{2} + \binom{n+1}{2} = n^2$

   (b) $\binom{n}{k}\binom{k}{j} = \binom{n}{n-j}\binom{n-j}{k-j}$

6. (*) You have 10 different colored balls in a single pile. You perform the following procedure:

   i. Split the balls into two piles (in any way)

   ii. Select a pile containing at least 2 balls (if it exists)

   iii. Split that pile into two piles (in any way)

   iv. Repeat steps ii & iii until you cannot

   How many steps will you take until the procedure is completed? In how many different ways can this be done?

7. A 4-digit number is called a palindrome if it is the same when the digits are read in reverse. For example, 7337 and 3333 are 4-digit palindromes, but 1337 and 0990 are not. Note that 0990 doesnt count because its actually a 3-digit number.

   A 4-digit number is called an almost-palindrome if there is a way to change exactly one digit so that the result is a 4-digit palindrome. For example, 1337, 1501, and 1990 are 4-digit almost-palindromes (they could become

1331 or 7337, 1001 or 1551, and 1991), but 1234, 0991, and 1331 are not. The issue with 0991 is again that it is actually a 3-digit number, and the issue with 1331 is that if you change any digit, then it becomes a non-palindrome.

How many 4-digit almost-palindromes are there?